On the Number of Equilateral Triangles in Euclidean Spaces I

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Abstract

The following problem was posed by Erdős and Purdy: "What is the maximum number of equilateral triangles determined by a set of n points in $\mathbb{R}^d$?" New bounds for this problem are obtained for dimensions 2, 4 and 5. In addition it is shown that for $d = 2$ the maximum is attained by subsets of the regular triangle lattice.

1 Introduction

In 1946 Paul Erdős [5] posed the seemingly innocent problem of finding the maximum number of unit segments that can be determined by a set of n points in the plane. This question turned out to be one of the most challenging unsolved problems ever proposed by Erdős. Even though the solution has long been out of reach, the quest for solving the problem generated a considerable amount of research in combinatorial geometry.

Since then many problems sharing the same spirit have been raised; problems concerning the maximum number of geometric objects that can be determined by an n-point set in an euclidean space. In this article we address one of them.

> "What is the maximum number of equilateral triangles that can be determined by n points in the plane?"

This problem was proposed several times by Erdős and Purdy ([8],[9],[10],[11]), and it is also mentioned in [3] not as a famous unsolved problem but as a problem that deserves to be studied.

For an arbitrary finite set P denote by $E(P)$ the number of triplets in P that are the vertices of an equilateral triangle, we define

$$E(n) = \max_{\substack{|P|=n \\ P\subseteq\mathbb{R}^2}} E(P).$$

It is easy to see that $E(n) \leq \frac{2}{3}\binom{n}{2}$ since every pair of points in the plane are the vertices of at most two equilateral triangles; also by considering a suitable set of the regular triangular

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grid one can easily obtain $E(n) > cn^2$. This has been noted before by Elekes and Erdős [4], in fact they proved that for every triangle T, the maximum number of triangles similar to T determined by n points in the plane is $\Theta(n^2)$.

The aim of this paper is to refine these bounds. We show in Section 2 that

$$0.1955 \leq \frac{1}{3} - \frac{\sqrt{3}}{4\pi} \leq \liminf_{n\to\infty} \frac{E(n)}{n^2} \leq \frac{1}{4} = 0.25,$$

next we prove in Section 3 that $E(n)$ is achieved by subsets of the equilateral triangle lattice. Finally, in Section 4, we discuss the same problem on higher dimensions where we find lower bounds for dimensions 4 and 5, and a new upper bound for the 5-dimensional space.

Throughout the paper we use the following notation. For any P finite set, $G = (P, \Delta(P))$ denotes the 3-uniform hypergraph with vertex set P, where $\Delta(P)$ is the set of triplets in P determining equilateral triangles. For $x \in P$ we denote by $\deg_\Delta(x)$ the number of triangles in $\Delta(P)$ with x as one of its vertices.

2 Bounds on the Number of Triangles

Although the order of magnitude of the function $E(n)$ is known to be quadratic, it is not even known whether $\lim_{n\to\infty} \frac{E(n)}{n^2}$ exists. The aim of this section is to give new bounds for $E(n)$. Theorem 1 gives a non trivial upper bound and its proof allows us to determine $E(n)$ for the first few values of n. On the other hand we obtain in Theorem 4 the best known lower bound for $E(n)$ which we believe is very close to the true value of $E(n)$.

We start by pointing out the following simple but very useful observation: For every $x \in P$ define $N_x = P \cap R_x(P) \setminus \{x\}$ where R_x denotes the rotation of a $\frac{\pi}{3}$ angle with center at x. We have that

$$\deg_\Delta(x) = |N_x| = |P \cap R_x(P)| - 1. \tag{1}$$

Indeed, xyz is an equilateral triangle in P if and only if $R_x(y) = z$, i.e. $z \in P \cap R_x(P) \setminus \{x\}$.

We make use of this observation to prove the next theorem.

Theorem 1 $E(n) \leq \left\lfloor \left(\frac{n-1}{2}\right)^2 \right\rfloor$.

