The 2-page crossing number of K_n

Bernardo M. Ábrego
California State University, Northridge
bernardo.abrego@csun.edu,

Oswin Aichholzer
Graz University of Technology
oaich@ist.tugraz.at

Silvia Fernández-Merchant
California State University, Northridge
silvia.fernandez@csun.edu,

Pedro Ramos
Universidad de Alcalá
pedro.ramos@uah.es

Gelasio Salazar
Universidad Autónoma de San Luis Potosí
gsalazar@ifisica.uaslp.mx.

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Abstract

Around 1958, Hill conjectured that the crossing number $\text{cr}(K_n)$ of the complete graph K_n is $Z(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$ and provided drawings of K_n with exactly $Z(n)$ crossings. Towards the end of the century, substantially different drawings of K_n with $Z(n)$ crossings were found. These drawings are *2-page book drawings*, that is, drawings where all the vertices are on a line ℓ (the spine) and each edge is fully contained in one of the two half-planes (pages) defined by ℓ. The *2-page crossing number* of K_n, denoted by $\nu_2(K_n)$, is the minimum number of crossings determined by a 2-page book drawing of K_n. It was generally conjectured that $\text{cr}(K_n) = Z(n)$ and since $\text{cr}(K_n) \leq \nu_2(K_n) \leq Z(n)$, the conjecture $\nu_2(K_n) = Z(n)$ appeared as a milestone in the way to find the correct values of $\text{cr}(K_n)$. In this paper we develop a novel and innovative technique to investigate crossings in drawings of K_n, and use it to prove that $\nu_2(K_n) = Z(n)$. To this end, we extend the inherent geometric definition of k-edges for finite sets of points in the plane to topological drawings of K_n. We also introduce the concept of $\leq\leq k$-edges as a useful generalization of $\leq k$-edges. Finally, we extend a powerful theorem that expresses the number of crossings in a rectilinear drawing of K_n in terms of its number of k-edges to the topological setting.

1 Introduction

In a *drawing* of a graph G in the plane, each vertex is represented by a point and each edge $e = uv$ is represented by a simple curve, in such a way that the endpoints of this simple curve are the points representing u and v. The *crossing number* $\mathrm{cr}(D)$ of a drawing D is the number of pairwise intersections of edges (in a point other than a common endpoint) in D, and the *crossing number* $\mathrm{cr}(G)$ of G is the minimum $\mathrm{cr}(D)$, taken over all drawings D of G.

A drawing is *good* if (i) no three edges meet at a point other than a vertex; (ii) two edges share at most one point (including vertices); and (iii) if two edges share a point and they are not both incident to the same vertex, then they cross at that point; that is, such a common point is a *crossing*, rather than tangential. It is well-known (and easy to prove) that every graph has a crossing-minimal drawing which is good. Thus, when our aim (as in this paper) is to estimate the crossing number of a graph, we may assume that all drawings under consideration are good.

Around 1958, Hill conjectured that

$$\mathrm{cr}(K_n) = Z(n) := \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor. \tag{1}$$

This conjecture appeared in print a few years later in papers by Guy [10] and Harary and Hill [11]. Hill described drawings of K_n with $Z(n)$ crossings, which were later corroborated by Blažek and Koman [2]. These drawings show that $\mathrm{cr}(K_n) \le Z(n)$. The best known general lower bound is $\lim_{n\to\infty} \mathrm{cr}(K_n)/Z(n) \ge 0.8594$, due to de Klerk et al. [9]. For more on the history of this problem we refer the reader to the excellent survey [3].

One of the major motivations for investigating crossing numbers is their application to VLSI design. With this motivation in mind, Chung, Leighton and Rosenberg [5] analyzed embeddings of graphs in books: the vertices lie on a line (the *spine*) and the edges lie on the *pages* of the book. If the book has k pages, and crossings among edges are allowed, the result is a *k-page book drawing*.

In this work we concentrate on 2-page book drawings. Thus, we may regard the pages as the closed half-planes defined by the spine. The *2-page crossing number* $\nu_2(G)$ of a graph G is the minimum of $\mathrm{cr}(D)$ taken over all 2-page book drawings D of G. Alternative terminologies for the 2-page crossing number are *circular crossing number* [12] and *fixed linear crossing number* [6].

In 1996, Shahrokhi et al. [15] found 2-page book drawings of K_n with $Z(n)$ crossings, thus showing that $\nu_2(K_n) \le Z(n)$. This yielded the natural conjecture (popularized by Vrt'o [16]) $\mathrm{cr}(K_n) = \nu_2(K_n) = Z(n)$.

