ON THE CONSTRUCTION OF NEW NORMAL NUMBERS

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Abstract. In this paper, we will discuss, through beta-expansions the concept of normality in other bases than integer. Our aim is to generalize Champernowne's construction, which results in a normal number, working in a very special case: the golden mean.

1. First definitions and preliminary results

For $\beta > 1$, consider the transformation $T_\beta : [0,1) \to [0,1)$ defined by

$$T_\beta(x) = \{\beta x\} \text{ where } \{y\} = y - \lfloor y \rfloor .$$

Rényi [2] proved that if $x \in [0,1)$ and $a_n = \lfloor \beta\, T_\beta^{n-1}(x) \rfloor \in \{0, \dots, \lceil \beta \rceil - 1\}$, then

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \frac{a_3}{\beta^3} + \cdots ,$$

and this is called the *β-expansion of x*. Rényi also proved that T_β is ergodic:

Theorem 1.1 (Theorem 2, [2]). *There is an unique probability measure ν on $[0,1)$, equivalent to Lebesgue measure, such that T_β is ergodic with ν.*

From the ergodicity of T_β we immediately get the following two corollaries:

Corollary 1.2. *There exists $E \subset [0,1)$ with $\nu(E) = 0$, such that for all ν-integrable functions f and $\alpha \notin E$ we have that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T_\beta^n(\alpha)) = \int_0^1 f(x) d\nu .$$

Corollary 1.3. *There exists $E \subset [0,1)$ with $\nu(E) = 0$, such that for all $\alpha \notin E$ and $[a,b] \subset [0,1)$ we have that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[a,b]}(T_\beta^n(\alpha)) = \int_0^1 \chi_{[a,b]}(x) d\nu = \nu([a,b]).$$

Definition 1.4. We say that a vector $(b_1, \dots b_k) \in \{0, \dots, \lceil \beta \rceil - 1\}^k$ is *admissible* if there is an $x \in [0,1)$ such that its β-expansion starts with the digits $b_1, \dots, b_k$:

$$x = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \cdots + \frac{b_k}{\beta^k} + \cdots .$$

Date: August 19, 2007.

The authors would like to thank the Departmento de Mathematica at Universidade de São Paulo, São Carlos for its hospitality, and their advisors Dr. Ali Tahzibi and Dr. Federico Rodriguez Hertz. M. S. Borman was supported by NSF grant OISE-0526008, and T. M. Fragoso, J. K. Mengue, and G. M. Silva Neto were supported by CNPq Processo 450913/2007-4.

Definition 1.5. Given an admissible vector $(b_1,\dots b_k)$ we will define the *cylinder* $\mathrm{cyl}(b_1,\dots b_k) \subset [0,1)$ to be the set

$$\mathrm{cyl}(b_1,\dots b_k) = \left\{ x \in [0,1) \mid x = \sum_{i=1}^{\infty} \frac{a_i}{\beta^i} \text{ and } (a_1,\dots a_k) = (b_1,\dots b_k) \right\}.$$

Definition 1.6. Let $\alpha \in [0,1)$ and $\vec{v} = (b_1,\dots b_k)$ be an admissible vector, we define the *frequency* of $\vec{v}$ in α to be

$$F_\alpha(\vec{v}) = \lim_{N\to\infty} \frac{\#\{n \in \{0,\dots,N-1\} \mid T_\beta^n(\alpha) \in \mathrm{cyl}(\vec{v})\}}{N}$$

provided that the limit exists.

Definition 1.7. Let $\alpha \in [0,1)$. We say that α is *β-normal* if for every admissible vector $\vec{v} = (b_1,\dots b_k)$ we have that

$$F_\alpha(\vec{v}) = \nu(\mathrm{cyl}(\vec{v})).$$

In other words, α is β-normal if in its expansion the frequency of each admissible vector is the ν-measure of the cylinder associated to that vector.

Example 1.8. For $\beta = 10$, we have that ν is the Lebesgue measure and α is called normal if in its decimal expansion, every vector $(b_1,\dots,b_k) \in \{0,\dots,9\}^k$ has frequency 10^{-k}.

The first example of normal number has given by Champernowne [1] in 1933. Champernowne showed that listing all vectors and then concatenating them

$$\begin{aligned} C_{10} &= .(0)(1)\dots(8)(9)(00)(01)\dots(98)(99)(000)(001)\dots(998)(999)(0000)\dots \\ &= .01\dots890001\dots9899000001\dots9989990000\dots \end{aligned}$$

gives a number that is 10-normal.