Proof. It is enough to prove that any n-point set P has a point w with

$$\deg_\Delta(w) \leq \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{2}$$

Assuming this we prove the theorem by induction on n. The theorem is clearly true for $n \leq 3$. For $n \geq 4$ consider an n-point set P and let w be a point in P satisfying (2). Then by induction hypothesis

$$E(P) = E(P \setminus \{w\}) + \deg_\Delta(w) \leq \left\lfloor \left(\frac{n-2}{2}\right)^2 \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \left(\frac{n-1}{2}\right)^2 \right\rfloor.$$

Now we prove (2). Consider points $x, y \in P$ whose distance realizes the diameter of P. First assume there is a point $z \in P$ such that xyz is an equilateral triangle. By maximality of the segment xy, P must be contained in the Reuleáux triangle xyz. Then if X, Y, Z

are the shaded regions on Figure 1 we have that $N_x \subset X, N_y \subset Y, N_z \subset Z$ and since $x \notin N_x, y \notin N_y, z \notin N_z$ we have that N_x, N_y and N_z are pairwise disjoint. So

$$\deg_{\Delta}(x) + \deg_{\Delta}(y) + \deg_{\Delta}(z) = |N_x| + |N_y| + |N_z| \leq n,$$

hence at least one of the points x, y or z has $\deg_{\Delta} \leq \left\lfloor \frac{n}{3} \right\rfloor \leq \left\lfloor \frac{n-1}{2} \right\rfloor$.

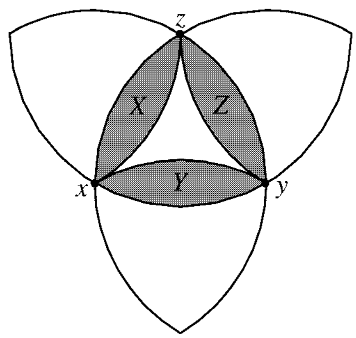

Figure 1: The regions X, Y and Z are intersections of Reuleáux triangles.

Now suppose no point in P forms an equilateral triangle together with x and y. We shall prove by contradiction that $|N_x \cap N_y| \leq 1$. So we assume there are two points u and v in $N_x \cap N_y$. This means there are points $u_x, v_x, u_y, v_y \in P$ such that all triangles $xu_xu, xv_xv, yu_yu, yv_yv$ are equilateral (see Figure 2). Notice that the segments u_yu_x and v_yv_x are obtained from yx by a $\frac{\pi}{3}$-rotation with center in u and v respectively. So v_xv_y and u_xu_y are parallel and have the same length, i.e. $u_xv_xv_yu_y$ is a parallelogram. This gives us a contradiction since one of the diagonals of the parallelogram is longer than any of it sides, including those which by assumption have maximal length in P. Hence $|N_x \cap N_y| \leq 1$.

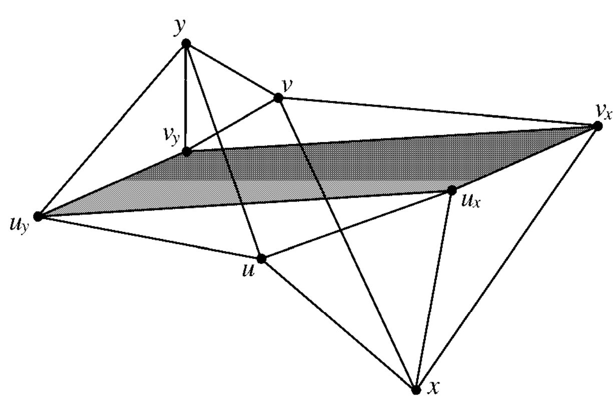

Figure 2: Proof of $|N_x \cap N_y| \leq 1$.

Finally since $\{u, v\} \cap \{x, y\} = \emptyset$ by hypothesis, neither x nor y are in $N_x \cup N_y$. Thus

$$|N_x| + |N_y| - 1 \leq |N_x \cup N_y| \leq |P \backslash \{x, y\}| = n - 2,$$

so either $|N_x| \leq \lfloor \frac{n-1}{2} \rfloor$ or $|N_y| \leq \lfloor \frac{n-1}{2} \rfloor$, i.e. one of x or y satisfies $\deg_\Delta \leq \lfloor \frac{n-1}{2} \rfloor$. ■

Using property (2) it is easy to determine the exact values of $E(n)$ for $1 \leq n \leq 8$, as well as all the sets that achieve equality for $1 \leq n \leq 7$ (see Figure 3).