Up until a few months ago, essentially the same results (lower bounds) were known for $\mathrm{cr}(K_n)$ as for $\nu_2(K_n)$; the only additional result was the exact calculation of $\nu_2(K_{13})$ and $\nu_2(K_{14})$ [4] (confirming Hill's Conjecture, i.e., Identity (1)). In [4], Buchheim and Zhang reformulated the problem of finding $\nu_2(K_n)$ as a maximum cut problem on associated graphs, and then solved exactly this maximum cut problem. Very recently, De Klerk and Pasechnik [8] used this max cut reformulation to find the exact value of $\nu_2(K_n)$ for all $n \le 21$ and $n = 24$, and also, by using semidefinite programming techniques, to obtain the asymptotic bound $\lim_{n\to\infty} \mathrm{cr}(K_n)/Z(n) \ge 0.9245$. Both the results in [4] and the results in [8] are computer-aided.

In this paper we prove that $\nu_2(K_n) = Z(n)$. The main technique for the proof is the extension of the concept of *k-edge* of a finite set of points to topological drawings of the complete graph. We do this in a way such that the identity proved by Ábrego and Fernández-Merchant [1] and Lovász et al. [14], that expresses the crossing number of a rectilinear drawing of K_n in terms of the k-edges of its set of vertices, is also valid in the topological setting.

We recall that a drawing D is *rectilinear* if the edges of D are straight line segments, and the *rectilinear crossing number* $\overline{\text{cr}}(G)$ of a graph G is the minimum of $\text{cr}(D)$ taken over all rectilinear drawings D of G. An edge pq of D is a *k-edge* if the line spanned by pq divides the remaining set of vertices into two subsets of cardinality k and $n-2-k$. Thus a k-edge is also an $(n-2-k)$-edge. Denote by $E_k(D)$ the number of k-edges of D. The following identity has been key to the recent developments on the rectilinear crossing number of K_n.

$$\overline{\text{cr}}\,(D) = 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2\rfloor-1} k\,(n-2-k)\,E_k\,(D)\,. \tag{2}$$

In Section 2 we generalize the concept of k-edge to arbitrary (that is, not necessarily rectilinear) drawings of K_n. This allows us to extend Identity (2) to topological (good) drawings of K_n. The key observation to extend the definition of k-edge to the new setting is to observe that, although half-planes are not well defined, we can use the orientation of the triangles defined by three points: the edge pq will be a k-edge of the topological drawing if the set of triangles adjacent to pq is divided, according to its orientation, into two subsets with cardinality k and $n-k-2$. In Section 3 we use this tool to show that $\nu_2(K_n) = Z(n)$. In order to do that, we need to introduce the new concept of $\leq\leq k$-edges, because for topological drawings the lower bound for $\leq k$-edges, $E_{\leq k}(D) \geq 3\binom{k+2}{2}$ does not hold. In Section 4 we analyze optimal 2-page drawings of K_n, and in Section 5 we present some open questions and directions for future research.

2 Crossings and k-edges

In this section we generalize the concept of k-edges, which has so far only been used in the geometric setting of finite sets of points in the plane, to topological drawings of K_n. Let D be a good drawing of K_n, let $\overrightarrow{pq}$ be a directed edge of D and r a vertex of D other than p or q. We say that *r is on the left (respectively, right) side of $\overrightarrow{pq}$* if the topological triangle pqr traced in that order (its vertices and edges correspond to those in D) is oriented counterclockwise (respectively, clockwise). Note that this is well defined as the three edges pq, qr, and rp in D do not self intersect and do not intersect each other, as D is a good drawing. We say that the edge pq is a *k-edge of D* if it has exactly k points of D on the same side (left or right), and thus $n-2-k$ points on the other side. Hence, as before, a k-edge is also an $(n-2-k)$-edge. Note that the direction of the edge pq is no longer relevant and every edge of D is a k-edge for some unique k such that $0 \leq k \leq \lfloor n/2\rfloor - 1$. Let $E_k(D)$ be the number of k-edges of D.

Theorem 1. *For any good drawing D of K_n in the plane the following identity holds,*

$$\mathrm{cr}\,(D) = 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k\,(n-2-k)\,E_k\,(D)\,.$$

Proof. In a good drawing of K_n, we say that an edge pq *separates* the vertices r and s if the orientations of the triangles pqr and pqs are opposite. In this case, we say that the set $\{pq, r, s\}$ is a *separation*. It is straightforward to check that any good drawing of K_4 is isomorphic to one of the three drawings shown in Figure 1.

