Minimizing crossings of 2-page drawings of K_n with prescribed number of edges in each page

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Abstract

We consider the problem of determining the 2-page book crossing number of the complete graph K_n when the number of edges in each page is given. We find upper and lower bounds of the right order of magnitude depending on the number of edges in the page with the least number of edges.

1 Introduction

A *k-page book* is the union of k-half planes in the space with common boundary and disjoint interiors. The common boundary is a straight line called the *spine* and the half planes are called *pages*. We are concerned only with 2-page book drawings of K_n, that is, drawings of K_n such that the vertices are placed on the spine and each edge (except by its endpoints) is completely contained in the interior of a single page. Ábrego et al. [1] proved Harary-Hill's conjecture (see [3] and [2]) for 2-page drawings, that is, they proved that a 2-page drawing of the complete graph has at least $\frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$ crossings. This inequality is tight; in fact the authors of [1] partially classified the drawings that achieve equality and, as expected, these drawings have essentially the same number of edges in each page. In this paper we consider the problem of minimizing $\nu_2(K_n, r)$, the number of crossings of the complete graph K_n on n vertices where r edges are drawn on one page (the red edges) and the remaining $\binom{n}{2} - r$ edges are drawn on the other page (the blue edges). Given n and r, this problem is equivalent to determining the minimum number of monochromatic crossings (red-red or blue-blue) over all different ways to color exactly r red and $\binom{n}{2} - r$ blue diagonals of the regular n-gon.

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2 Results

Let D be a 2-page drawing of the complete graph on n vertices with exactly r edges in one page and $\binom{n}{2} - r$ on the other page. We call R the set of r red edges, and B the set of the remaining $\binom{n}{2} - r$ blue edges. In this paper we concentrate on the case when $r = o(n^2)$. The full range of values of r will be included in the full version. For simplicity, we assume that n is even. Our main result follows.

Theorem 1. *If $r = o(n^2)$, then*

$$\frac{1}{2}r^2 - o(r^2) \leq \nu_2(K_n, r) - \binom{n}{4} + \frac{1}{4}r(n-2)^2 \leq 0.6109r^2 + o(r^2).$$

The proof of this theorem is outlined in what follows. The proofs of the lemmas are omitted for lack of space. Using the model of the regular n-gon, we define $\mathrm{cr}(X, Y)$ as the number of crossings of edges in X with edges in Y. If $X = \{e\}$, we write $\mathrm{cr}(e, Y)$ instead of $\mathrm{cr}(\{e\}, Y)$. In addition, every edge e separates $(n-2)/2 - i(e)$ points from the remaining $(n-2)/2 + i(e)$ points, where $0 \leq i(e) \leq (n-2)/2$ is twice the difference of the absolute value of the difference of points in each side.

Theorem 2. *If D is a 2-page drawing of the complete graph on n vertices with exactly r edges in one page, then*

$$\mathrm{cr}(D) = \binom{n}{4} - \frac{1}{4}r(n-2)^2 + \sum_{e \text{ red}} \left(\mathrm{cr}(e, R) + i(e)^2\right).$$

Proof. In the model of 2-colored edges in the regular n-gon, every red edge e has $(n-2)/2 - i(e)$ vertices on one side and $(n-2)/2 + i(e)$ vertices on the other, and any edge joining vertices on different sides crosses e. Thus

$$\frac{1}{4}(n-2)^2 - i(e)^2 = \mathrm{cr}(e, R) + \mathrm{cr}(e, B).$$

Adding this equation over all red edges yields

$$\frac{1}{4}r(n-2)^2 - \sum_{e \text{ red}} i(e)^2 = \sum_{e \text{ red}} \mathrm{cr}(e, R) + \mathrm{cr}(R, B).$$

Because $\text{cr}(D) = \binom{n}{4} - \text{cr}(R, B)$, it follows that

$$\text{cr}(D) = \binom{n}{4} - \frac{1}{4} r(n-2)^2 + \sum_{e \text{ red}} \left(\text{cr}(e, R) + i(e)^2\right).$$

□

For the rest of the paper, we assume that D is a 2-page drawing of the complete graph on n vertices with exactly $r = |R|$ edges in one page and with $\nu_2(K_n, r)$ crossings.