n	$E(n)$
1	0
2	0
3	1
4	2
5	3
6	5
7	8
8	10

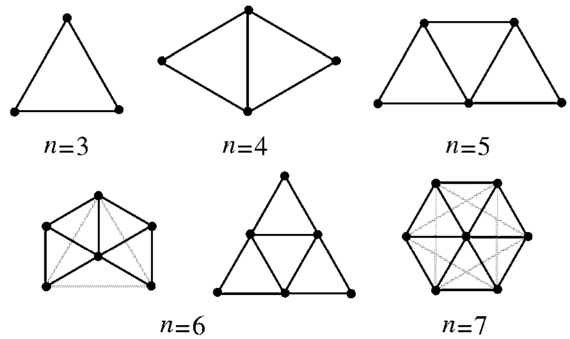

Figure 3: Optimal sets for the first values of $E(n)$.

When trying to find sets in the plane with large number of equilateral triangles, the first examples that come to mind are subsets of the equilateral triangle lattice. Our goal for the rest of the section is to prove right this intuitive idea, to do so we estimate the number of equilateral triangles determined by specific examples. Denote by Λ the unit equilateral triangle lattice generated by 1 and $\gamma = e^{i\pi/3}$. We refer to the elements of Λ as lattice points.

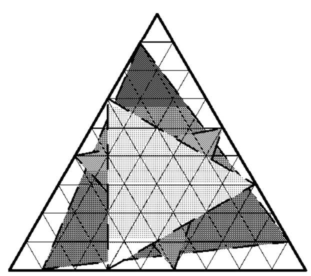

Figure 4: Some equilateral triangles in T_m.

Let us start by considering for each $m \in \mathbb{N}$ the set T_m consisting of all lattice points in a solid equilateral triangle of side m, and sides parallel to the lattice as shown on Figure 4. The following is a well-known result.

Theorem 2 $E(T_m) = \binom{m+3}{4}$.

Proof. Notice that every equilateral triangle in T_m is inscribed in an equilateral triangle with sides parallel to the original triangle. Since there are exactly $\binom{m+2-j}{2}$ such triangles with side $1 \leq j \leq m$, and each inscribes exactly j triangles, we have that

$$E(T_m) = \frac{1}{2}\sum_{j=1}^{m}(m+2-j)\,(m+1-j)\,j = \binom{m+3}{4}.$$

■

Since $|T_m| = \binom{m+2}{2}$, this result implies that $E(n) \geq \frac{1}{6}n^2 + O(n^{3/2})$. Now, we study a second example where we also obtain the precise number of equilateral triangles. Let H_m denote the set of lattice points contained in a regular hexagon of side m and sides parallel to the lattice. We have the following.

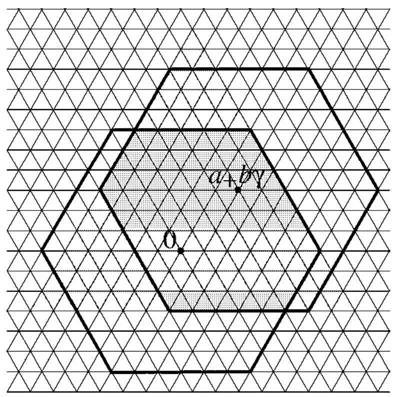

Figure 5: Intersection of H_m and $H(a+b\gamma)$.

Theorem 3 $E(H_m) = \frac{1}{4}m\left(7m^3 + 14m^2 + 9m + 2\right)$.