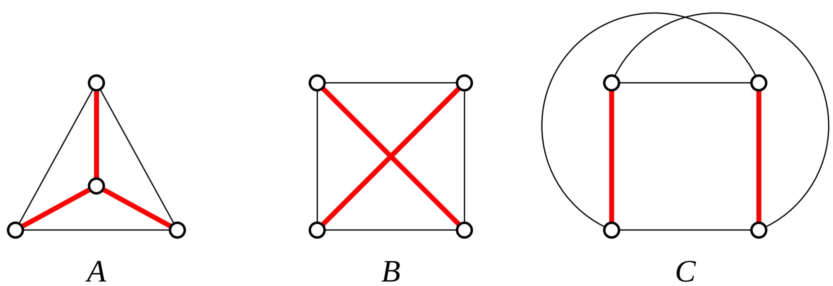

Figure 1: The three different types A, B, and C of nonisomorphic good drawings of K_4, with 3, 2, and 2 separations. The edge of each separation is shown bold.

We denote by T_A, T_B, and T_C the number of induced subdrawings of D of type A, B, and C, respectively. Then

$$T_A + T_B + T_C = \binom{n}{4}, \tag{3}$$

and since the subdrawings of types B or C are in one-to-one correspondence with the crossings of D, it follows that

$$\mathrm{cr}\,(D) = T_B + T_C. \tag{4}$$

We count the number of separations in D in two different ways: First, each subdrawing of type A has 3 separations (the edge in each separation is bold in Figure 1), and each subdrawing of types B or C has 2 separations. This gives a total of $3T_A + 2T_B + 2T_C$ separations in D. Second, each k-edge belongs to exactly $k(n-2-k)$ separations. Summing over all k-edges for $0 \le k \le \lfloor n/2 \rfloor - 1$ gives a total of $\sum_{k=0}^{\lfloor n/2 \rfloor - 1} k(n-2-k)E_k(D)$ separations in D. Therefore

$$3T_A + 2T_B + 2T_C = \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k\,(n-2-k)\,E_k(D). \tag{5}$$

Finally, subtracting Identity (5) from three times Identity (3) we get

$$T_B + T_C = 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k\,(n-2-k)\,E_k(D),$$

and thus by Identity (4) we obtain the claimed result. □

For $0 \leq k \leq \lfloor n/2 \rfloor - 1$ and D a good drawing of K_n, we define the set of *$\leq k$-edges* of D as all j-edges in D for $j = 0, \ldots, k$. The number of $\leq k$-edges of D is denoted by

$$E_{\leq k}(D) := \sum_{j=0}^{k} E_j(D).$$

Similarly, we denote the number of *$\leq\leq k$-edges* of D by

$$E_{\leq\leq k}(D) := \sum_{j=0}^{k} E_{\leq j}(D) = \sum_{j=0}^{k} \sum_{i=0}^{j} E_i(D) = \sum_{i=0}^{k} (k+1-i) E_i(D).$$

To avoid special cases, we define $E_{\leq\leq -1}(D) = E_{\leq\leq -2}(D) = 0$.

The following result restates Theorem 1 in terms of the number of $\leq\leq k$-edges.

Proposition 2. *Let D be a good drawing of K_n. Then*

$$cr(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 2} E_{\leq\leq k}(D) - \frac{1}{2}\binom{n}{2}\left\lfloor \frac{n-2}{2} \right\rfloor - \frac{1}{2}\left(1 + (-1)^n\right) E_{\leq\leq \lfloor n/2 \rfloor - 2}(D).$$

Proof. First note that for $2 \leq k \leq \lfloor n/2 \rfloor - 1$ we have that $E_{\leq\leq k}(D) - E_{\leq\leq k-1}(D) = E_{\leq k}(D)$ and $E_{\leq k}(D) - E_{\leq k-1}(D) = E_k(D)$. Thus

$$E_k(D) = E_{\leq\leq k}(D) - 2E_{\leq\leq k-1}(D) + E_{\leq\leq k-2}(D).$$

We rewrite the last term in Theorem 1.

