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On the Maximum Number of Equilateral Triangles II

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ABSTRACT

Erdős and Purdy raised the problem of finding the maximum number of equilateral triangles determined by a set of n points in $\mathbb{R}^d$. This question is investigated in the first part of this series. Here we study some variations where the sets in consideration are in convex or general position. Non trivial bounds are given for these problems, as well as for the corresponding questions where the triangles at issue have unit length side.

1 Introduction

In this article, the second and last part of this series, we explore some interesting variations of the following question raised by Erdős and Purdy [5],[7]-[9], (see also [3]).

> "What is the maximum number of equilateral triangles that can be determined by n points in the plane?"

This problem, together with its natural generalizations to higher dimensions, is treated in the first part of this series [1]. For the second part we impose additional conditions on the point sets, e.g. sets in convex or general position.

These type of restricted problems have been studied before for the famous "unit distance" and "different distances" problems (see [4],[12]). But even with these extra conditions most of these problems remain open.

For an arbitrary finite set $P \subseteq \mathbb{R}^d$, we say P is in *convex position* if it forms the vertex set of a convex polytope, we also say that P is in *general position* if there are no $d+1$ points in a $(d-1)$-dimensional hyperplane and no $d+2$ points in a $(d-1)$-dimensional sphere. Denote by $E(P)$ $(F(P))$ the number of triplets in P that are the vertices of an equilateral (unit equilateral) triangle. We define

$$E_d(n) = \max_{\substack{|P|=n \\ P\subseteq\mathbb{R}^d}} E(P),\ F_d(n) = \max_{\substack{|P|=n \\ P\subseteq\mathbb{R}^d}} F(P) \tag{1}$$

and $E_d^{conv}(n)$, $F_d^{conv}(n)$, $E_d^{gen}(n)$, $F_d^{gen}(n)$ as generalizations of the functions $E_d(n)$ and $F_d(n)$ where the maxima are taken over point sets in convex or general position. For simplicity when $d = 2$ we avoid writing the subscript on all these functions (e.g. $E^{conv}(n) = E_2^{conv}(n)$).

In section 2 we find the correct order of magnitude for $E^{conv}(n)$ and $F^{conv}(n)$, as well as a lower bound for $F_3^{conv}(n)$. Section 3 is devoted to the construction of sets in general position with a large number of equilateral triangles. These examples provide non trivial lower bounds for the functions $E_d^{gen}(n)$ and an even better bound for $E^{gen}(n)$. Finally in section 4 we obtain the asymptotic value of $F_d^{gen}(n)$ for all dimensions d, and the exact values of $F^{gen}(n)$ and $F_3^{gen}(n)$.

Let us remark that in [1] it was proved that $E(n)$ is a discrete function in the sense that the maximum in (1) can be taken just over subsets of a fixed regular triangle lattice. The proof of this fact can be easily modified to conclude that the same statement holds for the functions $E^{conv}(n)$ and $E^{gen}(n)$.

Throughout the paper we use the following notation. For any P finite set, $G = (P, \Delta(P))$ denotes the 3-uniform hypergraph with vertex set P, where $\Delta(P)$ is the set of triplets in P determining equilateral triangles. For $\Delta' \subseteq \Delta(P)$ and $x \in P$ we denote by $\deg_{\Delta'}(x)$ the number of triangles in Δ' with x as one of its vertices. For sake of brevity $\deg_\Delta(x) = \deg_{\Delta(P)}(x)$, and in the special case when $\Delta_u(P)$ denotes the set of unit equilateral triangles determined by P we write $\deg_{\Delta_u(P)}(x) = \deg_\Delta^u(x)$. All triangles and angles in this paper are named in counter-clockwise order. Also we often identify the euclidean plane with the field of complex numbers.

2 Sets in Convex Position

In 1970 Erdős and Moser [6] conjectured that the maximum number of unit distances $f^{conv}(n)$ among n points in convex position is at most cn for some constant c. Erdős conjectured even further that there is a natural number k with the property that every convex n-gon has a vertex with at most k other vertices equidistant from it [12].

The best result along these lines is by Füredi [11] who proved that $f^{conv}(n) = O(n \log n)$. From this we deduce that $F^{conv}(n) = O(n \log n)$. In this section we refine this bound and prove that both $F^{conv}(n)$ and $E^{conv}(n)$ are functions of linear order. To accomplish this we use the following lemma, which is closely related to the second Erdős conjecture mentioned before.

Lemma 1 *For any finite point set P in convex position and $\Delta' \subseteq \Delta(P)$, if T is a triangle in Δ' with smallest possible sides then T has a vertex with* $\deg_{\Delta'} = 1$.

Proof. Let $\mathcal{C}$ be the solid convex polygon determined by P and let $T = p_1p_2p_3$. Suppose by contradiction that $\deg_{\Delta'}(p_j) \geq 2$ for $j = 1, 2, 3$. Distinguish two cases.

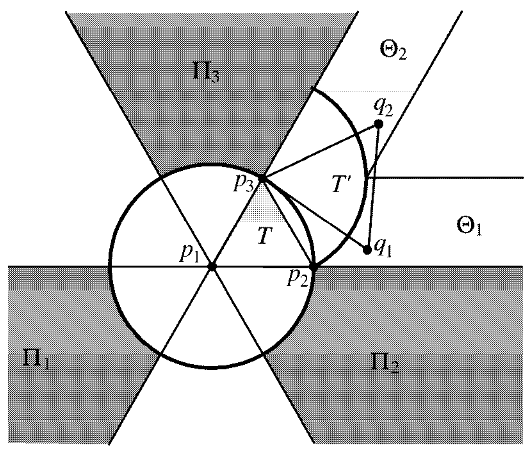

Figure 1: Illustration for proof of Case 1, Lemma 1.

Case 1. There is a triangle $T' \in \Delta'$ such that the solid triangles T and T' have only one point in common. Suppose $T' = p_3q_1q_2$, also assume by symmetry that T' is contained in the cone spanned by the angle $\measuredangle p_3p_1p_2$. Define the closed regions Π_1, Π_2, Π_3 and Θ_1, Θ_2 according to Figure 1. By minimality of T we have that $q_1 \in \Theta_1$ and $q_2 \in \Theta_2$.