Proof. By observation (1) we have that

$$E(H_m) = \frac{1}{3}\sum_{x\in H_m}|N_x| = \frac{1}{3}\sum_{x\in H_m}\left(|H_m \cap R_x(H_m)| - 1\right). \tag{3}$$

Assume 0 is the center of H_m, observe that $R_x(H_m)$ is an hexagon with sides parallel to the lattice and center $R_0^{-1}(x)$. Since the function $R_0^{-1} : H_m \to H_m$ is bijective we obtain from (3)

$$E(H_m) = \frac{1}{3}\sum_{x\in H_m}\left(|H_m \cap H(x)| - 1\right) \tag{4}$$

where $H(x)$ is the set of lattice points in the hexagon of side m and center x. Now suppose $x = a + b\gamma$ where $\gamma = e^{i\pi/3}$ and $1 \leq a \leq m$, $0 \leq b \leq m-a$. Then by completing the corresponding intersection (see Figure 5) to an equilateral triangle with sides of length $3m-a-b$, we find that

$$\begin{aligned}|H_m \cap H(x)| &= |T_{3m-a-b}| - |T_{m-a-1}| - |T_{m-b-1}| - |T_{m-1}| \\ &= \binom{3m-a-b+2}{2} - \binom{m-a+1}{2} - \binom{m-b+1}{2} - \binom{m+1}{2}.\end{aligned}$$

Then after using the hexagonal symmetry of the sum in (4), and the fact that $|H_m| = |H(0)| = 3m^2 + 3m + 1$, we get the expression

$$E(H_m) = \left(m^2 + m\right) + \tfrac{6}{3} \sum_{\substack{1\le a\le m \\ 0\le b\le m-a}} \left(\binom{3m-a-b+2}{2} - \binom{m-a+1}{2} - \binom{m-b+1}{2} - \binom{m+1}{2} - 1\right),$$

simplifying we obtain $E(H_m) = \frac{1}{4}m\left(7m^3 + 14m^2 + 9m + 2\right)$. ■

We recall that $|H_m| = 3m^2 + 3m + 1$, hence as a consequence we have $E(n) \ge \frac{7}{36}n^2 + O(n^{3/2})$. This is already an improvement from our previous lower bound, and it suggests that 'round' clusters of lattice points should give better bounds. This is in fact the case as we see on the next theorem.

Theorem 4 $E(n) \ge \left(\frac{1}{3} - \frac{\sqrt{3}}{4\pi}\right) n^2 + O(n^{3/2})$.

Proof. Consider the set C_m of all lattice points inside a circle of radius m centered at the origin. From (1), we know that

$$E(C_m) = \frac{1}{3}\sum_{x\in C_m} |N_x| = \frac{1}{3}\sum_{x\in C_m} \left(|C_m \cap R_x(C_m)| - 1\right). \tag{5}$$

Notice that $|C_m \cap R_x(C_m)|$ is equal to the number of lattice points in the intersection of two circles of radius m, one centered at 0, and the other at x. Using a simple area calculation we get that the area of such intersection is

$$2m^2 \arccos\left(\frac{|x|}{2m}\right) - \frac{|x|\,m}{2}\sqrt{4 - \left(\frac{|x|}{m}\right)^2}.$$

Now, since the number of lattice points in the region is roughly $\frac{2}{\sqrt{3}}$ of the area we obtain

$$|C_m \cap R_x(C_m)| = \left(4\arccos\left(\frac{|x|}{2m}\right) - \frac{|x|}{m}\sqrt{4 - \left(\frac{|x|}{m}\right)^2}\right)\frac{m^2}{\sqrt{3}} + O\left(m\right). \tag{6}$$

Let $f(y) = 4\arccos\left(\frac{y}{2}\right) - y\sqrt{4-y^2}$, according to (5) and (6) we have that

$$E(C_m) = \frac{1}{3}\sum_{x\in C_m}\left(\frac{m^2}{\sqrt{3}} f\left(\frac{|x|}{m}\right) + O\left(m\right)\right).$$

Since all the linear errors in (6) are uniformly bounded (all of them are less than $8m$) we get

$$E(C_m) = \frac{2m^4}{9}\sum_{x\in C_m}\frac{\sqrt{3}}{2m^2} f\left(\frac{|x|}{m}\right) + O(m^3).$$

Let $g : D_1 \to \mathbb{R}$ be defined as $g(x) = f\left(|x|\right)$, where D_1 stands for the unit disk in $\mathbb{C}$. Observe that by considering the dual lattice of $\frac{1}{m}\Lambda$ (i.e. the lattice of regular hexagons of side $\frac{1}{\sqrt{3}m}$ whose set of centers is $\frac{1}{m}\Lambda$), we have that each hexagon has area $\frac{\sqrt{3}}{2m^2}$ and the set $\left\{\frac{x}{m} : x \in C_m\right\}$ consists precisely of the centers of the hexagons inside D_1. Hence $\sum_{x\in C_m} \frac{\sqrt{3}}{2m^2} f\left(\frac{|x|}{m}\right)$ is a Riemann sum for the function g on the unit disk, i.e.

