The rectilinear local crossing number of $K_{n,m}$

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Abstract

We bound $\overline{\text{lcr}}(K_{n,m})$, the rectilinear local crossing number of the complete bipartite graph $K_{n,m}$, for every n and m. We completely determine $\overline{\text{lcr}}(K_{n,m})$ whenever $\min(n,m) \leq 4$.

1 Introduction

We are concerned with rectilinear drawings of the complete graph $K_{n,m}$. That is, drawings with n red vertices and m blue vertices in the plane, where every edge joining two different color vertices is drawn as a straight line segment. We also assume that any two of these edges share at most one point.

In general, the local crossing number of a graph G was defined by Ringel as follows (see Guy et al. [2], Kainen [3], and Schaefer [5]). The *local crossing number* of a drawing D of a graph G, denoted $\text{lcr}(D)$, is the largest number of crossings on any edge of D. The *local crossing number* of G, denoted $\text{lcr}(G)$, is the minimum of $\text{lcr}(D)$ over all drawings D of G. This is also known as the *cross-index* (Thomassen [6]). The equivalent definition for rectilinear drawings is the *rectilinear local crossing number* of G, denoted $\overline{\text{lcr}}(G)$, as the minimum of $\text{lcr}(D)$ over all rectilinear drawings D of G. Recently, Ábrego and Fernández-Merchant [1] completely determined $\overline{\text{lcr}}(K_n)$ using a *Separation Lemma* (see Lemma 2 in [1]).

The *crossing number* of a graph G, denoted by $\text{cr}(G)$, is the smallest number of crossings among all drawings of G. When this minimum is restricted to rectilinear drawings, we obtained the *rectilinear crossing number* of G, denoted by $\overline{\text{cr}}(G)$. The value of $\overline{\text{cr}}(G)$ can be used to bound $\overline{\text{lcr}}(G)$ (as done in [2] for drawings of K_n on the torus). Namely, adding the number of crossings of every edge over all edges of a graph G counts precisely twice the number of crossings of G. In our problem, this means that It follows that

$$\overline{\text{lcr}}(K_{m,n}) \geq \frac{2\,\overline{\text{cr}}(K_{m,n})}{mn}.$$

The Zarankiewicz Conjecture (Paul Turán, 1944), states that $\overline{\text{cr}}(K_{m,n}) = \text{cr}(K_{m,n}) = Z(m,n) := \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$, but this has only been proved when $\min(m,n) \leq 6$, and for $m = 7$ and $n \leq 10$. The current best published lower bound on $\overline{\text{cr}}(K_{m,n})$ is $0.86\,Z(m,n)$ by de Klerk et al. [4] and recently, Norine and Zwols announced the lower bound $0.905\,Z(m,n)$, but this has not been published. This would yield

$$\overline{\text{lcr}}(K_{m,n}) \geq \frac{0.905}{8}mn + \Theta(mn) > 0.113125\,mn + \Theta(mn).$$

If the Zarankiewicz Conjecture were true, we would have

$$\overline{\text{lcr}}(K_{m,n}) \geq mn/8 + \Theta(mn).$$

The Zarankiewicz drawing of $K_{m,n}$ with $Z(n,m)$ crossings (see Figure 1) has local crossing number $\left(\left\lfloor \frac{m}{2} \right\rfloor - 1\right)\left(\left\lfloor \frac{m}{2} \right\rfloor - 1\right)$ showing that

$$\overline{\text{lcr}}(K_{m,n}) \leq mn/4 + \Theta(mn).$$

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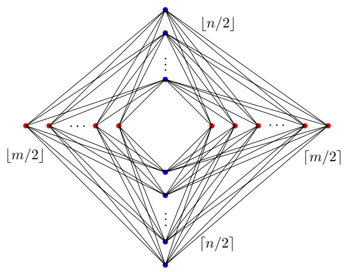

Figure 1: Zarankiewicz drawing of $K_{m,n}$ with $Z(m,n)$ crossings.