Now, by assumption $\deg_{\Delta'}(p_1) \geq 2$, thus there is a triangle $p_1r_1r_2 \in \Delta'$. Since, by convexity, there are no points of $\mathcal{C}$ in Π_1; then either r_1 or r_2 is in $\Pi_2 \cup \Pi_3$. Let r be the point with such property.

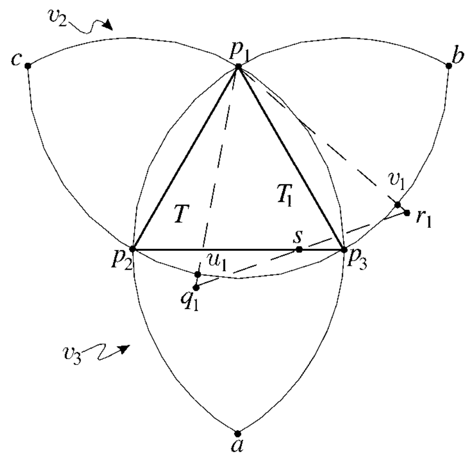

Figure 2: Illustration for the proof of Case 2, Lemma 1.

If $r \in \Pi_3$ then p_3 would be contained in the solid closed triangle rp_1q_2. Similarly, when $r \in \Pi_2$ we would have p_2 contained in the solid closed triangle rq_1p_1 (except when $q_1 = p_2$, in which case p_2 would be contained in rq_2p_1). In any case we get a contradiction.

Case 2. Any solid triangle $T' \in \Delta'$ with one of its vertices in $\{p_1, p_2, p_3\}$ intersects the interior of T. By assumption there are triangles $T_j = p_jq_jr_j$, $j = 1, 2, 3$ in Δ' intersecting T in its interior. Let C_1, C_2, C_3 be the $|p_1 - p_2|$ radius circles with centers p_1, p_2 and p_3 respectively. Define a, b, c as the intersections of these circles according to Figure 2.

Let $u_j = p_jq_j \cap C_j$ and $v_j = p_jr_j \cap C_j$ $(j = 1, 2, 3)$, these points are well defined by minimality of T. By symmetry we can assume that u_1 lies in the arc p_2p_3 and v_1 is in the arc p_3b. There is no point $x \in \mathcal{C}$ in the arc ap_3, otherwise p_3 would be an interior point of xv_1p_2. Thus u_2 and v_2 are in the arcs p_3p_1 and p_1c respectively. By analogy u_3 and v_3 are in the arcs p_1p_2 and p_2a respectively.

By construction $\measuredangle v_3p_2p_3 > \pi/3$. On the other hand

$$(\measuredangle q_1p_2p_3 + \pi/3) + \pi/3 < \measuredangle q_1p_2p_1 + \measuredangle p_1p_3q_1 = \measuredangle r_1p_3p_1 + \measuredangle p_1p_3q_1 = \measuredangle r_1p_3q_1$$

and by convexity $\measuredangle r_1p_3q_1 < \pi$. Hence $\measuredangle q_1p_2p_3 < \pi/3 < \measuredangle v_3p_2p_3$. Again by convexity, q_1 should be outside the solid triangle $p_2p_3v_3$. Thus

$$\measuredangle p_1p_3q_3 = \measuredangle p_2p_3v_3 < \measuredangle p_2p_3q_1. \tag{2}$$

Let $s = q_1r_1 \cap p_2p_3$. Since $\measuredangle\ sp_2p_1 = \measuredangle\ sq_1p_1 = \frac{\pi}{3}$ then p_1, p_2, q_1, s are cocyclic points therefore

$$\measuredangle\ p_2p_3q_1 < \measuredangle\ p_2sq_1 = \measuredangle\ p_2p_1q_1. \tag{3}$$

From (2) and (3) we conclude that $\measuredangle\ p_1p_3q_3 < \measuredangle\ p_2p_1q_1$; by symmetry we also have $\measuredangle\ p_2p_1q_1 < \measuredangle\ p_3p_2q_2$ and $\measuredangle\ p_3p_2q_2 < \measuredangle\ p_1p_3q_3$ which is a contradiction. ■

Theorem 1 *For every $n \geq 3$*

1. $E^{conv}(n) \leq n - 2$
2. $\left\lfloor \frac{2}{3}(n-1) \right\rfloor \leq F^{conv}(n) \leq n - \frac{n}{4\pi \log_2 n}$.

Proof. Let P be an n-point set in convex position.

(1) Let $x \in P$ be the vertex given by Lemma 1 with $\Delta' = \Delta(P)$. Then $E(P) = 1 + E(P - \{x\}) \leq 1 + E^{conv}(n-1)$ for all P, and thus the result follows by induction.

(2) We prove the upper bound by induction on n. True for $n = 3$. Also clear by induction if there is a vertex $x \in P$ with $\deg^u_\Delta(x) = 0$. Hence we may assume that $\deg^u_\Delta(x) \geq 1$ for all $x \in P$. By Lemma 1 we know that all triangles in $\Delta_u(P)$ have a vertex of $\deg^u_\Delta = 1$.

If there is a triangle with at least two of its vertices of $\deg^u_\Delta = 1$, then by deleting these two vertices the result would follow by induction, so we can further assume that all triangles in $\Delta_u(P)$ have exactly one vertex of $\deg^u_\Delta = 1$. Construct the directed subgraph $H = (V(H), A(H))$ where $V(H)$ is the set of points in P with $\deg^u_\Delta \geq 2$ and for every triangle xyz in $\Delta_u(P)$, being x the vertex with $\deg^u_\Delta = 1$, we include the arc $\overrightarrow{yz}$ in $A(H)$. Observe that $|A(H)| = |\Delta_u(P)|$ and $|V(H)| = n - |\Delta_u(P)|$. Füredi proved [11] that among the vertices of a convex n sided polygon the unit distance occurs at most $\pi n (2\log_2 n - 1)$ times, applying this to the convex polygon obtained from $V(H)$ we find that

$$|\Delta_u(P)| = |A(H)| \leq 2\pi (n - |\Delta_u(P)|)(2\log_2(n - |\Delta_u(P)|) - 1)$$

therefore $F(P) = |\Delta_u(P)| \leq n - (1/4\pi) n / \log_2 n$.

