On the convex crossing number

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Abstract

We improve the bounds for the minimum number $\text{bkcr}_1(n, m)$ of crossings among convex drawings of graphs with n vertices and m edges. We show that $\frac{1}{3.9551} \leq \text{bkcr}_1(n, m)n^2/m^3 \leq \frac{1}{3}$, whenever $n \ll m \ll n^2$. We also present conjectures about the precise and asymptotic value of $\text{bkcr}_1(n, m)$.

1 Convex geometric graphs

A *convex drawing* of a graph is a drawing of a graph where the vertices are located on a convex closed curve and the edges are closed curves that lie entirely within that curve. The goal of this paper is to bound the number of crossings in such drawings depending on its number of vertices and edges, n and m, respectively. In 1982, Ajtai, Chvátal, Newborn, and Szemerédi [1], and independently Leighton [2], proved the so called *Crossing Lemma*. It states that any drawing of any graph with n vertices and $m > 4n$ edges has at least $c \cdot m^3/n^2$ edge-crossings, where c is a universal constant.

This result was refined for convex drawings by Shahrokhi et al. [6]. Specifically, for any graph G, if $\text{bkcr}_1(G)$ denotes the minimum number of crossings over all convex drawings of G, then $\text{bkcr}_1(G) \geq \frac{1}{27}m^3/n^2$ for any graph G with n vertices and $m \geq 3n$ edges. (The notation $\text{bkcr}_1(G)$ follows [5] because the convex crossing number is equivalent to the book crossing number of graphs drawn in a single page)

In this paper we improve Shahrokhi et al. result as follows

Theorem 1. *If G is a graph with n vertices and $m \geq (61/16)n$ edges, then*

$$\text{bkcr}_1(G) \geq \frac{512}{2025}\frac{(m-n)^3}{n^2} > \frac{1}{3.9551}\frac{(m-n)^3}{n^2}$$

To complement this result, we exhibit drawings of graphs with n vertices, m edges, and few crossings.

Theorem 2. *For every $n \geq 3$ and $m \geq n-3$, there are graphs G on n vertices and m edges such that*

$$\text{bkcr}_1(G) \leq \frac{1}{3}\frac{(m+1)^3}{(n-2)^3}.$$

2 Crossing inequalities

The proof of the main theorem is based on a technique developed by Pach et al. [4, 3] for general graphs: the idea is to prove tight inequalities for the crossing number of sparse graphs, and then use the probabilistic method to establish a general result. These inequalities are interesting on their own as they settle the minimum value of $\text{bkcr}_1(G)$ for some classes of sparse graphs.

Because the edges joining consecutive vertices in the boundary do not have any crossings, from now on we consider only drawings with no edges among consecutive vertices. We call these drawings *strictly convex* (this only affects m by at most n, which is negligible when $m \gg n$).

Theorem 3. *If D is a strictly convex drawing of a graph on n vertices and m edges, then*

1. $\text{cr}(D) \geq (m+1) - (n-2)$,
2. $\text{cr}(D) \geq \frac{7}{3}(m+1) - 3(n-2)$,
3. $\text{cr}(D) \geq \frac{25}{6}(m+1) - \frac{20}{3}(n-2)$,
4. $\text{cr}(D) \geq 6(m+1) - \frac{45}{4}(n-2)$.

The first three inequalities are tight for $n-2 \leq m+1 \leq \frac{3}{2}(n-2)$, $\frac{3}{2}(n-2) \leq m+1 \leq 2(n-2)$, and $2(n-2) \leq m+1 \leq \frac{5}{2}(n-2)$; respectively.

The proof is omitted due to space limitations.

