The maximum number of halving lines and the rectilinear crossing number of K_n for $n \leq 27$

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Abstract

For $n \leq 27$ we present exact values for the maximum number $h(n)$ of halving lines and $\widetilde{h}(n)$ of halving pseudolines, determined by n points in the plane. For this range of values of n we also present exact values of the rectilinear $\overline{\text{cr}}(K_n)$ and the pseudolinear $\widetilde{\text{cr}}(K_n)$ crossing numbers of the complete graph K_n. $\widetilde{h}(n)$ and $\widetilde{\text{cr}}(K_n)$ are new for $n \in \{14, 16, 18, 20, 22, 23, 24, 25, 26, 27\}$, $h(n)$ is new for $n \in \{16, 18, 20, 22, 23, 24, 25, 26, 27\}$, and $\overline{\text{cr}}(K_n)$ is new for $n \in \{20, 22, 23, 24, 25, 26, 27\}$.

Keywords: Halving lines, rectilinear crossing number, complete graphs

1 Introduction

Let S be an n-point set in $\mathbb{R}^2$ in general position. A *k-set* of S is a set P of k points in S that can be separated from $S \backslash P$ using a straight line. The so called "k-set problem" asks for the maximum number of k-sets that an

n-element set can have. In a similar fashion we say that a directed segment $\overrightarrow{s_1 s_2}$ in S is a *k-edge* if there are exactly k points in S to the right side of $s_1 s_2$. It is easy to see that there is a 1-to-1 correspondence between k-sets and $(k-1)$-edges, so an equivalent problem is to find the maximum number of $(k-1)$-edges determined by n points in the plane. When n is even and $k = (n-2)/2$ the k-edges are called *halving lines,* that is, lines through two points in S leaving $(n-2)/2$ points of S on each side. When n is odd the halving lines leave $(n-3)/2$ and $(n-1)/2$ points of S on each side.

An important open problem in discrete geometry is to find the maximum number $h(n)$ of halving lines that can be determined by n points in the plane. This was first raised by Erdős, Lovász, Simmons, and Straus [11], [14]. Another important and related problem was proposed by Erdős and Guy: find the minimum number of convex quadrilaterals in a set of n points in general position. Equivalently, determine $\overline{\mathrm{cr}}(K_n)$, the *rectilinear crossing number* of K_n [10], that is, the smallest number of crossings in a drawing of the complete graph K_n, in which every edge is drawn as a straight segment. Further references and related problems can be found in [8].

All these problems can be formulated in the more general setting of generalized configurations of points [13]. A *generalized configuration* or a *pseudoconfiguration* consists of a set of points in the plane together with an *arrangement of pseudolines*, such that every pair of points has exactly one pseudoline passing through them. A *pseudoline* is a curve in $\mathbb{P}^2$, the projective plane, whose removal does not disconnect $\mathbb{P}^2$. An *arrangement of pseudolines* is a collection of pseudolines with the property that every two of them intersect each other exactly once. In this new setting we can define by analogy *k-pseudoedges*, *halving pseudolines*, and *pseudolinear crossing number* $\widetilde{\mathrm{cr}}(K_n)$ of K_n. We denote by $\widetilde{h}(n)$ the maximum number of halving pseudolines spanned by generalized configurations of n points in the plane. We also let $\mathcal{N}_k(n)$ and $\mathcal{N}_{\leq k}(n)$ denote the maximum number of $(k-1)$-pseudoedges, $\leq (k-1)$-pseudoedges respectively, determined by pseudoconfigurations of n points. Trivially, $\widetilde{\mathrm{cr}}(K_n) \leq \overline{\mathrm{cr}}(K_n)$ and $h(n) \leq \widetilde{h}(n)$.

Here we report improved lower bounds for $\widetilde{h}(n)$. This improvement is enough to match the geometric constructions that serve as upper bounds in the range $n \in \{14, 16, 18, 20, 22, 23, 24, 25, 26, 27\}$. We also obtain new lower bounds for $\mathcal{N}_{\leq \lfloor n/2 \rfloor - 1}(n)$. As a consequence we determine the exact values of $\widetilde{\mathrm{cr}}(K_n), \overline{\mathrm{cr}}(K_n)$, $\widetilde{h}(n)$, and $h(n)$ for the same range. The new values are summarized in Table 1. It is important to note that all of these bounds are shown to be tight thanks to the remarkable (indeed, as we show, optimal) geometric constructions obtained by Aichholzer et al. [4].

n	14	16	18	20	22	23	24	25	26	27
$h(n) = \widetilde{h}(n)$	22	27	33	38	44	75	51	85	57	96
$\mathcal{N}_{\leq \lfloor n/2 \rfloor -1}(n)$	69	93	120	152	187	178	225	215	268	255
$\overline{cr}(n) = \widetilde{cr}(n)$	324	603	1029	1657	2528	3077	3699	4430	5250	6180

Table 1
New exact values for $h(n), \widetilde{h}(n), \overline{\mathrm{cr}}(K_n)$, and $\widetilde{\mathrm{cr}}(K_n)$.