$$\begin{aligned}
\sum_{k=0}^{\lfloor n/2 \rfloor - 1} k(n-2-k) E_k(D) &= \sum_{k=2}^{\lfloor n/2 \rfloor - 1} k(n-2-k)\left(E_{\leq\leq k}(D) - 2E_{\leq\leq k-1}(D) + E_{\leq\leq k-2}(D)\right) \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor - 3} \left(k(n-2-k) - 2(k+1)(n-3-k) + (k+2)(n-4-k)\right) E_{\leq\leq k}(D) \\
&\qquad + \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\left(n - 1 - \left\lfloor \frac{n}{2} \right\rfloor\right) E_{\leq\leq \lfloor n/2 \rfloor - 1}(D) \\
&\quad + \left(-2\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\left(n - 1 - \left\lfloor \frac{n}{2} \right\rfloor\right) + \left(\left\lfloor \frac{n}{2} \right\rfloor - 2\right)\left(n - \left\lfloor \frac{n}{2} \right\rfloor\right)\right) E_{\leq\leq \lfloor n/2 \rfloor - 2}(D) \\
&= -2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq\leq k}(D) + \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\left(n - 1 - \left\lfloor \frac{n}{2} \right\rfloor\right) E_{\leq\leq \lfloor n/2 \rfloor - 1}(D) \\
&\qquad + \left(-2\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\left(n - 1 - \left\lfloor \frac{n}{2} \right\rfloor\right) + \left(\left\lfloor \frac{n}{2} \right\rfloor - 2\right)\left(n - \left\lfloor \frac{n}{2} \right\rfloor\right)\right) E_{\leq\leq \lfloor n/2 \rfloor - 2}(D).
\end{aligned}$$

Because $E_{\leq\leq \lfloor n/2 \rfloor - 1}(D) = E_{\leq\leq \lfloor n/2 \rfloor - 2}(D) + E_{\leq \lfloor n/2 \rfloor - 1}(D) = E_{\leq\leq \lfloor n/2 \rfloor - 2}(D) + \binom{n}{2}$, it follows by

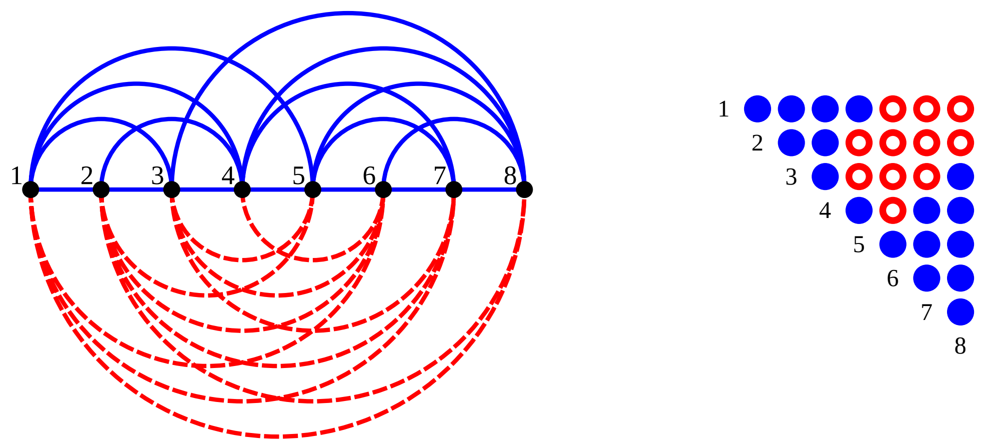

Figure 2: Two-colored diagram for a 2-page book drawing D of K_8 and the corresponding 2-page matrix. Solid dots and lines represent blue edges. Open dots and dashed lines represent red edges.

Theorem 1 that

$$\begin{aligned}\operatorname{cr}(D) &= 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2\rfloor-1} k(n-2-k)E_k(D) = 3\binom{n}{4} + 2\sum_{k=0}^{\lfloor n/2\rfloor-3} E_{\leq\leq k}(D) \\ &+ \left(n+1-2\left\lfloor\frac{n}{2}\right\rfloor\right)E_{\leq\leq\lfloor n/2\rfloor-2}(D) - \left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\left(n-1-\left\lfloor\frac{n}{2}\right\rfloor\right)\binom{n}{2} \\ &= 2\sum_{k=0}^{\lfloor n/2\rfloor-3} E_{\leq\leq k}(D) - \frac{1}{2}\binom{n}{2}\left\lfloor\frac{n-2}{2}\right\rfloor + \begin{cases} E_{\leq\leq\lfloor n/2\rfloor-2}(D) & \text{if } n \text{ is even} \\ 2E_{\leq\leq\lfloor n/2\rfloor-2}(D) & \text{if } n \text{ is odd,}\end{cases}\end{aligned}$$

which is equivalent to the claimed result. □

3 The 2-page crossing number of K_n

Consider a 2-page book drawing D of K_n with horizontal spine and label the vertices $1, 2, \ldots, n$ from left to right. Color the edges above or on the spine blue and below the spine red, respectively. We construct an upper triangular matrix which corresponds to the coloring of these edges, see Figure 2. We call this the *2-page matrix* of D. Label the columns of the 2-page matrix with $2, \ldots, n$ from left to right and the rows with $1, 2, \ldots, n-1$ from top to bottom. For $i < j$ an entry ij (row-column) in the 2-page matrix is a point with the same color as the edge ij in the drawing D.