$$\sum_{x\in C_m} \frac{\sqrt{3}}{2m^2} f\left(\frac{|x|}{m}\right) \geq \int_{D_1} g(x)\,dx + O\left(\frac{1}{m}\right).$$

Using a system of polar coordinates we calculate the integral

$$\begin{aligned}
\int_{D_1} g(x)\,dx &= \int_0^{2\pi}\left(\int_0^1 r f(r)\,dr\right)d\theta \\
&= 2\pi \int_0^1 r f(r)\,dr \\
&= 2\pi \left(\pi r^2 - \tfrac{\sqrt{4-r^2}(2r+r^3)}{4} - 2(r^2-1)\arcsin\left(\tfrac{r}{2}\right)\right)\Big|_0^1 \\
&= 2\pi\left(\pi - \tfrac{3\sqrt{3}}{4}\right),
\end{aligned}$$

therefore

$$E(C_m) \geq \left(\frac{4\pi^2}{9} - \frac{\sqrt{3}\pi}{3}\right) m^4 + O(m^3)$$

and again, since the number of lattice points in C_m is approximately $\frac{2}{\sqrt{3}}$ of the area, we have that if $n = |C_m|$ then $n = \frac{2\pi}{\sqrt{3}} m^2 + O(m)$. Hence

$$E(n) \geq \left(\frac{1}{3} - \frac{\sqrt{3}}{4\pi}\right) n^2 + O(n^{3/2})$$

and this inequality easily extends for arbitrary n. ■

Summarizing the results from this section we have

$$\left(\frac{1}{3} - \frac{\sqrt{3}}{4\pi}\right) n^2 + O(n^{3/2}) \leq E(n) \leq \frac{1}{4} n^2$$

and in fact we conjecture that

Conjecture 1 $\lim\limits_{n\to\infty} \frac{E(n)}{n^2}$ *exists, and its value is* $\frac{1}{3} - \frac{\sqrt{3}}{4\pi}$.

In other words we think that C_m provides an asymptotically best example. We also believe that the following stronger conjecture is true.

Conjecture 2 *For any n-point set P, there is $x \in P$ with* $\deg_\Delta(x) \leq \left(\frac{2}{3} - \frac{\sqrt{3}}{2\pi}\right)(n + \sqrt{n})$.

The validity of this conjecture would immediately imply $E(n) \leq \left(\frac{1}{3} - \frac{\sqrt{3}}{4\pi}\right) n^2 + O(n^{3/2})$, and together with Theorem 4 it would constitute a proof of Conjecture 1.

3 Extremal Sets

The best evidence supporting Conjecture 1 is the next theorem , where we prove that the problem of determining $E(n)$ is the same if, to compute the maximum, we just consider subsets P of the equilateral triangle lattice Λ. This in turn indicates that the problem of determining $E(n)$ is discrete in nature, as it is stated in Corollary 1.

Theorem 5 *$E(n)$ is attained by subsets of the regular triangular lattice.*

Proof. Consider a system of coordinates with axis forming a $\frac{\pi}{3}$ angle (see Figure 6). Take a set $P = \{p_1, p_2, \ldots, p_n\}$ of n points in the plane ($p_j = (x_j, y_j)$ for all $j = 1, 2, \ldots, n$). Assume without loss of generality that the minimum distance between points of P is $\frac{\sqrt{3}}{2}$.

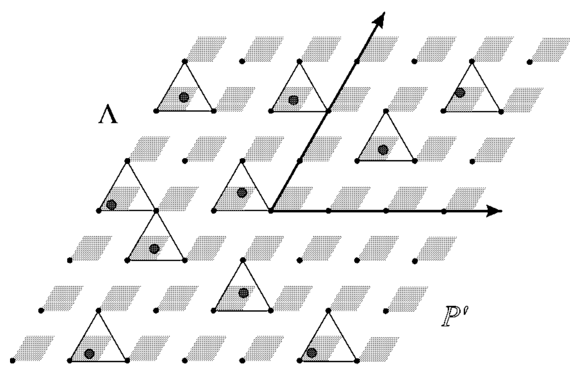

Figure 6: Λ and the shaded regions.