Here is the previous history about the quest for (small values of) $h(n)$. For $2 \leq n \leq 8$ $h(n)$ is easily obtained, and since all generalized configurations of points with $n \leq 8$ are stretchable [12], then $h(n) = \widetilde{h}(n)$ in this range. Eppstein [9] found point sets with even $10 \leq n \leq 18$ and a large number of halving lines. In particular he matched the upper bound found by Stöckl [16] for $\widetilde{h}(10)$. Andrzejak et al. [6] proved $h(12) = 18$. Later Beygelzimer and Radziszowski [7] extended this to $\widetilde{h}(12) = 18$ and they also proved that $h(14) = 22$. With respect to the odd values, Aichholzer et al. [5] found tight upper bounds for $h(n)$ with n odd, $11 \leq n \leq 21$.

Previous to this work, the exact value of $\overline{\mathrm{cr}}(K_n)$ was known for $n \leq 19$ and for $n = 21$ ([5]). For these values of n, it was recently proved that $\widetilde{\mathrm{cr}}(K_n) = \overline{\mathrm{cr}}(K_n)$ [3]. For general lower and upper bounds see [5], [3], and [2].

2 The Central Bound

In what follows Π denotes a circular sequence on n elements, that is, a doubly infinite sequence $(..., \pi_{-1}, \pi_0, \pi_1, ...)$ of permutations on n elements, such that any two consecutive permutations π_i and π_{i+1} differ by a transposition τ_i of neighboring elements, and such that for every j, π_j is the reversed permutation of $\pi_{j+\binom{n}{2}}$. Goodman and Pollack [13] established a one-to-one correspondence between circular sequences and generalized configurations of points. Thus we say that a circular sequence Π is *associated* to a set of n points S. When Π corresponds to a geometric drawing of K_n (i.e., each pseudoline is a straight line) we say that Π is *stretchable*. In this case S is a set of n points in general position in the plane. Any subsequence of Π consisting of $\binom{n}{2}$ consecutive permutations is an *n-halfperiod*. If τ_j occurs between elements in positions i and $i+1$ we say that τ_j is an *i-transposition*. If $i \leq n/2$ then any i-transposition or $(n-i)$-transposition is called *i-critical*. If Π is a finite subsequence of Π

then $\mathcal{N}_k(\Pi)$ and $\mathcal{N}_{\leq k}(\Pi)$ denote the number of k-critical and $(\leq k)$-critical transpositions in Π respectively. A k-transposition corresponds to a $(k-1)$-pseudoedge which also coincides with a $(k-1)$-edge if Π is stretchable.

We make use of two known results (A in [1] and [15], B in [5] and [3]):

$$\text{(A) } \widetilde{\mathrm{cr}}(\Pi) = \sum_{k=1}^{\lfloor n/2 \rfloor} (n-2k-1)\mathcal{N}_k(\Pi) - (3/4)\binom{n}{3} + (1/8)(1+(-1)^{n+1})\binom{n}{2}.$$

$$\text{(B) } \mathcal{N}_{\leq k}(\Pi) \geq 3\binom{k+1}{2} + 3\binom{k+1-\lfloor n/3 \rfloor}{2} - \max\{0, (k-\lfloor n/3 \rfloor)(n-3\lfloor n/3 \rfloor)\}.$$

Our main new tool is the following.

Theorem 2.1 *Let Π be a circular sequence associated to a generalized configuration of n points. Then*

$$\mathcal{N}_{\lfloor n/2 \rfloor}(\Pi) \leq \begin{cases} \left\lfloor \frac{1}{2}\binom{n}{2} - \frac{1}{2}\mathcal{N}_{\leq \lfloor n/2 \rfloor - 2}(\Pi) \right\rfloor, \text{if } n \text{ is even,} \\ \left\lfloor \frac{2}{3}\binom{n}{2} - \frac{2}{3}\mathcal{N}_{\leq \lfloor n/2 \rfloor - 2}(\Pi) + \frac{1}{3} \right\rfloor, \text{if } n \text{ is odd.} \end{cases}$$

Proof. For even n we prove that there must be at least one $(n/2-1)$-critical transposition between any two consecutive $n/2$-transpositions τ_i and τ_j $(i<j)$. Suppose τ_i transposes a and b. Then before τ_j takes place, at least one element a or b must leave the center (two middle positions, $n/2$ and $n/2+1$). This corresponds to having at least one $(n/2-1)$-critical transposition between τ_i and τ_j. In a given halfperiod the same holds for the last and first $n/2$-transpositions. Thus $\mathcal{N}_{n/2}(\Pi) \leq \mathcal{N}_{n/2-1}(\Pi)$. Since $\mathcal{N}_{\leq \lfloor n/2 \rfloor}(\Pi) = \binom{n}{2}$ then $2\mathcal{N}_{\lfloor n/2 \rfloor}(\Pi) \leq \binom{n}{2} - \mathcal{N}_{\leq \lfloor n/2 \rfloor - 2}(\Pi)$ and the result follows.