We start by proving some basic properties of the 2-page matrix.

Lemma 3. *Let D be a 2-page book drawing of K_n. For $1 \leq i < j \leq n$, let k be the sum of the number of points to the right plus the number of points above the entry ij in the 2-page matrix of D, which have the same color as ij. Then the edge ij is a k-edge. (It is possible to have $k > \lfloor n/2\rfloor - 1$.)*

Proof. Let $1 \le i < j \le n$ and assume that the edge ij is blue (red). We count the number of points l in D to the left (right) of ij. For $l \ne i, j$ the triangle ijl is oriented counter-clockwise (clockwise) if and only if either $l < i$ and the edge lj is blue (red), or $l > j$ and the edge il is blue (red). In the first case these edges correspond to blue (red) points above the entry ij, and in the second case to blue (red) points to the right of the entry ij, respectively. □

In view of Lemma 3 we say that the point in the cell ij of the 2-page matrix of D *represents* a k-edge if ij is a k-edge (or an $(n-2-k)$-edge) in D.

Lemma 4. *For $k < n/2 - 1$ and for $1 \le j \le k+1$, in the 2-page matrix of a drawing D of K_n there are at least $2\,(k+2-j)$ points in row j representing $\le k$-edges. Similarly, for $n-k \le j \le n$ there are at least $2\,(k+1-n+j)$ points in column j representing $\le k$-edges.*

Proof. For $1 \le j \le k+1$, in row j the rightmost $k+2-j$ points of each color represent $\le k$-edges as they have at most $k+1-j$ points of their color to the right and at most $j-1$ on top. So if each color appears at least $k+2-j$ times in row j, we have guaranteed $2\,(k+2-j)$ $\le k$-edges in row j. If one of the colors appears fewer than $k+2-j$ times, so that there are $k+2-j-e$ blue points in row j for some $1 \le e \le k+2-j$, then there are $n-j-(k+2-j-e) = n-2-k+e$ red points in this row. In this case we claim that also the leftmost e red points in this row represent $\le k$-edges. In fact, for $1 \le i \le e$, the i-th red point (from the left) in row j, has exactly $n-2-k+e-i$ red points to the right and perhaps more red points on top. Since for $n \ge 2$ we have $n-2-k+e-i \ge n/2-k$, this i-th red point also represents a $\le k$-edge. The equivalent result holds for the rightmost $k+1$ columns. □

Lemma 5. *For $0 \le j < n/2 - 1$, in the 2-page matrix of a drawing D of K_n there are two points in column n which correspond to j-edges in D. For n even there exists one such point in column n corresponding to an $(n/2-1)$-edge in D.*

Proof. We follow the lines of the proof of Lemma 4. Consider the points in column n in order from top to bottom. By Lemma 3 the i-th vertex of a color corresponds to an $(i-1)$-edge. Thus, if there are at least $j+1$ vertices for each color we are done. Otherwise assume without loss of generality that there are $j+1-e$ blue points in column n for some $1 \le e \le j+1$. Then there are $n-1-(j+1-e) = n-j+e-2$ red points in this column. For $1 \le i \le \lfloor n/2 \rfloor$ the i-th red point corresponds to an $(i-1)$-edge, and for $\lfloor n/2 \rfloor + 1 \le i \le n-j+e-2$ the i-th red point corresponds to an $(i-1) = (n-i-1)$-edge. Thus we get two red points corresponding to j-edges for $i = j+1$ and $i = n-j-1$. Finally, observe that these two points are different for $j < n/2-1$. For n even we get only one such point for $j = n/2-1$. □

The next theorem gives a lower bound on the number of $\le\le k$-edges, which will play a central role in deriving our main result.

Theorem 6. *Let $n \ge 3$. For every 2-page book drawing D of K_n and $0 \le k < n/2 - 1$, we have*

$$E_{\le\le k}(D) \ge 3\binom{k+3}{3}.$$

Proof. We proceed by induction on n. The induction base $n = 3$ holds trivially. For $n \geq 4$, consider a 2-page book drawing D of K_n with horizontal spine and label the vertices from left to right with $1, 2, \ldots, n$. Remove the point n and all incident edges to obtain a 2-page book drawing D' of K_{n-1}. To bound $E_{\leq\leq k}(D)$, recall that

$$E_{\leq\leq k}(D) = \sum_{j=0}^{k} (k + 1 - j) E_j(D). \tag{6}$$

All edges incident to n are in D but are not in D'. In fact, by Lemma 5, there are two j-edges adjacent to the vertex n for each $0 \leq j \leq k \leq \lfloor n/2 \rfloor - 2$. These edges contribute with $2\sum_{j=0}^{k}(k + 1 - j) = 2\binom{k+2}{2}$ to Identity (6). We next compare Identity (6) to