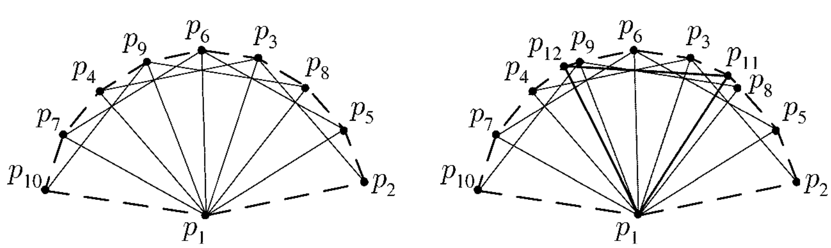

Figure 3: n sided polygons with $\lfloor \frac{2}{3}(n-1) \rfloor$ equilateral triangles.

For the lower bound we construct the following example (see Figure 3). Let $m = \lceil (n-1)/3 \rceil$. Choose a unit segment p_1p_3 and points $p_6, p_{12}, \ldots, p_{3m}$ such that

$$0 < \measuredangle p_3p_1p_6 < \measuredangle p_3p_1p_9 < \ldots < p_3p_1p_{3m} < \pi/3.$$

For $1 \leq k \leq \lceil (n-1)/3 \rceil$, construct the rhomb $p_1p_{3k-1}p_{3k}p_{3k+1}$ where the triangles $p_1p_{3k-1}p_{3k}$ and $p_1p_{3k}p_{3k+1}$ are equilateral. Then the set $\{p_1, p_2, \ldots, p_n\}$ is convex and it determines $\left\lfloor \frac{2}{3}(n-1) \right\rfloor$ unit equilateral triangles. ■

It is very likely that $F^{conv}(n) = \lfloor (2/3)\,(n-1) \rfloor$, but we could not prove this. One may be tempted to think that the multigraph obtained by undirecting the digraph H above, does not contain any cycles of length ≥ 3; but the example in Figure 4 disproves this. This set also achieves the lower bound in the last theorem. Notice that, according to the previous proof, the upper bound $F^{conv}(n) \leq (2c/(2c+1))n$ would follow from the conjecture $f^{conv}(n) \leq cn$.

For higher dimensions it is easy to see that, whenever the finite sets $A \subseteq \mathbb{R}^{d_1}, B \subseteq \mathbb{R}^{d_2}$ are in convex position and the origin belongs to the interior of their convex hulls, then the set $\{(a,0) : a \in A\} \cup \{(0,b) : b \in B\} \subseteq \mathbb{R}^{d_1+d_2}$ is also in convex position. From this observation the examples constructed in Theorem 5 in [1] are in convex position, thus $F_4^{conv}(n) = \Omega(n^2)$, $F_5^{conv}(n) = \Omega(n^{7/3})$, and for $d \geq 6$ the examples obtained from a generalized Lenz construction [5] are also in convex position, so $F_d^{conv}(n) = \Theta(n^3)$. In the 3-dimensional case it is possible to modify a construction by Erdős et al. to obtain the following theorem.

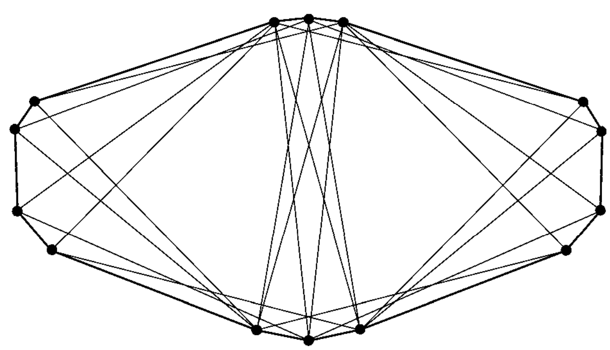

Figure 4: 14-point convex set determining 8 equilateral triangles.

Theorem 2 $F_3^{conv}(n) = \Omega(n \log^* n)$

Proof. In [10] (see also [12]) Erdős et al. constructed for every m an m-element set P in a unit sphere with the property that every point is at distance one from at least $c \log^* m$ other points (where $\log^*$ denotes the iterated logarithm function). Denote by S_1 such sphere and assume it is centered at 0. Let $\alpha = \sqrt{2-\sqrt{3}}$, for every $x \in S_1$ consider the set $Cap(x) = S_1 \cap \left\{y \in \mathbb{R}^d : |x-y| < \alpha\right\}$. The number α was chosen so that the points in S_1 at distance one from $Cap(x)$ are contained in a proper half sphere. Now let $C(x) = P \cap Cap(x)$, it is easy to see that the expected value of $|C(x)|$ equals $\frac{1}{4\pi}\int_{S_1} |C(x)|\, dx = ((2-\sqrt{3})/4)m$. Thus there is $x \in S_1$ satisfying $|C(x)| \geq ((2-\sqrt{3})/4)m$. Define Q as the set consisting of $\{0\} \cup C(x)$ together with all other points in P at a distance one from points in $C(x)$. By construction $Q \setminus \{0\}$ lies in a proper half sphere (hence Q is in convex position), it spans at least $c((2-\sqrt{3})/8)m\log^* m$ unit segments among points different from 0, and $((2-\sqrt{3})/4)m \le |Q| \le m$. Therefore by setting $n = |Q|$ we have that

$$F_3^{conv}(n) \ge F(Q) \ge c((2-\sqrt{3})/8)m\log^* m \ge c((2-\sqrt{3})/8)n\log^* n$$

as we wanted to prove. ■

3 Sets in General Position

In this section we find bounds for the function $E_d^{gen}(n)$. It is fairly easy to construct n-point subsets of $\mathbb{R}^d$ in general position with $\Theta(n)$ equilateral triangles. In contrast to the function $E_d(n)$ we can not use clusters of lattice points to provide quadratic lower bounds for $E_d^{gen}(n)$. Thus new techniques for the construction of sets in general position and with many equilateral triangles have to be developed. Here we give a recursive construction based on the Minkowski sum ($A+B = \{a+b : a \in A, b \in B\}$). This approach leads to the bounds $E^{gen}(n) = \Omega(n^{1.7078})$ and $E_d^{gen}(n) = \Omega(n\log n)$ when $d \ge 3$. With respect to the upper bounds we prove that $E_d^{gen}(n) = O(n^2)$.