We construct an upper triangular matrix $M(D)$ which corresponds to the coloring of the edges. We call this the *2-page matrix* of D. For $i < j$ an entry (i, j) (row,column) in the 2-page matrix $M(D)$ is a point with the same color as the edge ij in the drawing D. In order to find a lower bound for $\text{cr}(D)$, in the model of the 2-page matrix, it is convenient to have most of the r edges in the square submatrix with entries (a, b) with $1 \le a \le n/2$ and $n/2 + 1 \le b \le n$. The next lemma accomplishes this.

Lemma 3. *For every $1 \le j \le n/2$, let S_j be set of the edges*

$$\{(a(\text{mod} n), b(\text{mod} n)) : \\ j \le a \le j + n/2 - 1, j + n/2 \le b \le j + n - 1\}.$$

There is $1 \le j \le n/2$ such that

$$|S_j \cap R| \ge r\left(1 - \frac{2}{n}\sqrt{2(r-1)}\right) > r - \frac{(2r)^{3/2}}{n}.$$

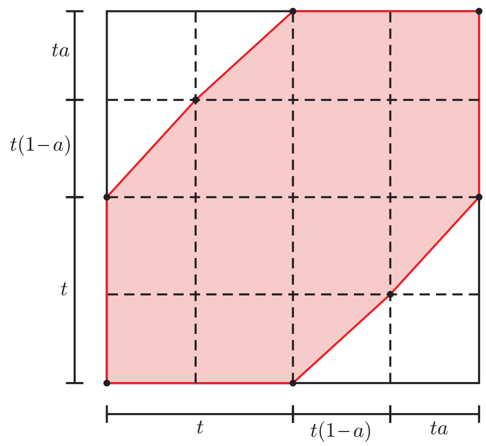
By using a suitable rotation of the indices, we can assume that the j from the previous lemma is equal to 1. Note that if $r < n^{2/3}/2$, then $|S_1 \cap R| = r$. Similarly, if $r = o(n^2)$, then $|S_1 \cap R| = r - o(r)$, so in effect S_1 contains asymptotically almost all of the red edges.

Lemma 4. *Let $s = |R \cap S_1|$. Then*

$$\sum_{e \text{ red}} \left(\text{cr}(e, R) + i(e)^2\right) \ge \frac{1}{2}s(s-1) - \frac{4\sqrt{2}}{3}(s^{3/2} - 1).$$

By Lemma 3, we can assume $s = r - o(r)$. Now the proof of the lower bound of Theorem 1 follows by using Theorem 2 and Lemma 4.

To prove the upper bound, we consider a drawing induced by a 2-page matrix $M(D)$ where the red points are those in the interior of a centrally symmetric octagon with the dimensions described in Figure 1. The octagon is also mirror-symmetric with respect to the diagonal of slope 1, and its center coincides with a point e in $M(D)$ with $i(e) = 0$. The area of the octagon is r and so $t = \sqrt{r/(4-2a)}$. The optimal value of a is the



real solution of $2a^3 - 12a^2 + 43a - 18 = 0$, which is about $a \approx 0.477$. The corresponding number of crossings of the drawing is

$$\text{cr}(D) = \binom{n}{4} - \frac{1}{4}r(n-2)^2 + h(a)r^2 + o(r^2),$$

where

$$h(a) = \frac{-2a^3 + 21a^2 - 41a + 32}{12(a-2)^2} \approx 0.6109.$$

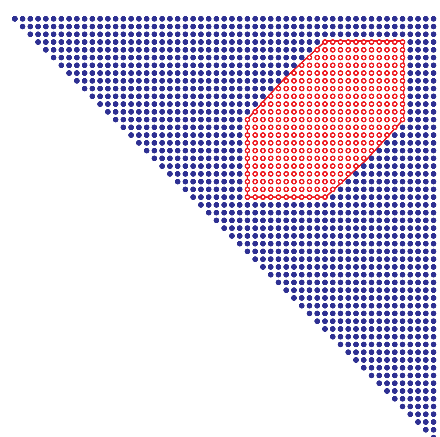


Figure 1: $M(D)$ of a drawing with few crossings

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