Problem 1.9. *Can Champernowne‘s construction be generalized? What is the set of $\beta > 1$ such that it‘s possible?*

In this paper we will extend Chapernowne’s construction to the case where β is the golden ratio

$$\beta = \phi = \frac{1+\sqrt{5}}{2}.$$

Rényi [2] proved that the ergodic measure ν is given by

$$\nu(E) = \int_E h(x)dm \text{ where } h(x) = \begin{cases} \frac{5+3\sqrt{5}}{10} & \text{if } x < \frac{\sqrt{5}-1}{2} = \phi^{-1} \\ \frac{5+\sqrt{5}}{10} & \text{if } x \geq \frac{\sqrt{5}-1}{2} = \phi^{-1} \end{cases}. \tag{1.1}$$

In this case, a vector $(b_1,\dots b_k) \in \{0,1\}^k$ is admissible if and only if $b_i = 1$ implies that $b_{i+1} = 0$. The measure ν of a cylinder is the Markov measure

$$\nu(b_1,\dots,b_k) = P_{b_1}\pi_{b_1b_2}\pi_{b_2b_3}\dots\pi_{b_{k-1}b_k},$$

where

$$P_0 = \nu([0,\phi^{-1})) = \frac{5+\sqrt{5}}{10}, \quad P_1 = 1 - P_0 = \frac{5-\sqrt{5}}{10}, \tag{1.2}$$

$$\pi_{00} = \frac{\nu([0,\phi^{-2}))}{\nu([0,\phi^{-1}))} = \phi^{-1} = \frac{\sqrt{5}-1}{2}, \quad \pi_{01} = 1 - \pi_{00} = \frac{3-\sqrt{5}}{2}, \quad \pi_{10} = 1. \tag{1.3}$$

Definition 1.10. Let m_i be the i-th admissible vector under the natural ordering for base ϕ:

$$m_1 = 0, m_2 = 1, m_3 = 00, m_4 = 01, m_5 = 10, m_6 = 000, \ldots$$

Then set C_ϕ to be the number

$$C_\phi = (.m_1 0 m_2 0 m_3 0 m_4 0 m_5 0 \ldots)_\phi$$

Our goal is to prove the following theorem:

Theorem 1.11. *C_ϕ is ϕ-normal.*

Remark 1.12. Through analogous arguments of Champernowne‘s we have that zeroes inserted in C_ϕ are not significative in the frequency calculations, for further reference see [1].

Lemma 1.13. *For every admissible vector of the form $\vec{v} = (0, b_2, \ldots, b_{k-1}, 0)$ we have that*

$$\nu(\mathrm{cyl}(0, b_2, \ldots, b_{k-1}, 0)) = P_0\, \pi_{00}^{k-1}.$$

Proof. We will prove this by induction on k, the length of the vector $\vec{v}$. If $k = 1$ then $\vec{v} = 0$, and by definition $\nu(\mathrm{cyl}(0)) = P_0$. For $k = 2$ we have $\vec{v} = (0, 0)$, and again by definition $\nu(\mathrm{cyl}(0, 0)) = P_0\, \pi_{00}$.

In general in we have that $\vec{v} = (0, b_2, \ldots, b_{k-1}, 0)$ and by definition

$$\nu(\mathrm{cyl}(\vec{v})) = P_0\, \pi_{0b_2} \left(\prod_{j=2}^{k-2} \pi_{b_j b_{j+1}} \right) \pi_{b_{k-1}0}.$$

Now either $b_{k-1} = 0$ or $b_{k-1} = 1$. If $b_{k-1} = 0$, then

$$\begin{aligned} \nu(\mathrm{cyl}(0, b_2, \ldots, b_{k-2}, 0, 0)) &= P_0\, \pi_{0b_2} \left(\prod_{j=2}^{k-3} \pi_{b_j b_{j+1}} \right) \pi_{b_{k-2}0}\, \pi_{00} \\ &= \nu(\mathrm{cyl}(0, b_2, \ldots, b_{k-2}, 0))\, \pi_{00} \\ &= P_0\, \pi_{00}^{k-2}\, \pi_{00} \quad \text{(by the inductive hypothesis)} \\ &= P_0\, \pi_{00}^{k-1}. \end{aligned}$$

On the other hand if $b_{k-1} = 1$, then $b_{k-2} = 0$ and

$$\begin{aligned} \nu(\mathrm{cyl}(0, b_2, \ldots, b_{k-3}, 0, 1, 0)) &= P_0\, \pi_{0b_2} \left(\prod_{j=2}^{k-4} \pi_{b_j b_{j+1}} \right) \pi_{b_{k-3}0}\, \pi_{01}\, \pi_{10} \\ &= \nu(\mathrm{cyl}(0, b_2, \ldots, b_{k-3}, 0))\, \pi_{01}\, \pi_{10} \\ &= P_0\, \pi_{00}^{k-3}\, \pi_{01}\, \pi_{10} \quad \text{(by the inductive hypothesis)} \\ &= P_0\, \pi_{00}^{k-1} \quad (\text{by } \pi_{01}\, \pi_{10} = \pi_{00}\, \pi_{00}). \end{aligned}$$

Therefore either way $\nu(\mathrm{cyl}(\vec{v})) = P_0\, \pi_{00}^{k-1}$. □

Lemma 1.14. *If for every admissible vector of the form $\vec{v} = (0, b_2, \ldots, b_{k-1}, 0)$ we have that $F(\vec{v}) = F_{C_\phi}(\vec{v}) = \nu(\mathrm{cyl}(\vec{v}))$, then C_ϕ is ϕ-normal.*

Proof. We will prove this by induction on k. For $k = 1$, we know that

$$F(1) = 1 - F(0) = 1 - \nu(\mathrm{cyl}(0)) = \nu(\mathrm{cyl}(1)),$$

so by by Eq. (1.2) we have $F(1)$ exists and is determined by $F(0)$.