$$E_{\leq\leq k-1}(D') = \sum_{j=0}^{k-1} (k - j) E_j(D'). \tag{7}$$

Any edge contributing to Identity (7) also contributes to Identity (6), but possibly with a different value. A j-edge in D' is a j-edge or a $(j + 1)$-edge in D. More precisely, if for $a < b$ the point ab in the 2-page matrix of D' represents a blue j-edge in D' (the equivalent argument holds if ab is red), with $0 \leq j \leq \lfloor (n - 1)/2 \rfloor - 1$, then the sum of blue points above it or to its right is either j or $n - 3 - j$. If the edge an is blue, then the point ab has $j + 1$ or $n - 3 - j + 1$ blue points above it or to its right in the 2-page matrix of D. This means that the edge ab is a $(j + 1)$- or an $n - 2 - (n - 3 - j + 1) = j$-edge in D, respectively. If the edge an is red, then the point ab in the 2-page matrix of D has the same number of blue points above it or to its right as in the 2-page matrix of D'. So it is a j- or an $n - 2 - (n - 3 - j) = (j + 1)$-edge in D, respectively. A j-edge in D' contributes to Identity (7) with $k - j$. A j-edge and a $(j + 1)$-edge in D contribute to Identity (6) with $k + 1 - j$ and $k - j$, respectively. This is a gain of $+1$ or 0, respectively, towards $E_{\leq\leq k}(D)$ when compared to $E_{\leq\leq k-1}(D')$. Finally, a k-edge in both D and D' does not contribute to Identity (7) and contributes to Identity (6) with $+1$. Therefore

$$E_{\leq\leq k}(D) = E_{\leq\leq k-1}(D') + 2\binom{k+2}{2} + \sum_{j=0}^{k} \#(j\text{-edges in } D' \text{ that are } j\text{-edges in } D).$$

By induction hypothesis, $E_{\leq\leq k-1}(D') \geq 3\binom{k+2}{3}$ and thus

$$\begin{aligned} E_{\leq\leq k}(D) &\geq 3\binom{k+2}{3} + 2\binom{k+2}{2} + \sum_{j=0}^{k} \#(j\text{-edges in } D' \text{ that are } j\text{-edges in } D) \\ &= 3\binom{k+3}{3} - \binom{k+2}{2} + \sum_{j=0}^{k} \#(j\text{-edges in } D' \text{ that are } j\text{-edges in } D). \end{aligned}$$

We finally prove that

$$\sum_{j=0}^{k} \#(j\text{-edges in } D' \text{ that are } j\text{-edges in } D) \geq \binom{k+2}{2}. \tag{8}$$

In fact, we prove that for each $1 \leq j \leq k + 1$ there are at least $k + 2 - j$ points in row j of the 2-page matrix of D that represent i-edges in both D' and D for some $0 \leq i \leq k$. Suppose that the edge jn is blue (the equivalent argument holds when jn is red). Then any red point in row j with $i \leq k$ red points above it or to its right represents an i-edge in both D and D' for some $0 \leq i \leq k$; and any blue point in row j with $i \geq n-2-k$ blue points above it or to its right represents an i'-edge in both D and D' for some $0 \leq i' \leq k$. Thus, the first $k+2-j$ red points from the right in row j (if they exist) represent $\leq k$-edges, because they have at most $k+2-j-1$ red points to the right and at most $j-1$ red points above, and therefore they all contribute to the left-hand-side of Inequality (8). If there are fewer than $k+2-j$ red points in row j, say $k+2-j-e$ for some $1 \leq e \leq k+2-j$, then the first e blue points in row j from the left represent $\leq k$-edges, because they have at least $n-j-e \geq n-j-k-2+j = n-k-2$ blue points to their right. Hence there are at least $k+2-j-e$ red points and at least e blue points (for a total of at least $k+2-j$ points) that represent $\leq k$-edges in row j that contribute to the left-hand-side of Inequality (8). Summing over all $1 \leq j \leq k+1$, we get that

$$\sum_{j=0}^{k} \#(j\text{-edges in } D' \text{ that are } j\text{-edges in } D) \geq \sum_{j=1}^{k+1} (k+2-j) = \binom{k+2}{2}. \qquad \square$$

We are now ready to prove our main result, namely that the 2-page crossing number of K_n is $Z(n)$.

