Tight sets of triangles in R^2

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Abstract

A 3-uniform hypergraph is called tight when for any 3-coloring of its vertex set a heterochromatic edge can be found. In the paper tightness of 3-graphs with vertex set R^2 and edge sets arising from simple geometrical considerations are studied. Basically we show that 3-graphs with "fat shadows" are tight and also that some interesting 3-graphs with "thin shadows" are tight too.

1 Introduction

3.2 A *k-graph* is a couple $G = (V, E)$ of its vertex set V and its edge set E. Edges are by definition subsets of V with cardinality k. A k-graph G is called *tight* whenever for any map f from the vertex set onto a set of cardinality k (the colors) there is an edge e of G such that $|f(e)| = k$ (e is heterochromatic). This notion was introduced in [1] as a generalization of connectedness of graphs (graphs are 2-graphs and they are tight if and only if they are connected).

In [1] and [2] it is studied the main question for finite 3-graphs, namely how "small" can be a tight 3-graph. In [3] some general results about tightness of infinite k-graphs are obtained. However this paper is the first attempt to study a concrete class of infinite k-graphs from the point of view of their classification into tight and untight k-graphs.

Actually, there is another motivation for this paper. When tightness for a k-graph has to be shown, one must prove that for any "appropriate" coloring there is an heterochromatic edge. On the other hand, it is said that an hypergraph is Ramsey whenever there is a monochromatic edge for any "appropriate" coloring . So, the

3-graph, with vertex set $V = \{$the edges of $K_6\}$ and edge set $E = \{$ the triangles in $K_6\}$, is well known to be Ramsey (coloring with two colors). Therefore Ramsey Theory is in some sense opposite to the Theory of tight hypergraphs. One of the must interesting branches of Ramsey Theory is the Euclidean Ramsey Theory (see [4,5]) where theorems are proved about Ramsey properties of hypergraphs arising from geometrical considerations in n-dimensional euclidean space. From this point of view the results below are some first small steps of a theory that could be called "Euclidean Antiramsey Theory".

Below, we study tightness of sets of triangles (three non collinear points in R^2) in the euclidean plane R^2. From now on T will be a set of triangles and we will say that T is tight when the 3-graph (R^2, T) is tight.

For the study of finite 3-graphs is fundamental the notion of the *trace* of a set of vertices (see [1]). However, we found out that for the problems treated in this paper it is most useful to introduce the concept of *shadow* of a segment (two different points in R^2). Let T be a set of triangles and let AB a segment. The set $ShAB$) of all points C in R^2 such that ABC is a triangle in T is called the *shadow* of AB in T or equivalently the T-shadow of AB. Always will be clear which is the set of triangles T. Also, when we talk that any property holds for the shadows, it will mean that this property holds for every segment in R^2.

2 Almost tight sets of triangles

Let T be the set of all equilateral triangles in R^2. By coloring red a single point with red, coloring blue a circle with center in the red point and coloring green any other point in R^2 (see fig. 1), we obtain a coloring which shows that T is not tight. In spite of that, in this coloring a weaker interesting property holds, namely, there are trichromatic triangles as near as required to an equilateral triangle.

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In this section we characterize through their shadows the sets of triangles for which the above property holds for any coloring of the plane. Moreover, it turns out that this characterization is useful to prove some shadow's criteria for tightness in the next section.

For a fixed coloring of the plane, a triangle ABC is said to be *almost trichromatic* if for every $\varepsilon > 0$ there exist a trichromatic triangle t such that each of the balls with radius ε and centers in A, B and C contains some vertex of t. A set of triangles is said to be *almost tight* if for any coloring of the plane it contains an almost trichromatic triangle.

Theorem 1 *A set of triangles is almost tight if and only if it has non empty shadows.*

Proof Suppose that the set of triangles T has non empty shadows and let us consider some green, blue, red-coloring of the plane. By a point of the type blue-green (blue-red, green-red) we mean a limit point of blue and green (blue and red, green and red) points. Consider P, Q and R three non collinear points with different colors (it is easy to see that they exist). Thus on the union of the segments PQ, QR and RP there exist points of at least two different types. So, we may assume that A and B are two points of different type (say A is blue-red and B is green-red). Let C be a blue point (the other cases are analogous) on the shadow of A and B. Then for any sufficiently small $\varepsilon > 0$ there exist a red point $A_\varepsilon \in Ball_\varepsilon(A)$ and a green point $B_\varepsilon \in Ball_\varepsilon(B)$, therefore the triangle $A_\varepsilon B_\varepsilon C$ is trichromatic and hereby the " if part " of the theorem is proved.