By Dirichlet Simultaneous Approximation Theorem, there is an integer $1 \leq t \leq 4^n$ such that for every $1 \leq j \leq n$

$$0 \leq \left\{ tx_j + \frac{1}{4} \right\} < \frac{1}{2} \text{ and } 0 \leq \left\{ ty_j + \frac{1}{4} \right\} < \frac{1}{2}$$

where $\{x\}$ denotes the fractional part of x.

Let $p'_j = tp_j + \left(\frac{1}{4}, \frac{1}{4}\right)$ for $1 \leq j \leq n$, and set $P' = \{p'_1, p'_2, \ldots, p'_n\}$. By the choice of t all points of P' are in the shaded regions indicated in Figure 6. Moreover since any two points in P' are at distance at least $\frac{\sqrt{3}}{2}$ (by assumption on the minimum distance in P), then each connected region contains at most one point of P'. In particular, all points of P' lie in the interior of a minimal triangle of Λ pointing upwards. Now consider the set $Q = \{q_1, q_2, \ldots, q_n\}$ of n points in the plane where $q_j = (\lfloor tx_j + \frac{1}{4} \rfloor, \lfloor ty_j + \frac{1}{4} \rfloor)$. Notice that all points of Q lie on Λ.

To conclude the proof of the theorem we verify that $E(Q) \geq E(P)$. Clearly $E(P) = E(P')$. Now suppose that the points p'_j, p'_k, p'_l are the vertices of an equilateral triangle. Let T_j and T_k be the minimal triangles of Λ to which p'_j and p'_k belong (see Figure 7). Then the locus of all points of the plane forming an equilateral triangle with one interior point of T_j and one interior point of T_k is the union of the interiors of the triangles T and T' in Figure 7, where a and b are the unique two points forming equilateral triangles with q_j and

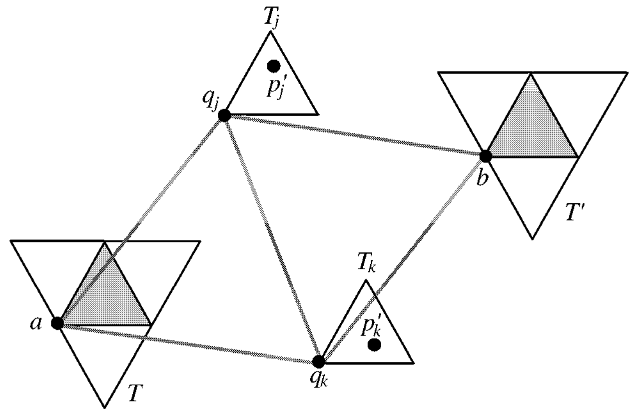

Figure 7: Illustration for the proof of Theorem 5.

q_k. Since p_l' must be in the interior of a minimal equilateral triangle of Λ pointing upwards, then p_l' must lie on the interior of the shaded region indicated in Figure 7. Therefore either $q_l = a$ or $q_l = b$, which means that the points q_j, q_k, q_l are the vertices of an equilateral triangle, and hence $E(Q) \geq E(P')$. ■

Observe that we proved in fact the following stronger statement: For any set in the plane P with n points, there is a lattice set $Q \subseteq \Lambda$ and a bijective map $h : P \to Q$ so that

1. $h(T) \in \Delta(Q)$ for every $T \in \Delta(P)$
2. $d(h(p_1), h(p_2)) \leq \left(\frac{\sqrt{3}}{2m}(4^n + 1)\right) d(p_1, p_2)$ for every $p_1, p_2 \in P$. Here $d(p_1, p_2)$ denotes the euclidean distance between p_1 and p_2, and m stands for the minimum distance among points in P.