For odd n, let $\tau_1', \tau_2', ..., \tau_w'$ with $w = \mathcal{N}_{(n-3)/2}(\Pi)$ be the $(n-3)/2$-critical transpositions of a halfperiod Π ordered by their occurrence within Π. Assume without loss of generality that the first transposition of Π is τ_1'. Let b_i be the number of $(n-1)/2$-critical transpositions that occur after τ_i' and before τ_{i+1}' (or the end of the halfperiod if $i=w$). Note that $b_i \leq 3$ since the three elements in the center (that is, those elements in the middle three positions $(n \pm 1)/2$ and $(n+3)/2$) remain in the center between two consecutive $(n-3)/2$-critical transpositions. We prove that if $b_i = b_j = 3$ for some $i<j$ and no other b in between equals 3, then there is some l between i and j such that $b_l \leq 1$. Thus either at most one $b_i = 3$ or the average of $b_1, b_2, ..., b_w$ is ≤ 2. Thus $\mathcal{N}_{(n-1)/2}(\Pi) = \sum_{i=1}^{w} b_i \leq 2w+1 = 2\mathcal{N}_{(n-3)/2}(\Pi)+1$.

Now note that $j \neq i+1$ since all three elements in the center were transposed between τ_i' and τ_{i+1}' and two of them remain in the center between τ_{i+1} and τ_{i+2} (or the end of Π). That is, $b_{i+1} \leq 2$. Assume by way of contradiction that $b_l = 2$ for all $i < l < j$. One of the three transpositions after τ'_j does not involve the new element brought to the center by τ'_j. Thus this transposition can take place right before τ'_j without modifying $\mathcal{N}_{\leq k}(\Pi)$ (the other two transpositions switched order). But now $b_{j-1} = 3$, i.e., the gap between "threes" was reduced. We can do the same until $b_{i+1} = 3$ which is impossible. Finally since $\mathcal{N}_{\leq \lfloor n/2 \rfloor}(\Pi) = \binom{n}{2}$ then $3\mathcal{N}_{\lfloor n/2 \rfloor}(\Pi) \leq 2\binom{n}{2} - 2\mathcal{N}_{\leq \lfloor n/2 \rfloor -2}(\Pi) + 1$ and the result follows. □

3 New exact values of $h(n), \widetilde{h}(n), \overline{\text{cr}}(K_n), \widetilde{\text{cr}}(K_n)$

Theorem 1 gives a new upper bound for $\widetilde{h}(n)$ if we use the bound (B) for $\mathcal{N}_{\leq \lfloor n/2 \rfloor -2}(\Pi)$. The numerical values of this bound in our range of interest are shown in Table 1. From Theorem 1 and the fact that $\mathcal{N}_{\leq \lfloor n/2 \rfloor -1}(\Pi) = \binom{n}{2} - \mathcal{N}_{\lfloor n/2 \rfloor}(\Pi)$ we obtain that $\mathcal{N}_{\leq \lfloor n/2 \rfloor -1}(\Pi) \geq \left\lceil \frac{1}{2}\binom{n}{2} + \frac{1}{2}\mathcal{N}_{\leq \lfloor n/2 \rfloor -2}(\Pi) \right\rceil$ if n is even, and $\mathcal{N}_{\leq \lfloor n/2 \rfloor -1}(\Pi) \geq \left\lceil \frac{1}{3}\binom{n}{2} + \frac{2}{3}\mathcal{N}_{\leq \lfloor n/2 \rfloor -2}(\Pi) - \frac{1}{3} \right\rceil$ if n is odd. Then, by applying the bound in (B) for $\mathcal{N}_{\leq \lfloor n/2 \rfloor -2}(\Pi)$, we get for $n \geq 10$

$$\mathcal{N}_{\leq \lfloor n/2 \rfloor -1}(n) \geq \begin{cases} \binom{n}{2} - \left\lfloor \frac{1}{24}n(n+30) - 3 \right\rfloor & \text{if } n \text{ is even,} \\ \binom{n}{2} - \left\lfloor \frac{1}{18}(n-3)(n+45) + \frac{1}{9} \right\rfloor & \text{if } n \text{ is odd.} \end{cases}$$

This lower bound is at least as good as (B) with $k = \lfloor n/2 \rfloor - 1$ for all even $n \geq 10$ and all odd $n \geq 21$. In Table 1 we show the bounds obtained for our range of n values. We also calculate a new lower bound for $\widetilde{\text{cr}}(K_n)$ using (A) with $\mathcal{N}_{\lfloor n/2 \rfloor}(n) = \binom{n}{2}$, the previous bound for $\mathcal{N}_{\leq \lfloor n/2 \rfloor -1}(n)$, and (B) for $k \leq \lfloor n/2 \rfloor - 2$. All the bounds shown in Table 1 are attained by Aichholzer's et al. constructions [4], and thus Table 1 actually shows the exact values of $\widetilde{h}(n)$, $h(n)$, $\mathcal{N}_{\leq \lfloor n/2 \rfloor -1}(n)$, $\widetilde{\text{cr}}(K_n)$, and $\overline{\text{cr}}(K_n)$ for n in the specified range.

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