For a finite set A in the plane let $i(A) = \log(3E(A) + |A|)/\log|A|$ be the index of A.

Theorem 3 *For any finite subset of the plane A in general position with $E(A) > 0$*

$$E^{gen}(n) \ge \Omega(n^{i(A)}). \tag{4}$$

Proof. We need two lemmas. Since the first one is rather technical we defer its proof to the Appendix.

Lemma 2 *For any A and B finite subsets in the plane in general position there exists a constant $c_{(A,B)} > 0$ such that for almost all $v \in \mathbb{C}$ with $|v| > c_{(A,B)}$ (with respect to the Lebesgue measure), the set $A + vB$ is in general position.* ■

Lemma 3 *If A and B are finite sets in the plane such that $A+B$ has exactly $|A||B|$ elements, then $E(A+B) \ge 3E(A)E(B) + |A|E(B) + |B|E(A)$.*

Proof. Given $T_A = a_1a_2a_3$ and $T_B = b_1b_2b_3$ equilateral triangles with vertices in A and B respectively, consider the following three types of triangles in $A+B$.
Type 1: Triangles of the form $a+b_1, a+b_2, a+b_3$ with $a \in A$.
Type 2: Triangles of the form a_1+b, a_2+b, a_3+b with $b \in B$.
Type 3: Triangles of the form $a_j+b_1, a_{j+1}+b_2, a_{j+2}+b_3$ with $j = 1,2,3$ ($a_4 = a_1, a_5 = a_2$).

Clearly all triangles of type 1 or 2 are equilateral. By setting $\gamma = e^{i\pi/3}$ we have

$$\begin{aligned}\gamma((a_{j+1}+b_2)-(a_j+b_1)) &= \gamma(b_2-b_1)+\gamma(a_{j+1}-a_j) \\ &= (b_3-b_1)+(a_{j+2}-a_j) \\ &= (a_{j+2}+b_3)-(a_j+b_1).\end{aligned}$$

Then all triangles of type 3 are equilateral. Since $|A+B| = |A||B|$, the relation $(a,b) \longleftrightarrow a+b$ for $a \in A$, $b \in B$ is bijective. So all triangles mentioned above are different. Therefore there are at least $3E(A)\,E(B)+|A|\,E(B)+|B|\,E(A)$ equilateral triangles in $A+B$. ■

Using the previous two lemmas we construct a large set in general position with many equilateral triangles as follows. Let A be a finite set in general position. From Lemma 2 we can obtain suitable sets $A_1, A_2, \ldots, A_m$ similar to A such that $A^* = A_1+A_2+\cdots+A_m$ is in general position and furthermore it has exactly $|A|^m$ elements. Now by a recursive application of Lemma 3 we have that for every k

$$3E(A_1+\ldots+A_k)+|A|^k \geq (3E(A)+|A|)^k$$

hence

$$E(A^*) \geq \frac{1}{3}\left((3E(A)+|A|)^m - |A|^m\right)$$

and by letting $n=|A|^m$, it yields

$$E^{gen}(n) \geq E(A^*) \geq \frac{1}{3}\left(n^{i(A)}-n\right) = \Omega\left(n^{i(A)}\right)$$

(note that $i(A)>1$ because $E(A)>0$). ■

Now, to obtain non trivial lower bounds for $E^{gen}(n)$ using the previous theorem, it is enough to construct sets A with large index. In particular when A is the triangle and the rhomb in Figures 5a and 5b, the index of A is $\log 6/\log 3 > 1.6309$ and $\log 10/\log 4 > 1.6609$ respectively.

We show three other sets with larger indices.

- The first one is a configuration of 8 points determining 8 equilateral triangles, thus with index $5/3 > 1.666$ (see Figure 5c). To construct this set start with a triangle $p_1p_2p_3$. Consider the equilateral triangles $p_1p_4p_2$ and $p_2p_5p_3$. Reflect the points p_2, p_4 and p_5 with respect to the middle point of the segment p_1p_3 to get the points p_6, p_7 and p_8 respectively. For almost any triangle $p_1p_2p_3$ the set $\mathcal{C}=\{p_1,p_2,\ldots,p_8\}$ is in general position. The equilateral triangles in this configuration are (the triangle $p_ip_jp_k$ corresponds to (i,j,k)): $(1,4,2), (3,7,6), (2,5,3), (6,8,1), (1,5,7), (4,3,8)$, $(4,5,6)$ and $(7,8,2)$.

- The second set is a 27-point configuration determining 81 equilateral triangles, hence with index $\log(270)/\log(27) > 1.6986$. To construct this set (see Figure 5d) start with the 8-point configuration $\mathcal{C}=\{p_1,p_2,\ldots,p_8\}$ given above and choose any point Q not in $\mathcal{C}$. Let $\mathcal{C}'=\{p_9,p_{10},\ldots,p_{16}\}$ and $\mathcal{C}''=\{p_{17},p_{18},\ldots,p_{24}\}$ be obtained from $\mathcal{C}$ by the $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ counterclockwise rotations respectively with center in Q (here p_{i+8} and p_{i+16} are the images of p_i under those rotations, for $i=1,\ldots,8$). Let p_{25}, p_{26} and p_{27} be the unique points such that the triangles $p_4p_{25}p_{15}$, $p_{12}p_{26}p_{23}$ and $p_{20}p_{27}p_7$ are equilateral. Then for a suitable choice of Q

