Crossings in twisted graphs

B. M. Ábrego,* S. Fernández-Merchant* A. P. Figueroa,†
J. J. Montellano-Ballesteros,‡ E. Rivera-Campo§

Abstract

We consider twisted graphs, that is, topological graphs that are weakly isomorphic to subgraphs of the complete twisted graph. We determine the exact minimum number of crossings of edges among the set of twisted graphs with n vertices and m edges; state a version of the crossing lemma for twisted graphs and conclude that the mid-range crossing constant for twisted graphs is 1/6.

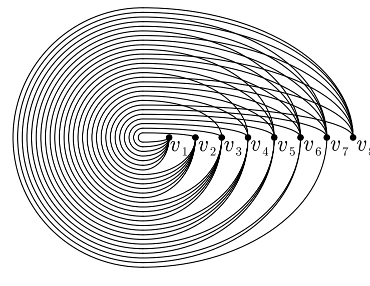

Figure 1: The complete twisted graph T_8.

1 Introduction

A *simple topological graph* is a drawing of a graph in the plane where the vertices are points, and the edges are simple continuous arcs satisfying that any two arcs have at most one point in common, which is either a common endpoint or a proper crossing, and no arc passes through any other vertex different from its endpoints. If all edges of a simple topological graph are straight-line segments, then it is called a *geometric graph*. A geometric graph whose vertices are in convex position, is called a *convex graph*. Two simple topological graphs G and G' are *weakly isomorphic* if there exists an isomorphism between G and G' such that two edges of G' cross if and only if the corresponding edges of G do.

The *complete twisted graph* T_n is a complete simple topological graph with vertices $v_1 v_2, \dots v_n$ such that two edges $v_i v_j$ and $v_{i'} v_{j'}$ cross if and only if $i < i' < j' < j$ or $i' < i < j < j'$. (See Figure 1.) A simple topological graph G is a *twisted graph* if G is weakly isomorphic to a subgraph of T_n. Twisted graphs were found in [3] as complete topological graphs with maximum number of edge-crossings but with no subgraph weakly isomorphic to the complete convex graph with 5 vertices. A version of the Erdős-Szereres Theorem for topological graphs appeared in [5]: every complete topological graph with n vertices has a topological subgraph with $m \geq c \log^{1/8} n$ edges, which is weakly isomorphic to a complete convex or twisted graph. Several problems previously studied for convex graphs were recently studied for twisted graphs [4].

We are interested in crossing numbers on twisted graphs. The *crossing number* of a topological graph D is the number of edge-crossings in D. This measure for the non-planarity of a graph has been extensively studied [7]. Some of the major motivations for investigating crossing numbers are their applications to VLSI design, to digital visual design, and to classical problems in discrete geometry. One of the fundamental results in this area is known as the *crossing lemma* [2]: for any topological graph D with n vertices and $m > 4n$ edges, $cr(D) \geq (1/64)m^3/n^2$, and this is tight except for the multiplicative constant 1/64. This constant has been progressively improved with the best current bound $cr(D) \geq (1/29)m^3/n^2$ (for $m > 7n$) [1]. It was proved in [6] that this multiplicative constant tends to a positive constant, called the *mid-range crossing constant*, when $n \to \infty$ and $n \ll m \ll n^2$.

We prove the *crossing lemma for twisted graphs* and conclude that the *mid-range crossing constant for twisted graphs* is 1/6. In fact, for every m and n, we determine the exact minimum crossing number within the class $T_{n,m}$ of twisted graphs with n vertices and m edges and provide a family of crossing optimal graphs.

2 Results

Let $cr_T(n, m)$ be the minimum number of crossings among all twisted graphs in $T_{n,m}$. Let $t_{n,m}$ be the unique integer such that $\binom{n-t_{n,m}-1}{2} \leq \binom{n}{2} - m <$

*California State Univ., Northridge, [bernardo.abrego, silvia.fernandez]@csun.edu.
†Dept. de Matemáticas, ITAM, apaulinafg@gmail.com
‡Inst. de Matemáticas, UNAM, juancho@im.unam.mx
§Dept. de Matemáticas, UAM-I, erc@xanum.uam.mx

$\binom{n-t_{n,m}}{2}$. Then
$t_{n,m} = n - \left\lfloor \frac{3}{2} + \sqrt{\left(n-\frac{1}{2}\right)^2 - 2m} \right\rfloor$.

Theorem 1 (Crossing number of twisted graphs)**.** *Let n and m be integers with $n \geq 1$ and $0 \leq m \leq \binom{n}{2}$. Then $cr_T(n,m) =$*

$$\binom{t_{n,m}}{2}m \; - \; 2\binom{t_{n,m}+1}{3}n \; + \; 3\binom{t_{n,m}+2}{4}.$$

This theorem is a direct consequence of the following two results.

Theorem 2 (Crossing inequalities)**.** *Let $G \in T_{n,m}$. For all $t \leq n$, $cr(G) \geq \binom{t}{2}m - 2\binom{t+1}{3}n + 3\binom{t+2}{4}$.*

This bound is actually tight for any n and m when $t = t_{n,m}$.