Reciprocally, let AB be a segment such that $Sh(AB)$ is empty. Let us color A with green, B with red and $R^2 \setminus \{A, B\}$ with blue. Suppose that $t \in T$ is an almost trichromatic triangle for this coloring. We have that $\{A, B\}$ is not contained in t. So, a point (say A) in $\{A, B\}$ is not in t and it is easy to see that for sufficiently small ε , A is not in the ε-neighborhood of any vertex of t. This is a contradiction. □

The elegant formulation of the preceding theorem is not suitable for its use in the next section. Actually, we proved an stronger fact in the "if part " of this theorem, namely the following.

Theorem 2 *Suppose the set of triangles has non empty shadows. Then there exist points A and B such that for every C in the shadow of A and B there exist two function $R^+ \ni \varepsilon \mapsto A_\varepsilon \in R^2$ and $R^+ \ni \varepsilon \mapsto B_\varepsilon \in R^2$ such that the distances between $A\varepsilon$ and $B\varepsilon$ to A and B respectively are less than ε and the triangle $A_\varepsilon B_\varepsilon C$ is trichromatic. Moreover, it is always possible to find those functions in a way that their images are monochromatic sets.*

3 Shadow's criteria

Unfortunately, there is a big difference between almost tight sets and tight sets. Namely, we were not able to found the characterization of the latter by properties of their shadows. However, in this section we show that if shadows are sufficiently "thin" ("fat") then the set of triangles is untight (tight).

By a shadow-closed set we mean a proper subset S of the plane with at least two points such that the shadow of every pair of points in S is contained in S.

Theorem 3 *Sets of triangles having shadow-closed sets are untight.*

Proof Let S be a shadow-closed set. Since S is a proper subset of the plane we may color it with blue and green and the rest of the plane with red. Thus every trichromatic triangle in T must have two vertices in S and the other not in S. But this is not posible by definition of shadow-closed set. □

Corollary 1 *Sets of triangles with numerable shadows are untight.*

Proof Take a segment AB in the plane and define the following sets:

$$C_1 = Sh(AB)\ ,\ C_i = \bigcup_{w_1,w_2 \in C_{i-1}} Sh(w_1w_2)\ ,\ S = \bigcup_{i=1}^{\infty} C_i\ .$$

Since $S \neq R^2$ it is shadow-closed. □

Theorem 4 *Sets of triangles with open shadows are tight.*

Proof Let us consider an arbitrary 3-coloring of the plane. By theorem 1 we know that there is in the set T an almost tricromatic triangle ABC. Since $Sh(BC)$ is an open set, there exist $\varepsilon' > 0$ such that $Ball_{\varepsilon'}(A) \subseteq Sh(BC)$. So, by theorem 2, there exist $A' \in Ball_{\varepsilon'}(A)$ and $B' \in Ball_{\varepsilon'}(B)$ such that $A'B'C$ is trichromatic. Since $A' \in Ball_{\varepsilon'}(A) \subseteq Sh(BC)$, we have that the triangle $A'BC$ is in T. As long as $Sh(A'C)$ is open, there exist $\varepsilon'' > 0$ such that $Ball_{\varepsilon''}(B) \subseteq Sh(A'C)$ (see fig. 2).

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Let B'' be a point in $Ball_{min(\varepsilon'',\varepsilon')}(B)$. Again, by theorem 2 the point B'' can be chosen with the same color as B' and therefore the triangle $A'B''C$ is trichromatic and in T. We conclude that T is tight. □

By theorem 1 every set of triangles T with non empty shadows is almost tight. This means that there are trichromatic triangles as near as you want to a triangle in T. So, we can suspect that if the set of triangles has some property of "stablility" under small movements then it will tight. We say that a set of triangles is *stable* whenever for every segment AB on the plane there exist $C \in Sh\,(AB)$ and $\varepsilon > 0$ such that $C \in Sh\,(A_0B_0)$ for every $A_o \in Ball_\varepsilon\,(A)$ and $B_o \in Ball_\varepsilon\,(B)$.

The following is a general criteria for tightness.