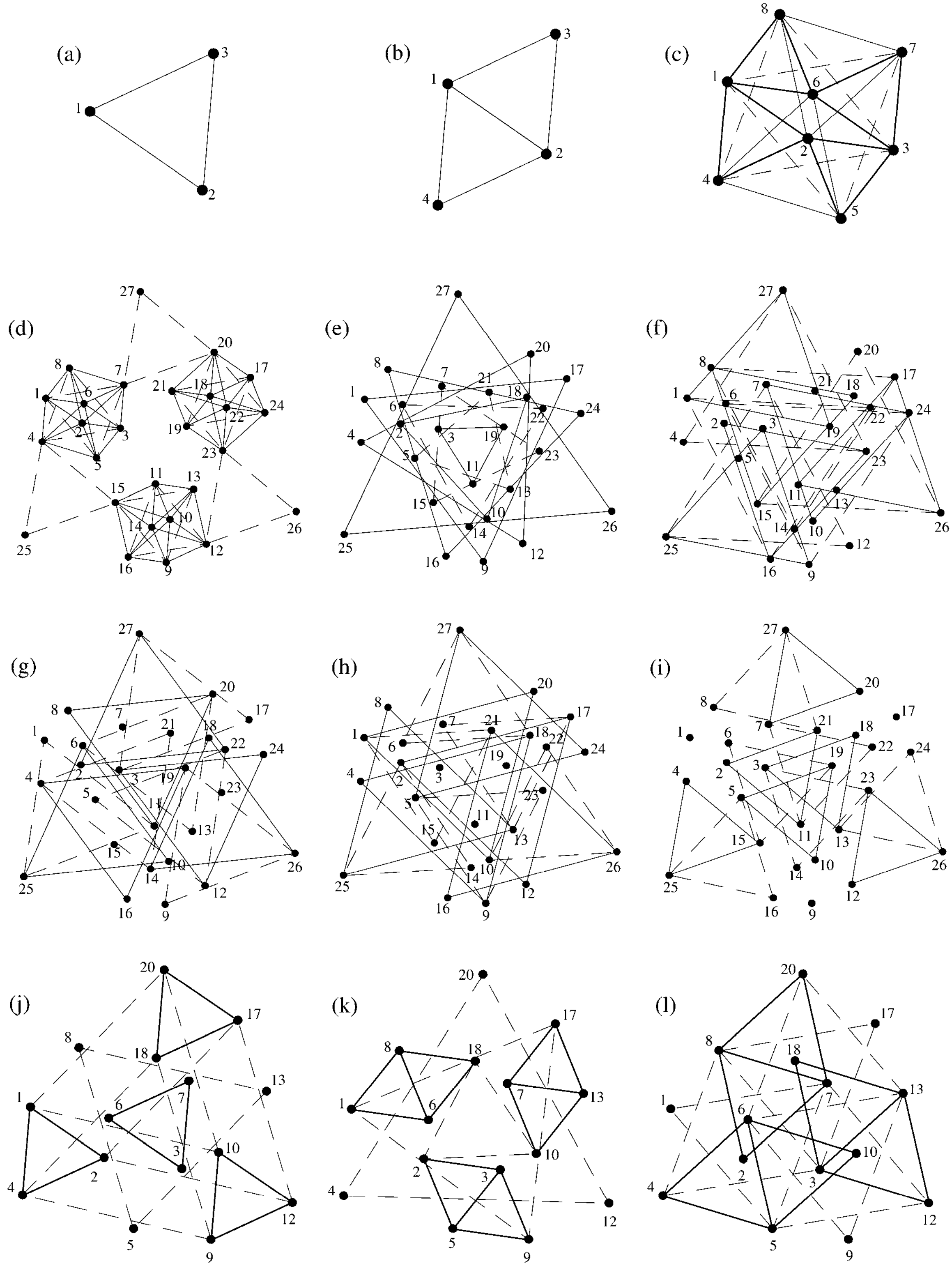

Figure 5: Starting set to give lower bounds of $E^{gen}(n)$.

(almost any point Q works), $p_i \neq p_j$ whenever $i \neq j$, and the set $\{p_1, p_2, ..., p_{27}\}$ is in general position. The equilateral triangles determined by this set are:

Figure 5d (in $\mathcal{C}$)	(1,4,2), (3,7,6), (2,5,3), (6,8,1) (1,5,7), (4,3,8), (4,5,6), (7,8,2)
Figure 5d (in $\mathcal{C}'$)	(9,12,10), (11,15,14), (10,13,11), (14,16,9) (9,13,15), (12,11,16), (12,13,14), (15,16,10)
Figure 5d (in $\mathcal{C}''$)	(17,20,18), (19,23,22), (18,21,19), (22,24,17) (17,21,23), (20,19,24), (20,21,22), (23,24,18)
Figure 5e (center in Q)	(1,9,17), (2,10,18), (3,11,19), (4,12,20) (17,21,23), (20,19,24), (20,21,22), (23,24,18),(25,26,27)
Figure 5f	(25,9,3), (10,24,7), (26,17,11), (18,8,15), (27,1,19), (2,16,23) (25,12,7), (14,17,8), (26,20,15), (22,1,16), (27,4,23), (6,9,24)
Figure 5g	(25,10,6), (12,24,3), (26,18,14), (20,8,11), (27,2,22), (4,16,19) (25,11,1), (13,20,6), (26,19,9), (21,4,14), (27,3,17), (5,12,22)
Figure 5h	(25,13,8), (10,20,1), (26,21,16), (18,4,9), (27,5,24), (2,12,17) (25,14,2), (9,23,5), (26,22,10), (17,7,13), (27,6,18), (1,15,21)
Figure 5i	(25,15,4), (10,19,5), (26,23,12), (18,3,13), (27,7,20), (2,11,21) (25,16,5), (15,19,6), (26,24,13), (23,3,14), (27,8,21), (7,11,22).

Even though this set does not provide the best lower bound for $E^{gen}(n)$, it is interesting because it avoids rhombs similar to the one in Figure 5b.