Theorem 3 (Tightness of crossing inequalities)**.** *For any $n \geq 1$ and $0 \leq m \leq \binom{n}{2}$, there exists a twisted graph G in $T_{n,m}$ such that*

$$cr(G) = \binom{t_{n,m}}{2}m - 2\binom{t_{n,m}+1}{3}n + 3\binom{t_{n,m}+2}{4}.$$

Namely, the set of edges of G is

$$\{(i,j) : 1 \leq i \leq n-1, i+1 \leq j \leq \min\{i+t,n\}\} \cup \{(i, t+1+i) : 1 \leq i \leq s\},$$

where $t = t_{n,m}$ and $s = \binom{n-t}{2} - \binom{n}{2} + m$.

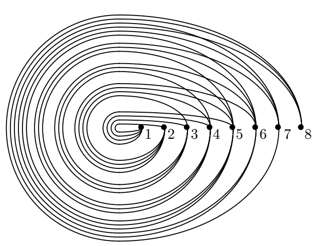

Figure 2: A twisted graph with 8 vertices, 20 edges, and $cr_T(8,20)$ crossings.

Here is how our bound relates to the classic crossing lemma and mid-range crossing constant [2, 6]. Let $n < m$ be positive integers; the notation $n \ll m \ll n^2$ stands for m is a function of n such that $lim_{n\to\infty}(n/m) = lim_{n\to\infty}(m/n^2) = 0$.

Theorem 4 (Crossing lemma for twisted graphs)**.** *Let $G \in T_{n,m}$ for integers n and m. If $n \ll m \ll n^2$, then*

$$cr(G) \geq \frac{1}{6} \cdot \frac{m^3}{n^2} - o\left(\frac{m^3}{n^2}\right).$$

If $m = cn$ for some constant $c \geq 2$ and $n > \frac{\lfloor c \rfloor^2 + 3\lfloor c \rfloor + 2}{\lfloor c \rfloor + 1 - c}$, then

$$cr(G) \geq \binom{\lfloor c \rfloor}{2}\left(\frac{3c - 2\lfloor c \rfloor - 2}{3c^3}\right)\frac{m^3}{n^2} + 3\binom{\lfloor c \rfloor + 2}{4}.$$

If $m = cn^2$ for some constant $0 < c \leq \frac{1}{2}$, then

$$cr(G) \geq \left(\frac{3\sqrt{1-2c}+1}{3(1+\sqrt{1-2c})^3} + o(1)\right)\frac{m^3}{n^2}.$$

Moreover, these inequalities are tight.

An immediate corollary of Theorem 4 is the exact value of this mid-range crossing constant.

Theorem 5 (Mid-range crossing constant for twisted graphs)**.** *Let n and m be integers such that $n \ll m \ll n^2$. Then*

$$\lim_{n\to\infty} cr_T(n,m)\frac{n^2}{m^3} = \frac{1}{6}.$$

There is a nice transition in the behavior of the constant before and after the mid-range. Namely, if $m = cn$ for some constant $c > 2$, then

$$\lim_{n\to\infty} cr_T(n,m)\frac{n^2}{m^3} = \binom{\lfloor c \rfloor}{2}\left(\frac{3c - 2\lfloor c \rfloor - 2}{3c^3}\right)$$

and

$$\lim_{c\to\infty} \binom{\lfloor c \rfloor}{2}\left(\frac{3c - 2\lfloor c \rfloor - 2}{3c^3}\right) = \frac{1}{6}.$$

Similarly, If $m = cn^2$ for some constant $0 < c \leq \frac{1}{2}$, then

$$\lim_{n\to\infty} cr_T(n,m)\frac{n^2}{m^3} = \frac{\sqrt{1-2c}+1/3}{(1+\sqrt{1-2c})^3}$$

and

$$\lim_{c\to 0^+} \frac{\sqrt{1-2c}+1/3}{(1+\sqrt{1-2c})^3} = \frac{1}{6}.$$

References

[1] E. Ackerman. On topological graphs with at most four crossings per edge. arXiv:1509.01932v1.

[2] M. Ajtai, V. Chvátal, M. Newborn, and A. Szemerédi. Crossing-free subgraphs, *Ann. Discrete Mathematics*, **12** (1982), 9-12.

[3] H. Harborth, I. Mengersen, Drawings of the complete graph with maximum number of crossings. In: Proc. 23rd Southeastern Conf. on Comb., Graph Th., and Computing, **88** (1992) 225–228.

[4] E. Omaña-Pulido, E. Rivera-Campo, Notes on the twisted graph. In: Márquez A., Ramos P., Urrutia J. (eds) EGC 2011. Lecture Notes in Computer Science **7579** (2012) 119–125.

[5] J. Pach, J. Solymosi, G. Tóth: Unavoidable configurations in complete topological graphs. *Discrete Comput. Geom.* **30** (2003) 311–320.

[6] J. Pach, J. Spencer and G. Tóth, New bounds on crossing numbers, *Discrete Comput. Geom.* **24** (2000), 623–644.

[7] M. Schaefer, The graph crossing number and its variants: A survey, *E. J. Combinatorics*, **DS21**, (2014).