Theorem 5 *Every stable set of triangles is tight.*

Proof Let T be a stable set set. Of course, T has no empty shadows and T is almost tight. Let A and B be two points which satisfy the conditions of almost tightness. Since T is stable then there exist $C \in Sh\,(AB)$ and $\varepsilon > 0$ such that $C \in Sh\,(A_0B_0)$ for every $A_o \in Ball_\varepsilon\,(A)$ and $B_o \in Ball_\varepsilon\,(B)$. On the other hand almost tightness states that there exist $A' \in Ball_\varepsilon\,(A)$ and $B' \in Ball_\varepsilon\,(B)$ such that $A'B'C$ is trichromatic. Finally as $C \in Sh\,(A'B')$ then $A'B'C$ is a triangle in T. Therefore T is tight. □

Recall, that a *similarity* is a composition of a translation a rotation and a dilation. If T is a set of triangles such that $\varphi(T) = T$ for any similarity φ then we will say that T is *closed under similarities*. The set of all triangles similar to a given triangle has this property and is untight by corollary 1. However, if shadows have non empty interior, then the set must be tight.

Theorem 6 *If a set of triangles closed under similarities has shadows with non empty interior, then it is tight.*

Proof Let T be a set of triangles closed under similarities which has shadows with non empty interior. We shall prove that T is stable. Let AB be a segment on the plane. Since $Sh(AB)$ has nonempty interior, then there exist $C \in Sh(AB)$ and $r > 0$ such that $Ball_r(C) \subseteq Sh(AB)$. Let ε be a positive real number, $A' \in Ball_\varepsilon(A)$, $B' \in Ball_\varepsilon(B)$ and φ the similarity such that $\varphi(A) = A'$, $\varphi(B) = B'$. Denote by C' and r' the point and the number such that $\varphi(Ball_r(C)) = Ball_{r'}(C')$. We have $\lim_{\varepsilon\to 0} C' = C$ and $\lim_{\varepsilon\to 0} r' = r$, so for a suficiently small fixed ε we obtain that $C \in Ball_{r'}(C') \subseteq Sh(A'B')$ and therefore T is stable. □

We shall remark that a set of triangles having shadows with non empty interior is not necessarily tight as can be seen from the following example. Take two open disjoint balls in the plane. Color them with two different colors and color the rest of the plane with a third color. Taking the set of all triangles which are not trichromatic in this coloring we see that it is untight and the shadow of every segment has non empty interior.

4 Sets of triangles with "thin" shadows

In preceding section we proved some theorems showing that families with sufficiently "fat" shadows are tight. For example, the set of triagles with an angle in the interval $[0.01\pi, 0.02\pi]$ is tight by theorem 6 and the set of triangles with area greater than a given number is tight by theorem 4.

However, we can not apply those theorems in the case, say, of the set of all rectangle triangles or in the case of all isoceles triangles. The point is that here the shadows have empty interiors (see fig. 3).

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In this section we shall prove that several interesting, from the geometrical point of view, sets of triangles with "thin" shadows are tight.

It is not dificult to show that the set of all rectangle triangles is tight. More attracting is the general case when the triangles have a fixed angle. For a real number $\alpha \in (0, \pi)$ an *α-angle triangle* is a triangle having one of its angles with measure α.

Theorem 7 *The set of α-angle triangles is tigth for every $\alpha \in (0, \pi)$.*

Proof Let us start by considering a trichromatic triangle ABC such that $\angle BCA > \alpha$ (the existence of such triangle is granted by theorem 6). Supose A, B and C are colored red, green and blue respectively. Let D and E denote points on the rays $\overrightarrow{BC}$ and $\overrightarrow{AC}$ respectively, such that $\angle BDA = \angle BEA = \alpha$. If D is to be colored blue or green then ACD or ABD would become a trichromatic α-triangle, thus we will assume D is colored red, and by the same reason E is colored green. If any point X on the $\overrightarrow{DA}$ ray is to be colored green or blue then either XDC or XDB would be a trichromatic triangle with an angle α, thus we will suppose that every point on the $\overrightarrow{DA}$ ray is colored red and by the same reason the whole $\overrightarrow{EB}$ ray is colored green.