Theorem 7. *For every positive integer n, $\nu_2(K_n) = Z(n)$.*

Proof. The cases $n=1$ and $n=2$ are trivial. Let $n \geq 3$. As we mentioned before, 2-page book drawings with $Z(n)$ crossings were constructed by Shahrokhi, et al. [15] showing $\nu_2(K_n) \leq Z(n)$. Let D be a 2-page book drawing of K_n. To prove the lower bound, we use Theorem 6 and Proposition 2.

$$\begin{aligned} \operatorname{cr}(D) &\geq 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 2} 3\binom{k+3}{3} - \frac{1}{2}\binom{n}{2}\left\lfloor \frac{n-2}{2} \right\rfloor - \frac{3}{2}\left(1+(-1)^n\right)\binom{\lfloor \frac{n}{2} \rfloor + 1}{3} \\ &= 6\binom{\lfloor \frac{n}{2} \rfloor + 2}{4} - \frac{1}{2}\binom{n}{2}\left\lfloor \frac{n-2}{2} \right\rfloor - \frac{3}{2}\left(1+(-1)^n\right)\binom{\lfloor \frac{n}{2} \rfloor + 1}{3} \\ &= \begin{cases} \frac{1}{64}(n-1)^2(n-3)^2 & \text{if } n \text{ is odd} \\ \frac{1}{64}n(n-2)^2(n-4) & \text{if } n \text{ is even} \end{cases} = Z(n). \end{aligned}$$

$\square$

4 Optimal configurations

It was proved in [1] and [14] that the inequality $E_{\leq k}(P) \geq 3\binom{k+2}{2}$ holds for every set P of n points in general position in the plane and for every k such that $0 \leq k \leq \lfloor n/2 \rfloor - 2$. This inequality used with the rectilinear version of Theorem 1 gives $Z(n)$ as a lower bound for the rectilinear crossing number of K_n [1]. In contrast to the rectilinear case, the inequality $E_{\leq k}(D) \geq 3\binom{k+2}{2}$ does not hold in general for topological drawings D of K_n, not even for general 2-page book drawings as can be seen in Figure 3. This shows the relevance of introducing the number $E_{\leq\leq k}(D)$ and Theorem 6.

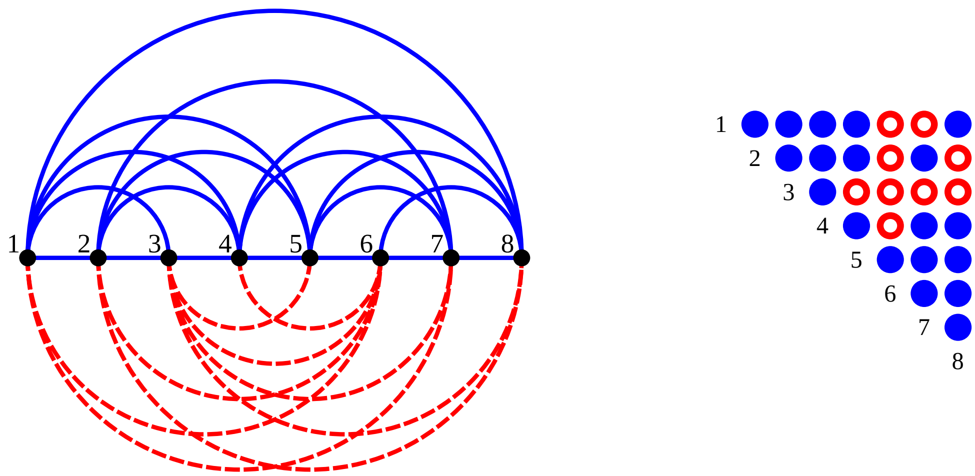

Figure 3: This 2-page book drawing of K_8 has four 0-edges, $(1,7)$, $(1,8)$, $(2,7)$, and $(2,8)$, and four 1-edges, $(1,5)$, $(1,6)$, $(3,8)$, and $(4,8)$.

However, the inequality $E_{\leq k}(D) \geq 3\binom{k+2}{2}$ does hold for crossing optimal drawings of K_n, where in fact the following stronger result is true.

Proposition 8. *Let D be a 2-page book drawing of K_n and $I_n = \{k \in \mathbb{Z} : 0 \leq k \leq \lfloor n/2 \rfloor - 2\}$. The following are equivalent: (i) $\operatorname{cr}(D) = Z(n)$, (ii) $E_k(D) = 3(k+1)$ for all $k \in I_n$, (iii) $E_{\leq k}(D) = 3\binom{k+2}{2}$ for all $k \in I_n$, and (iv) $E_{\leq\leq k}(D) = 3\binom{k+3}{3}$ for all $k \in I_n$.*