- The third example is obtained as a particular case in the construction of the 27-point set given above. If the point Q is precisely the circumcenter of the triangle $p_3p_7p_6$, then some points overlap, namely: $p_3 = p_{14} = p_{23}$, $p_5 = p_{16} = p_{25}$, $p_6 = p_{15} = p_{19}$, $p_7 = p_{11} = p_{22}$, $p_8 = p_{21} = p_{27}$ and $p_{13} = p_{24} = p_{26}$. Therefore some of the triangles among the original 81 are repeated. Only 29 different remain (see Figures 5j, 5k, 5l):

  $(1,4,2)$, $(3,7,6)$, $(2,5,3)$, $(6,8,1)$, $(1,5,7)$, $(4,3,8)$, $(4,5,6)$, $(7,8,2)$,

  $(9,12,10)$, $(10,13,7)$, $(3,5,9)$, $(9,13,6)$, $(12,7,5)$, $(12,13,3)$, $(6,5,10)$

  $(17,20,18)$, $(18,8,6)$, $(7,13,17)$, $(17,8,3)$, $(20,6,13)$, $(20,8,7)$, $(1,9,17)$,

  $(2,10,18)$, $(4,12,20)$, $(5,13,8)$, $(4,9,18)$, $(2,12,17)$, $(1,10,20)$.

  This set has 15 points, 29 equilateral triangles, and thus its index is $\log 102/\log 15$. This gives the best known bound which is stated in the following theorem.

Theorem 4 $E^{gen}(n) = \Omega(n^{\log 102/\log 15}) \geq \Omega(n^{1.7078})$.

■

With respect to the upper bound, it was recently proved [2] that $E^{gen}(n) = o(n^2)$ as a corollary of a general result concerning the structure of sets with many similar subsets.

Unfortunately the construction obtained from Theorem 3 does not work to the same extent in higher dimensions. It is impossible to obtain the equilateral triangles of type 3 in Lemma 3 without violating the general position assumption.. Next we show the best bounds we could attain. For this purpose we define a *d-simplex* to be a configuration of $d+1$ points in $\mathbb{R}^d$ all at distance 1 from each other.

Theorem 5 *For any $d \geq 3$ there is a constant $c_d > 0$ so that*

$$c_d n \log n \leq E_d^{gen}(n) \leq (d/3)\binom{n}{2}$$

Proof. For the lower bound proceed in the same way as in Theorem 3. By an analogue of Lemma 2 for higher dimensions, we can find a sequence of d-simplices: $A_1, A_2, \ldots, A_m$ so that $A^* = A_1 + A_2 + \cdots + A_m$ is in general position and furthermore it has exactly $|A|^m = (d+1)^m$ elements. Thus the relation $(a_1, a_2, \ldots, a_m) \in A_1 \times A_2 \times \cdots \times A_m \leftrightarrow a_1 + a_2 + \cdots + a_m \in A^*$ is a bijection. Now notice that for any $(a_2, a_3, \ldots, a_m) \in A_2 \times A_3 \times \cdots \times A_m$, and for any triangle xyz in A_1, the triangle corresponding to the m-tuples

$$(x, a_2, a_3, \ldots, a_m), (y, a_2, a_3, \ldots, a_m) \text{ and } (z, a_2, a_3, \ldots, a_m)$$

is equilateral. Since this is also true when the triangle xyz varies over any coordinate, we have that

$$E(A^*) \geq \binom{d+1}{3} m (d+1)^{m-1}$$

and by letting $n = (d+1)^m$ we get $E_d^{gen}(n) \geq (d(d-1)/6\log(d+1)) n \log n$.

For the upper bound, observe that given a pair of points in $\mathbb{R}^d$ the locus of a third point forming an equilateral triangle with this pair is a $(d-2)$-dimensional sphere. Thus by general position assumption there are at most d points on such a sphere, hence the bound $E_d^{gen}(n) \leq (d/3)\binom{n}{2}$ follows. ■

As a final remark, note that the technique used in Theorems 3-5 can be generalized (by proving the appropriate version of Lemma 2) to find sets under some other geometric restrictions and with many equilateral triangles, e.g. sets with no four points on a line or sets with no seven points on a conic.

4 Unit Triangles in General Position

Now we restrict our attention to subsets of $\mathbb{R}^d$ in general position and unit equilateral triangles. Theorems 6 and 7 give recursive formulas for the exact value of $F_d^{gen}(n)$ provided n is big enough. As a corollary we obtain the asymptotic behavior of $F_d^{gen}(n)$. We also provide the exact values of $F^{gen}(n)$ and $F_3^{gen}(n)$ for arbitrary n.

Consider an n-point set $P \subseteq \mathbb{R}^d$ in general position. Since by assumption there are no $d+2$ points on a $(d-1)$-dimensional sphere, and there is no configuration of $d+2$ points in $\mathbb{R}^d$ all at distance 1 from each other, then any point in P is in at most

$\binom{d+1}{2} - 1$ unit equilateral triangles determined by P, i.e., $\deg_\Delta^u(x) \le \binom{d+1}{2} - 1$ for all $x \in P$. From this we immediately obtain the upper bound

$$F_d^{gen}(n) \le \frac{1}{3}\left(\binom{d+1}{2} - 1\right) n \qquad (5)$$

which gives the correct order of magnitude. Now we construct a lower bound and prove that it is asymptotically best possible.

Theorem 6 *For any natural numbers $d \ge 2$ and $n \ge d+2$*

$$F_d^{gen}(n) \ge 2\binom{d}{2} + \binom{d}{3} + F_d^{gen}(n - (d+2)). \qquad (6)$$

Proof. Consider two regular d-simplices joined by a $(d-1)$-face. This configuration $\mathcal{S}_d$ has exactly $d+2$ points in $\mathbb{R}^d$ and determines exactly $2\binom{d}{2} + \binom{d}{3}$ unit equilateral triangles. Now let C be an extremal configuration in $\mathbb{R}^d$ with $n-(d+2)$ points in general position, i.e., $F(C) = F_d^{gen}(n-(d+2))$. It is always possible to construct an n-point configuration $P \subseteq \mathbb{R}^d$ as a disjoint union of a copy of $\mathcal{S}_d$ and a copy of C so that the points in P are in general position. Thus we get

$$F_d^{gen}(n) \ge F(P) \ge F(\mathcal{S}_d) + F(C) = 2\binom{d}{2} + \binom{d}{3} + F_d^{gen}(n-(d+2)).$$

■

Next we show that (6) is actually an identity for n large enough.