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Let F denote the intersection of the lines AD and BE (notice that we may assume AD is not parallel to BE by a suitable choice on the initial triangle). If F happens to be the intersection of the rays $\overrightarrow{EB}$ and $\overrightarrow{DA}$ then we are already done, otherwise F is such that $\angle BFA < \alpha$ (see fig. 4). By "moving" A' and B' on the rays $\overrightarrow{DA}$ and $\overrightarrow{EB}$

in such a way that $A'B'$ increases its length and remains parallel to AB we find that the angle $\angle B'CA'$ decreases continually, having as its limit value the angle $\angle BFA$, but as $\angle BFA < \alpha$ and $\angle BCA > \alpha$ we may assert by the intermediate value theorem, that there exist $A^* \in \overrightarrow{DA}$ and $B^* \in \overrightarrow{EB}$ such that $\angle B^*CA^* = \alpha$, thus obtaining the desired trichromatic triangle. □

Now, we will deal with sets of isosceles triangles. First of all, let us point out that the family of all isosceles triangles is tight; this can be easily seen by considering the circumcenter of an arbitrary trichromatic triangle. In fact there are several subsets of the isosceles triangles set wich are also tight. The following theorems refer to some of them.

Lemma 1 *For every r-coloring of the plane ($r > 1$), there always exist a different color pair of points at a given distance apart.*

Proof Let k be the given distance and A and B be points with different colors such that the length of AB is less than k. Consider a point C such that $AC = BC = k$ and note that either AC or BC is bichromatic in spite of the C color. □

Theorem 8 *The family of isosceles triangles with a side (any of its sides) of fixed length is tight.*

Proof Let k be an arbitrary positive number, consider an arbitrary blue, red, green coloring of the plane. By the above lemma, let P and Q be points at distance k and assume P is colored blue and Q is colored green. Let $F = Ball_{2k}(P) \cap Ball_{2k}(Q)$. Consider the following cases.

1.-There is a red point R on the interior of F. Consider the circumferences $C_k(R)$ and $C_k(P)$ and denote by P' a common point of these circumferences which does not lie on the line PQ (note that this point exist because $R \in int(F)$). If P' is colored red or green then $PP'Q$ or $PP'R$ would be a triangle as desired, therefore we will suppose P' is colored blue, and by the same reason, the analogue point Q' is colored green. Finally notice that the triangle $P'RQ'$ is trichromatic, isosceles and with two sides of length k.

2.- There are no red points on the interior of F.

Let P' and Q' be points on the line PQ such that $P'P = QQ' = \frac{k}{10}$(see fig. ?). Let $F_r = Ball_r(P') \cap Ball_r(Q')$. Let $S = \{r \in \mathbf{R}^+ : F_r \text{ contains red points}\}$ and denote by s the infimum of S. Let $t \geq s$ be a number arbitrary close to s such that there are red points on the F_t boundary. Denote by R a red point on the boundary of F_t. Take a point X in $C_k(R) \cap int(F_s)$ and note that X is not red, as there are no red points in $int(F_s)$. Assume witout losing generality that X is colored blue. If another point Y in $C_k(R) \cap int(F_s)$ is colored green then we are finished, thus we will suppose that every point in $C_k(R) \cap int(F_s)$ is colored blue.

Now consider the locus of the perpendicular bisectors of the segments RZ where Z is a point describing the arc $C_k(R) \cap int(F_s)$. Note that both P and Q belong to some of these perpendicular bisectors, because the only regions which are not covered by the perpendicular bisectors are those shown in the figure.

Therefore there is a point $Z \in C_k(R) \cap int(F_s)$ such that RZP or RZQ is a trichromatic triangle as desired. □

If we strength the conditions and ask for the family of isosceles triangles with both equal sides of fixed length then the result is false, this happens because the shadow of every sufficiently large segment is empty; the same holds for the family of isosceles triangles with the "different" side of fixed length, this time by considering the shadow of a sufficiently short segment.

The set of $(\alpha \pm \varepsilon)$-isosceles triangles will denote the set of all isosceles triangles such that the angle between the two equal sides belongs to the open interval $(\alpha - \varepsilon, \alpha + \varepsilon)$

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Theorem 9 *If $\alpha = 120^\circ$ or 90° then the set of $(\alpha \pm \varepsilon)$-isosceles triangles is tight.*

Proof Let $\alpha = 120^\circ$ and $\varepsilon > 0$, since the set of all equilateral triangles is almost tight then there exists a trichromatic triangle ABC such that

$$|\angle CAB - 60^\circ|, |\angle BCA - 60^\circ|, |\angle ABC - 60^\circ| < \frac{\varepsilon}{2}$$

Consider the circumcenter D of ABC and note that

$$\frac{\angle CDB}{\angle CAB} = \frac{\angle ADC}{\angle ABC} = \frac{\angle BDA}{\angle BCA} = 2$$

i.e.

$$|\angle CDB - 120^\circ|, |\angle BDA - 120^\circ|, |\angle ADC - 120^\circ| < \varepsilon$$

therefore, independently of the D color, any of the triangles CDB, ADC or BDA would be trichromatic and with the desired properties.