Proof. In order to satisfy (i) equality must be achieved in the proof of Theorem 7, which is precisely (iv). Reciprocally, (iv) implies equality in the proof of Theorem 7, which proves the equivalency of (i) and (iv). The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) follow directly from the definitions of $E_{\leq k}(D)$ and $E_{\leq\leq k}(D)$, using the identity $\sum_{m=0}^{r}\binom{m}{s} = \binom{r+1}{s+1}$. It remains to show that (iv) implies (ii), which we do by applying induction on k. For the induction base note that $E_{\leq\leq 0}(D) = E_{\leq 0}(D) = E_0(D) = 3$. For $1 \leq k \leq \lfloor n/2 \rfloor - 2$, the identities $E_j(D) = 3(j+1)$ for all $0 \leq j \leq k-1$ and $E_{\leq\leq k}(D) = 3\binom{k+3}{3}$ imply that

$$3\binom{k+3}{3} = E_{\leq\leq k}(D) = \sum_{j=0}^{k}(k+1-j)\,E_j(D) = E_k(D) + 3\sum_{j=0}^{k-1}(k+1-j)(j+1),$$

and thus

$$E_k(D) = 3\binom{k+3}{3} - 3\sum_{j=0}^{k-1}(k+1-j)(j+1) = 3\binom{k+3}{3} - \frac{1}{2}k\,(k+1)\,(k+5) = 3\,(k+1)\,. \quad \square$$

An elementary counting argument (see for instance [7]) shows that if n is even and $\operatorname{cr}(K_{n-1}) = Z(n-1)$, then $\operatorname{cr}(K_n) = Z(n)$. Moreover, under these assumptions any topological drawing of K_n with $Z(n)$ crossings satisfies that every induced subdrawing with $n-1$ points has $Z(n-1)$ crossings. Our technique allows a different proof of this fact for 2-page book drawings of K_n, but more importantly it gives for the first time a similar property when n is odd.

Proposition 9. *Let D be a 2-page book drawing of K_n with $Z(n)$ crossings and D' an induced subdrawing of D with $n-1$ vertices. Then D' satisfies that $E_k(D') = 3(k+1)$ for all k such that $0 \leq k \leq \lfloor n/2 \rfloor - 3$. In other words, if D is crossing optimal, then D' is crossing optimal when n is even, and D' is* almost *crossing optimal when n is odd, in the sense that Proposition 8(ii) holds for every $k \in I_{n-1}$ except perhaps for $k = (n-5)/2$.*

Proof. In order to achieve $\text{cr}(D) = Z(n)$, Proposition 8(iv) must hold and thus equality must be achieved throughout the proof of Theorem 6 for all $k \in I_n$. In particular, $E_{\leq\leq k}(D') = 3\binom{k+3}{3}$ for every

$$0 \leq k \leq \lfloor n/2 \rfloor - 3 = \begin{cases} \lfloor (n-1)/2 \rfloor - 2 \text{ if } n \text{ is even,} \\ \lfloor (n-1)/2 \rfloor - 3 \text{ if } n \text{ is odd.} \end{cases}$$

This means that D' satisfies Proposition 8(iv) and thus $\text{cr}(D') = Z(n-1)$ if n is even. However, if n is odd, then D' satisfies Proposition 8(iv) for all $k \in I_{n-1}$ except perhaps for $k = \lfloor (n-1)/2 \rfloor - 2 = (n-5)/2$. This is equivalent to satisfying Proposition 8(ii) for all k in the same range and thus being almost crossing optimal in the sense of the statement. □

A more detailed analysis on the crossing optimal 2-page book drawings will be included in the journal version of this work.

5 Concluding remarks

In this paper we proved the conjecture $\nu_2(K_n) = Z(n)$. One key technique used is the extension of the geometric concept of k-edge to topological drawings of the complete graph. In this way we "discretized" the problem and applied the identity proved by Ábrego and Fernández-Merchant [1] and Lovász et al. [14], that expresses the crossing number of a rectilinear drawing of K_n in terms of k-edges.

Our approach to determine k-edges in the topological setting is to define the orientation of three vertices by the orientation of the corresponding triangle in a good drawing of the complete graph. It is naturally to ask whether this defines an abstract order type. To this end the setting would have to satisfy the axiomatic system of [13](page 4). But it is easy to construct an example which does not fulfill these axioms, that is, our setting does not constitute an abstract order type in the sense of [13]. It is an interesting question for further research how this new concept compares to the classic order type, both in terms of theory (realizability, etc.) and applications.

We believe that the developed techniques of generalized orientation, k-edge for topological drawings, and $\leq\leq k$-edges are of interest in their own. We will investigate their usefulness for related problems in future work. For example, they might also play a central role to approach the crossing number problem for general drawings of the complete and complete bipartite graphs.

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