Theorem 7 *For $d \ge 3$ and $n \ge d(d-1)(d+2)(d+4)/12$ we have*

$$F_d^{gen}(n) \le 2\binom{d}{2} + \binom{d}{3} + F_d^{gen}(n-(d+2)). \qquad (7)$$

Proof. Consider an n-point set $P \subseteq \mathbb{R}^d$ in general position. For any $Q \subseteq P$ define $S(Q)$ to be the set of unit segments belonging to triangles in $\Delta_u(Q)$. Also, for any $x \in Q$ define $\deg_Q(x)$ as the number of $y \in Q$ such that $\overline{xy} \in S(Q)$. Recall that $\deg_\Delta^u(x) \le \binom{d+1}{2} - 1$ for all $x \in P$. We consider three cases.

Case 1. For all $x \in P$, $\deg_\Delta^u(x) \le \binom{d+1}{2} - 3$. Then $F(P) \le \frac{n}{3}\left(\binom{d+1}{2} - 3\right)$, and since $n \ge d(d-1)(d+2)(d+4)/12$ then

$$\begin{aligned} \tfrac{n}{3}\left(\binom{d+1}{2} - 3\right) &\le \left(\tfrac{n}{d+2} - 1\right)\left(2\binom{d}{2} + \binom{d}{3}\right) \\ &\le \left\lfloor \tfrac{n}{d+2} \right\rfloor \left(2\binom{d}{2} + \binom{d}{3}\right) + F_d^{gen}\left(n - (d+2)\left\lfloor \tfrac{n}{d+2} \right\rfloor\right) \\ &\le 2\binom{d}{2} + \binom{d}{3} + F_d^{gen}(n-(d+2)) \end{aligned} \qquad (8)$$

(the last inequality is given by a recursive application of Theorem 6).

Case 2. For some $x \in P$, $\deg_\Delta^u(x) = \binom{d+1}{2} - 1$. Let $P' = \{x\} \cup \{y \in P : \overline{xy} \in S(P)\}$. Note that $|P'| = d+2$, and all but one pair of points in P', say u and v, are at distance one from each other. Thus P' is a copy of the configuration $\mathcal{S}_d$ defined on the proof of Theorem 6, and moreover any triangle in $\Delta_u(P)$ with a vertex in P' is a triangle in $\Delta_u(P')$ (since all vertices in $P' \setminus \{u,v\}$ have already $d+1$ points in P' at distance one, $\deg_{P'}(u) = \deg_{P'}(v) = d$, and $|u-v| \neq 1$). Hence

$$F(P) \leq F(P') + F_d^{gen}(n-(d+2)) = 2\binom{d}{2} + \binom{d}{3} + F_d^{gen}(n-(d+2)).$$

Case 3. For some $x \in P$, $\deg_\Delta^u(x) = \binom{d+1}{2} - 2$. Let $P' = \{x\} \cup \{y \in P : \overline{xy} \in S(P)\}$. Again $|P'| = d+2$ but in this case $S(P')$ consists of all possible segments among points in P' except two. Suppose those two segments are $\overline{u_1u_2}, \overline{u_3u_4}$ for some $u_1, u_2, u_3, u_4 \in P'$.

Then any triangle in $\Delta_u(P)$ having a vertex in P' is either a triangle in $\Delta_u(P')$ or a triangle with at least one vertex in $\{u_1, u_2, u_3, u_4\}$ and at least one vertex not in P' (since all vertices in $P' \setminus \{u_1, u_2, u_3, u_4\}$ have already $d+1$ points in P' at distance 1).

a) If u_1, u_2, u_3, u_4 are all different then $\deg_{P'}(u_j) = d$ for all $1 \leq j \leq 4$. Thus for each $1 \leq j \leq 4$, there is at most one point w_j in $P \setminus P'$ at distance one from u_j. Then the only possible triangles in $\Delta_u(P) \setminus \Delta_u(P')$ involving points in P' are

$$\begin{aligned} u_1u_3w_1 \text{ if } w_1 &= w_3,\ u_1u_4w_1 \text{ if } w_1 = w_4, \\ u_2u_3w_2 \text{ if } w_2 &= w_3, \text{ and } u_2u_4w_2 \text{ if } w_2 = w_4. \end{aligned} \tag{9}$$

Thus

$$\begin{aligned} F(P) &\leq F(P') + 4 + F_d^{gen}(n-(d+2)) \\ &= \binom{d+2}{3} - 2d + 4 + F_d^{gen}(n-(d+2)) \\ &\leq 2\binom{d}{2} + \binom{d}{3} + F_d^{gen}(n-(d+2)) \end{aligned}$$

for all $d \geq 4$. For the remaining case, $d = 3$, notice that the four triangles given in (9) can be in $\Delta_u(P)$ at the same time only if $w_1 = w_2 = w_3 = w_4$, which means that w_1 would be at distance 1 from each of the points u_1, u_2, u_3 and u_4. But this is not possible by general position assumption. So in this case

$$\begin{aligned} F(P) &\leq F(P') + 3 + F_d^{gen}(n-(d+2)) \\ &= \binom{d+2}{3} - 2d + 3 + F_d^{gen}(n-(d+2)) \\ &= 7 + F_d^{gen}(n-5) = 2\binom{d}{2} + \binom{d}{3} + F_d^{gen}(n-(d+2)). \end{aligned}$$

b) If (without loss of generality) $u_1 = u_4$ then there are at most two points w_1, w_1' in $P \setminus P'$ at distance 1 from u_1 and for each $j \in \{2,3\}$ there is at most one point w_j in $P \setminus P'$ at distance 1 from u_j (since $\deg_{P'}(u_1) = d-1, \deg_{P'}(u_2) = \deg_{P'}(u_3) = d$).