Now for the second part. Let $\alpha = 90°$ and $\varepsilon > 0$, since the set of all isosceles rectangle triangles is almost tight then there exist A, B and C colored green, blue and red such that

$$|\angle CAB - 45°|\,,\,|\angle ABC - 45°|\,,\,|\angle BCA - 90°|\,, < \frac{\varepsilon}{2}$$

Consider the circumcenter D of ABC and note that

$$\frac{\angle CDB}{\angle CAB} = \frac{\angle ADC}{\angle ABC} = \frac{\angle BDA}{\angle BCA} = 2$$

i.e.

$$|\angle CDB - 90°|\,,\,|\angle ADC - 90°|\,,\,|\angle BDA - 180°|\,, < \varepsilon$$

If D is colored green or blue then either BCD or ADC would be a trichromatic triangle as required, thus we will suppose D is colored red. Let E denote a point on the bisector of $\angle BDA$ such that $DE = CD$ (see fig. 5), observe that

$$\angle EDA = \angle BDE = \frac{1}{2}\angle BDA = \angle BCA$$

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If E is colored green or blue then either BDE or DAE is an isosceles trichromatic triangle with an almost $90°$ angle. If, on the contrary, E is colored red then it is easy to observe that the trichromatic triangle BAE is isosceles. Besides $\angle BAE = 180° - \angle BCA$ (the points A, E, B and C lie on the circle with center D), therefore

$$|\angle BAE - 90°| = |90° - \angle BCA| < \frac{\varepsilon}{2}$$

and thus the triangle BAE meets the requirements. □

The *k-ratio* set of triangles will denote all the triangles with a given ratio k between the lengths of two of their sides (the 1-ratio set is the set of isosceles triangles).

The following result is an application of the above theorem.

Theorem 10 *If* $k \in \left(\sqrt{2}-1, 1\right]$ *then the* k*-ratio set is tight.*

Proof Let $k \in \left(\sqrt{2}-1, 1\right]$. Consider an isosceles almost $90°$ trichromatic triangle ABC (A, B and C colored green, blue and red) with $\angle BCA \simeq 90°$ and assume $CA = CB = 1$.

Let $C_k(C)$ denote the circle with center C and radius k. Let X be a point on $C_k(C)$ and not on the lines BC and AC, if X is colored blue or green then any of CAX or CBX is a trichromatic triangle with the required ratio between two of its sides. Thus, we will suppose every point on $C_k(C)$ (except perhaps four points) is colored red.

Now consider the circle $C_{k \cdot AB}(B)$, note that $AB \simeq \sqrt{2}$ and consequently

$$k > \sqrt{2} - 1 \Rightarrow$$
$$k + k \cdot AB \simeq k + \sqrt{2}k > 1 = BC$$

i.e. the circles $C_k(C)$ and $C_{k \cdot AB}(B)$ intersect each other in a point D, which allow us to affirm that the triangle ABD is trichromatic and with the given relation $\frac{BD}{AB} = k$. $\square$

We will say that a triangle is *steady* if one of its sides is equal to its corresponding altitude.

Theorem 11 *The set of steady triangles is tight.*

Proof Let us consider an isosceles trichromatic triangle ABC with unequal side AB, say A green, B blue and C red. Let D be the midpoint of AB. Consider the following cases.

Case 1. D is colored red.

Let us consider the rectangle $ABPQ$ with $PQ = \frac{AB}{2}$. Since P and Q are in $Sh(A, D) \cap Sh(D, B)$ then we may suppose they are colored red (otherwise we would have finished). Notice that $Sh(B, P) \cap Sh(Q, A) \cap Sh(A, B) \neq \emptyset$ and so, no matter what the color of a point in this intersection is, we are already done.

Case 2. D is not colored green (assume D is colored green).

In this case we will just consider the region determined by the rays DB and DC. Consider the points $E = Sh(A, B) \cap Sh(D, C)$ and $F = Sh(A, C) \cap Sh(D, C)$. We may suppose E is colored green and F is colored red (otherwise we would have finished). But the trichromatic triangle EFA is also steady, therefore the set of steady triangles is tight. □

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