Corollary 1 *For every $n \in \mathbb{N}$ we have that $E(n) = \max\limits_{\substack{P \subseteq \Lambda \\ |P|=n}} E(P)$.*

4 Higher Dimensions

In this section we consider a generalization of the function $E(n)$ to higher dimensions. For $d \geq 2$ let $E_d(n)$ denote the maximum number of equilateral triangles determined by n points in $\mathbb{R}^d$, i.e.

$$E_d(n) = \max_{\substack{P \subset \mathbb{R}^d \\ |P|=n}} E(P).$$

Similarly we define the function

$$F_d(n) = \max_{\substack{P \subset \mathbb{R}^d \\ |P|=n}} F(P)$$

where $F(P)$ denotes the number of triplets spanning a unit equilateral triangle.

Under this notation $E(n) = E_2(n)$ and trivially $E_d(n) \leq E_{d+1}(n), F_d(n) \leq F_{d+1}(n)$. We also have the trivial upper bound $F_d(n) \leq E_d(n) \leq \binom{n}{3}$ for all $d \geq 2$. A generalized Lenz's construction (Erdős [8]) shows that n^3 is actually the correct order of magnitude of both functions $E_d(n)$ and $F_d(n)$ for $d \geq 6$.

To see this consider $m = \left\lfloor \frac{n}{\lfloor \frac{d}{2} \rfloor} \right\rfloor$ pairs x_j, y_j with $x_j^2 + y_j^2 = \frac{1}{2}$. Let $P = \bigcup_{1 \leq k \leq \lfloor \frac{d}{2} \rfloor} A_k$ where

$$\begin{array}{rcl} A_1 & = & \{(x_j, y_j, 0, 0, 0, 0, ..., 0, 0) : 1 \leq j \leq m\} \\ A_2 & = & \{(0, 0, x_j, y_j, 0, 0, ..., 0, 0) : 1 \leq j \leq m\} \\ \vdots & & \vdots \\ A_{\lfloor \frac{d}{2} \rfloor} & = & \left\{ \begin{array}{l} \{(0, 0, ..., x_j, y_j) : 1 \leq j \leq m\} \text{ if } d \text{ even} \\ \{(0, 0, ..., x_j, y_j, 0) : 1 \leq j \leq m\} \text{ if } d \text{ odd.} \end{array} \right. \end{array}$$

Since any triangle with vertices in different A_j's is a unit equilateral triangle, we have that

$$\binom{\lfloor \frac{d}{2} \rfloor}{3} m^3 = \binom{\lfloor \frac{d}{2} \rfloor}{3} \left\lfloor \frac{n}{\lfloor \frac{d}{2} \rfloor} \right\rfloor^3 \leq F(P) \leq F_d(n) \leq E_d(n),$$

hence

$$F_d(n) = \Theta\left(n^3\right) \text{ and } E_d(n) = \Theta\left(n^3\right) \text{ for } d \geq 6.$$

We conjecture that in fact $F_d(n) \sim \frac{n^3}{6}\left(1 - \frac{1}{\lfloor \frac{d}{2} \rfloor}\right)\left(1 - \frac{2}{\lfloor \frac{d}{2} \rfloor}\right)$ for $d \geq 6$. The similar conjecture for the unit distance problem in dimension at least four was settled successfully by Erdős [7], unfortunately the proof relies heavily on the so called Erdős-Stone Theorem [12], and there is no similar Theorem for hypergraphs.

What are the best known bounds for $2 \leq d \leq 5$? For $d = 2$ we proved

$$\left(\frac{1}{3} - \frac{\sqrt{3}}{4\pi}\right) \leq \liminf_{n \to \infty} \frac{E_2(n)}{n^2} \leq \frac{1}{4}.$$

Concerning the unitary case

$$n^{1 + \frac{c}{\log \log n}} \leq F_2(n) \leq n^{\frac{4}{3}},$$

where the upper bound is a direct consequence of the Spencer, Szemerédi, Trotter Theorem [14], and the lower bound is given by a subset of the regular triangle lattice (Erdős [5]).

For $d = 3, 4$ the upper bounds $E_3(n) = O\left(n^{2.2}\right), F_3(n) = O\left(n^{1.8+\varepsilon}\right)$, and $F_4(n) = O\left(n^{65/23}\right)$ were proven by Akutsu, Tamaki and Tokuyama [1]. Purdy [13] proved as a special case of a more general result that $E_4(n) = O(n^{2.96})$.