Then the only possible triangles in $\Delta_u(P) \setminus \Delta_u(P')$ having a vertex in P' are $u_2u_3w_2$ (if $w_2 = w_3$) and $u_1w_1w_1'$. Thus for all $d \geq 3$

$$\begin{aligned} F(P) &\leq F(P') + 2 + F_d^{gen}(n-(d+2)) \\ &= \binom{d+2}{3} - 2d + 1 + 2 + F_d^{gen}(n-(d+2)) \\ &\leq 2\binom{d}{2} + \binom{d}{3} + F_d^{gen}(n-(d+2)). \end{aligned}$$

Finally, since P was arbitrary, the three cases above imply (7). ■

From Theorems 6 and 7, together with (5) we obtain the following corollary.

Corollary 1 *For all $d \geq 3$*

$$F_d^{gen}(n) = \left(2\binom{d}{2} + \binom{d}{3}\right) \left\lfloor \tfrac{n}{d+2} \right\rfloor + O(d^4).$$

■

When $d = 3$ the inequality (8) can be verified to be valid for all $n \geq 5$ using the fact that $F_d^{gen}(r) = \binom{r}{3}$ for all $r \leq d+1$. This, together with Theorem 6 yields the exact value

$$F_3^{gen}(n) = 3\left\lfloor \frac{n}{5} \right\rfloor + 3\left\lfloor \frac{n+1}{5} \right\rfloor + \left\lfloor \frac{n+2}{5} \right\rfloor.$$

A similar argument can be applied for $d = 2$ to obtain

$$F^{gen}(n) = \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor.$$

5 Appendix

Proof of Lemma 2. Denote by $Ball_\varepsilon(b)$ the set $\{z \in \mathbb{C} : |z - b| < \varepsilon\}$. Since B is in general position there exists $\varepsilon > 0$ such that $\bigcup_{b \in B} Ball_\varepsilon(b)$ is still in "general position" (i.e. there is no line intersecting three balls, or a circle intersecting four balls). Let ρ be such that $A \subseteq Ball_\rho(0)$ and set $c_{(A,B)} = 2\rho/\varepsilon$. Denote by S the set $\{v : |v| > 2\rho/\varepsilon \text{ and } A + vB \text{ not in general position}\}$. For $a \in A$, $b \in B$, and $T \subseteq A \times B$ let $(a, b)_v = a + vb$ and $T_v = \{(a, b)_v : (a, b) \in T\}$. Under this notation we have that

$$S = \left(\bigcup_{\substack{T \subseteq A \times B \\ |T| = 3}} \{v \in S : \exists \mathcal{L} \text{ line with } T_v \subseteq \mathcal{L}\} \right) \cup \left(\bigcup_{\substack{T \subseteq A \times B \\ |T| = 4}} \{v \in S : \exists \mathcal{C} \text{ circle with } T_v \subseteq \mathcal{C}\} \right)$$

To finish the proof we show that each term in this union has Lebesgue measure zero. First we argue that for $T \subseteq A \times B$, $|T| = 4$, if $\{v \in S : \exists \mathcal{C} \text{ circle with } T_v \subseteq \mathcal{C}\}$ is not empty then T has two points with the same B coordinate. Suppose not, then there is a circle $\mathcal{C}$ containing T_v for some v and $T = \{(a_j, b_j) : 1 \leq j \leq 4\}$ with all b_j's distinct. The circle $v^{-1}\mathcal{C}$ passes through the points $v^{-1}a_j + b_j$,$1 \leq j \leq 4$ but $|(v^{-1}a_j + b_j) - b_j| = |v^{-1}a_j| < \varepsilon/2$, hence $v^{-1}\mathcal{C}$ intersects $Ball_\varepsilon(b_j)$ for each $1 \leq j \leq 4$ which contradicts the definition of ε.

Similarly, for $T \subseteq A \times B$, $|T| = 3$, if $\{v \in S : \exists \mathcal{L} \text{ line with } T_v \subseteq \mathcal{L}\}$ is not empty then T has two points with the same B coordinate. From these two observations and the fact that A is in general position we have to examine only two cases.

Case 1. $T = \{(a_1, b_1), (a_2, b_1), (a_3, b_2)\}$ with $a_1 \neq a_2$ and $b_1 \neq b_2$. Suppose $\mathcal{L}$ is a line containing T_v then we have that $(a_2, b_1)_v - (a_1, b_1)_v = \lambda \left((a_3, b_2)_v - (a_2, b_1)_v\right)$ for some $\lambda \in \mathbb{R}$, thus $v = (b_2 - b_1)^{-1}(a_2 - a_3 + \lambda^{-1}(a_2 - a_1))$ which is the parametric equation of a line $\mathcal{L}'$. Hence $\{v \in S : \exists \mathcal{L} \text{ line with } T_v \subseteq \mathcal{L}\} \subseteq \mathcal{L}'$ and consequently $\{v \in S : \exists \mathcal{L} \text{ line with } T_v \subseteq \mathcal{L}\}$ has zero measure.

Case 2. $T = \{(a_1, b_1), (a_2, b_1), (a_3, b_2), (a_4, b_3)\}$ with $b_1 \neq b_2$ and $|T| = 4$. By considering the Möbius transformation $f(z) = (a_1 - a_2)^{-1}(a_1 + a_2 + 2(vb_1 - z))$ we can assume that $b_1 = 0$, $a_1 = -1$, and $a_2 = 1$. Now, let $|z - ih|^2 = 1 + h^2$ be the equation in z of a circle $\mathcal{C}$ containing T_v. Since $\{-1, 1\} \in \mathcal{C}$ then $a_3 + vb_2, a_4 + vb_3 \notin \mathbb{R}$, therefore

$$ih = \frac{1 - |a_3 + vb_2|^2}{a_3 - \overline{a_3} + vb_2 - \overline{vb_2}} = \frac{1 - |a_4 + vb_3|^2}{a_4 - \overline{a_4} + vb_3 - \overline{vb_3}};$$

which is a non trivial equation in v since $b_2 \neq 0$, in fact it is the equation of a circle if $b_1 = b_3$ or a cubic curve on two real variables otherwise. Therefore $\{v \in S : \exists \mathcal{C} \text{ circle with } T_v \subseteq \mathcal{C}\}$ has measure zero. ■

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