3 Proof of Theorem 1

Consider an arbitrary convex drawing D on n vertices and m edges. Remove the $b \leq n$ edges in

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the boundary. Then consider a random induced subgraph H obtained by selecting each vertex independently with probability p. If $n(H)$, $m(H)$, and $\mathrm{cr}(H)$ denote the number of vertices, edges, and crossings of H, respectively; then by Theorem 3(4), it follows that $\mathrm{cr}(H) + \frac{33}{2} \geq 6m(H) - \frac{45}{4}n(H)$. Furthermore, the expected value preserves this inequality and $\mathbb{E}(n(H)) = pn$, $\mathbb{E}(m(H)) = p^2(m-b)$, and $\mathbb{E}(\mathrm{cr}(H)) = p^4 \mathrm{cr}(D)$. Thus

$$\mathrm{cr}(D) \geq 6(m-b)p^{-2} - \frac{45}{4}np^{-3} - \frac{33}{2}p^{-4}.$$

Letting $p = 45n/(16(m-b))$, it follows that

$$\mathrm{cr}(D) \geq \frac{512}{2025}\frac{(m-n)^3}{n^2} + o(m^3/n^2).$$

4 Constructions

Theorem 4. *There exist strictly convex drawings D on n vertices and m edges such that*

$$\mathrm{cr}(D) \leq \frac{1}{3}\frac{(m+1)^3}{(n-2)^2}.$$

Proof. Let m and n be positive integers. Let D_j denote a strictly convex drawing of the complete graph on j vertices minus its j boundary edges. The constructions are obtained by connecting copies of D_k and D_{k+1} by an edge (between any 2 copies, see Figure 1). If there are a copies of D_k and b copies of D_{k+1}, then the resulting drawing has $n = a(k-2) + b(k-1) + 2$ vertices, $m = a(\binom{k}{2} - k + 1) + b(\binom{k+1}{2} - k) - 1$ edges, and $cr = a\binom{k}{4} + b\binom{k+1}{4}$ crossings. This implies that

$$cr = \frac{k(3k-5)}{12}(m+1) - \frac{k(k-1)^2}{12}(n-2).$$

This shows the tightness of the first three inequalities in Theorem 3. If $k = \lceil 2(m+1)/(n-2) \rceil$, then it can be verified that

$$\frac{k(3k-5)}{12}(m+1) - \frac{k(k-1)^2}{12}(n-2) \leq \frac{(m+1)^3}{(n-2)^2}.$$

□

5 Conjectures

Conjecture 5. *For any integer $k \geq 3$, and any strictly convex drawing D of a graph on n vertices and m edges*

$$\mathrm{cr}(D) \geq \frac{k(3k-5)}{12}(m+1) - \frac{k(k-1)^2}{12}(n-2).$$

This conjecture was proved for $k \in \{3,4,5\}$ in Theorem 3. If this conjecture is true, then by the proof of Theorem 4, the inequality would be tight

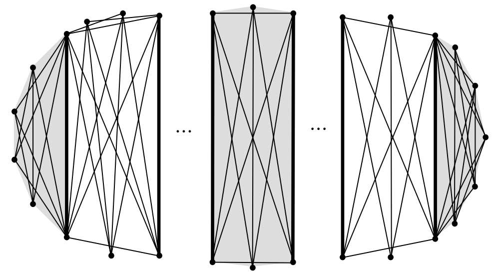

Figure 1: A construction illustrating Theorem 4 for $k = 6$. This construction is obtained by attaching copies of D_k and D_{k+1}.

when $(k-1)(n-2)/2 \leq (m+1) \leq k(n-2)$. Furthermore, the validity of this conjecture would provide the exact minimum number of crossings of an arbitrary strictly convex drawing.

Conjecture 6. *For any strictly convex drawing D of a graph on n vertices and m edges let $k = \lceil 2(m+1)/(n-2) \rceil$. We have that*

$$\mathrm{cr}(D) \geq \frac{k(3k-5)}{12}(m+1) - \frac{k(k-1)^2}{12}(n-2),$$

and this inequality is tight for some D.

Finally, this last conjecture would further imply the value of the mid-range crossing constant for convex graphs.

Conjecture 7. *If $\mathrm{bkcr}_1(n,m)$ denotes the minimum number of crossings among all convex drawings on n vertices and m edges, and if $n \ll m \ll n^2$, then*

$$\lim_{n\to\infty} \mathrm{bkcr}_1(n,m)\frac{n^2}{m^3} = \frac{1}{3}.$$

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