Now we give an upper bound for $E_5(n)$ which in turn improves Purdy's result. A special case of an Erdős's Theorem [6] states that there is a positive integer n_0 satisfying that any 3-uniform hypergraph with $n \geq n_0$ vertices and at least $n^{3 - \frac{1}{9}}$ triangles contains a subhypergraph isomorphic to $K_3^{(3)}$ (the 3-uniform hypergraph consisting of 9 vertices partitioned into three equal size classes, and whose set of 3-edges is formed by all triangles with exactly one vertex in each class). We use this result to prove the following theorem.

Theorem 6 $E_5(n) = O\left(n^{3-\frac{1}{9}}\right)$.

Proof. Suppose that for some $n \geq n_0$ there is an n-point set P determining at least $n^{3-\frac{1}{9}}$ equilateral triangles. Consider the 3-uniform hypergraph $G = (P, \Delta(P))$. Then G contains a subhypergraph H isomorphic to $K_3^{(3)}$.

Assume $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3$ are the vertices of H and all triplets of the form $\{x_j, y_k, z_l\}$ are the corresponding 3-edges. It can be shown that all sides of triangles in H have the same length (since all triangles in H are equilateral and have many sides in common). Assume without loss of generality that they have unit length.

Then for all $j \in \{1,2,3\}$ the set $\{y_1, y_2, y_3, z_1, z_2, z_3\}$ is contained in the 4-dimensional unit sphere S_{x_j} with center in x_j. This implies that the 3-dimensional spheres $S_{1,2} := S_{x_1} \cap S_{x_2}$ and $S_{1,3} := S_{x_1} \cap S_{x_3}$ also contain the set $\{y_1, y_2, y_3, z_1, z_2, z_3\}$. Note that the spheres $S_{1,2}$ and $S_{1,3}$ are distinct, since their centers are the middle points of the segments x_1x_2 and x_1x_3 respectively. Hence $S := S_{1,2} \cap S_{1,3}$ is a 2-dimensional sphere containing $\{y_1, y_2, y_3, z_1, z_2, z_3\}$. Let $V_{1,2}$ be the set of points in S equidistant to y_1 and y_2, and similarly define $V_{1,3}$. Then $V_{1,2}$ and $V_{1,3}$ are distinct circles contained in S. To get a contradiction observe that $\{z_1, z_2, z_3\} \subseteq V_{1,2} \cap V_{1,3}$ but $|V_{1,2} \cap V_{1,3}| \leq 2$. ■

To complete this section we present examples in dimensions 4 and 5 with large number of unit equilateral triangles.

Theorem 7 *$F_4(n) = \Omega(n^2)$ and $F_5(n) = \Omega(n^{7/3})$.*

Proof. Let $A \subseteq \mathbb{R}^2$ be the set of vertices of $\lfloor \frac{n}{8} \rfloor$ unit squares centered at the origin. Define $P \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ by $P = (A \times \{0\}) \cup (\{0\} \times A)$. Notice that if a_1a_2 is a unit segment in A then the triangle $(a, 0), (0, a_1), (0, a_2)$ is equilateral for any $a \in A$. Therefore

$$F_4(n) \geq F(P) \geq 32 \left\lfloor \frac{n}{8} \right\rfloor^2 = \Omega(n^2).$$

For the second part consider the sphere $S \subseteq \mathbb{R}^3$ of radius $\frac{1}{\sqrt{2}}$ centered at the origin. Clarkson et al. [2] proved the existence of an $\lfloor \frac{n}{2} \rfloor$-point set $B \subseteq S$ which determines $\Omega(n^{4/3})$ unit distances. Define the set $Q = (A \times \{0\}) \cup (\{0\} \times B) \subseteq \mathbb{R}^2 \times \mathbb{R}^3$. Again if b_1b_2 is a unit segment in B then the triangle $(a, 0), (0, b_1), (0, b_2)$ is equilateral for any $a \in A$. Therefore

$$F_5(n) \geq F(Q) \geq n \cdot \Omega(n^{4/3}) = \Omega(n^{7/3}).$$

■

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