2 Main results

Clearly $\overline{\text{lcr}}(K_{2,n})$=0. We determine $\overline{\text{lcr}}(K_{n,m})$ when $\min(n,m) \leq 4$ and improve the upper bound for all other cases.

Theorem 1. *For any integer $n \geq 3$,*

$$\overline{\text{lcr}}(K_{3,n}) = \left\lceil \frac{n-2}{4} \right\rceil \text{ and } \overline{\text{lcr}}(K_{4,n}) = \left\lceil \frac{n-2}{2} \right\rceil.$$

Proof. (Sketch) Figure 2 shows a drawing of $K_{3,n}$ such that each edge is crossed at most $\lceil \frac{n-2}{4} \rceil$ times and there is an edge with that exact number of crossings. This shows that $\overline{\text{lcr}}(K_{3,n}) \leq \lceil \frac{n-2}{4} \rceil$. The red vertices form an equilateral triangle. There are two special blue points d and e very close to the top red point, one above and one below. The rest of the blue points are (almost) evenly distributed among four arcs of circle. The Zarankiewicz construction of $K_{4,n}$

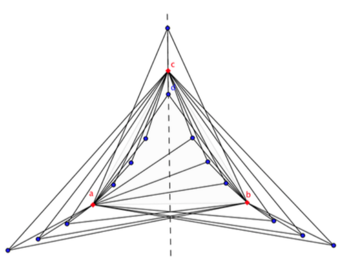

Figure 2: An optimal construction for $\overline{\text{lcr}}(K_{3,n})$.

for $m = 4$ and any n has local crossing number $\lceil \frac{n-2}{2} \rceil$ (see Figure 1) proving $\overline{\text{lcr}}(K_{4,n}) \leq \lceil \frac{n-2}{2} \rceil$. To prove that $\overline{\text{lcr}}(K_{3,n}) \geq \lceil \frac{n-2}{4} \rceil$, we consider several cases according to how the blue points are distributed among the regions determined by the red points. In each case, we identify 2 or 4 edges that must be crossed by a combined total of at least $\frac{n-2}{2}$ or $n-2$, respectively (see Figure 3). The proof of $\overline{\text{lcr}}(K_{4,n}) \geq \lceil \frac{n-2}{2} \rceil$ is more involved but follows similar lines. $\square$

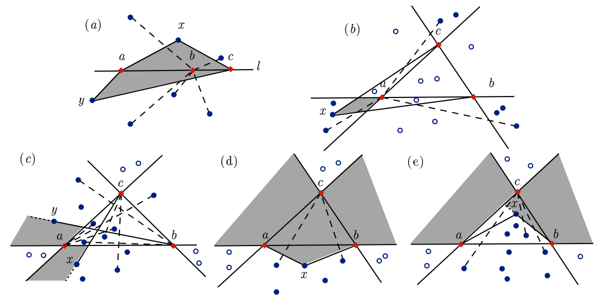

Figure 3: All shaded regions are empty of blue points.

Theorem 2. $\overline{\text{lcr}}(K_{m,n}) \leq \frac{2}{9}mn + \Theta(mn)$.

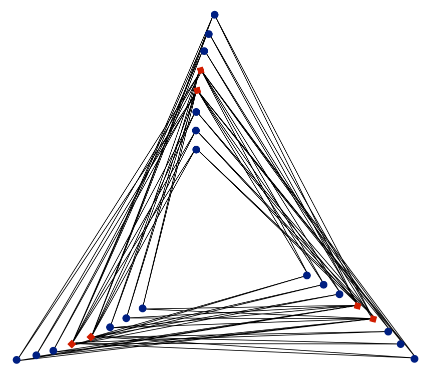

Figure 4: A drawing of $K_{m,n}$ with local crossing number $\frac{2}{9}mn + \Theta(mn)$. The red points are (almost) evenly distributed into 3 clusters and the blue into 6 clusters along 3 arcs of circle.

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