In general suppose that the frequencies are uniquely determined for all vectors with less than k digits. We know that $F(0, b_2, \ldots, b_{k-1}, 0)$ and $F(0, b_2, \ldots, b_{k-1})$ exist, and

$$F(0, b_2, \ldots, b_{k-1}, 0) + F(0, b_2, \ldots, b_{k-1}, 1) = F(0, b_2, \ldots, b_{k-1}),$$

so it follows that $F(0, b_2, \ldots, b_{k-1}, 1)$ exists. Furthermore

$$\begin{aligned} F(0, b_2, \ldots, b_{k-1}, 1) &= F(0, b_2, \ldots, b_{k-1}) - F(0, b_2, \ldots, b_{k-1}, 0) \\ &= \nu(\mathrm{cyl}((0, b_2, \ldots, b_{k-1}) - \nu(\mathrm{cyl}((0, b_2, \ldots, b_{k-1}, 0) \\ &= \nu(\mathrm{cyl}((0, b_2, \ldots, b_{k-1}, 1), \end{aligned}$$

so $F(0, b_2, \ldots, b_{k-1}, 1) = \nu(\mathrm{cyl}((0, b_2, \ldots, b_{k-1}, 1)$. A similar argument works for $F(1, b_2, \ldots, b_{k-1}, 0)$. Finally $F(1, b_2, \ldots, b_{k-1}, 1)$ exists because we know that both $F(1, b_2, \ldots, b_{k-1}, 0)$ and $F(1, b_2, \ldots, b_{k-1})$ exist, and

$$F(1, b_2, \ldots, b_{k-1}, 0) + F(1, b_2, \ldots, b_{k-1}, 1) = F(1, b_2, \ldots, b_{k-1}).$$

Likewise we have that

$$\begin{aligned} F(1, b_2, \ldots, b_{k-1}, 1) &= F(1, b_2, \ldots, b_{k-1}) - F(1, b_2, \ldots, b_{k-1}, 0) \\ &= \nu(\mathrm{cyl}((1, b_2, \ldots, b_{k-1}) - \nu(\mathrm{cyl}((1, b_2, \ldots, b_{k-1}, 0) \\ &= \nu(\mathrm{cyl}((0, b_2, \ldots, b_{k-1}, 1). \end{aligned}$$

Therefore by induction, we have that $F(\vec{v}) = \nu(\mathrm{cyl}(\vec{v}))$ for all admissible vectors $\vec{v}$ if it holds for admissible vectors of the form $\vec{v} = (0, b_2, \ldots, b_{k-1}, 0)$. □

Notation. We denote by (ϕ_n) the sequence

$$\phi_{-1} = 1, \phi_0 = 1, \phi_1 = 2, \phi_2 = 3, \ldots, \phi_n = \phi_{n-1} + \phi_{n-2}.$$

Observe that this is just the Fibonacci sequence shifted by two. Denote by S_n the number

$$S_n = \phi_n\phi_0 + \ldots + \phi_0\phi_n = \sum_{i=0}^{n} \phi_{n-i}\phi_i.$$

Lemma 1.15. *The number of digits of all admissible vectors with s digits is given by $s\phi_s$*

Proof. Again, we proceed by induction over the length

$s = 1$,

we have the blocks 0 and 1 that represents $2 = \phi_1$ digits.

$s = 2$,

we have the blocks $00, 01, 10$ thats represents ϕ_2 blocks, and each block has two digits, then we have $2\phi_2$ digits.

By induction, suppose that for $j = 1, \ldots, s$ there are ϕ_j admissible vectors with lenght j. Notice that every admissible vector in the form $(a_{s+1}a_s, \ldots, a_1)$ is of the form $(0, a_s, \ldots, a_1)$, where $(a_s, \ldots, a_1)$ is an admissible, or $1, 0, a_{s-1}, \ldots, a_1$ where $(a_s, \ldots, a_1)$ is admissible.

Then, there are ϕ_s vectors with the form $(a_s, \ldots, a_1)$ and ϕ_{s-1} admissible vectors in the form $(a_{s-1}, \ldots, a_1)$. Then, there are $\phi_s + \phi_{s-1} = \phi_{s+1}$ admissible vectors with lenght $(s+1)$. Which makes for the induction. Remember that these ϕ_s admissible vectors of lenght s represents $s\phi_s$ digits. $\square$

Lemma 1.16. *The number of times that an admissible of the form* $(0, b_2, ..., b_{k-1}, 0)$ *with lenght* k *occurs in admissibles of lenght* $s \geq k$ *is given by* S_{s-k}.

Proof. Suposse that the admissible vectors of lenght s have the form

$$(a_s, a_{s-1}, ..., a_1).$$

We calculate $\sum_{j=k}^{s} \psi_j$, , where ψ_j denote the number of occurrences of $(0, b_2, ..., b_{k-1}, 0)$ occupies the position of digits $(a_j, ..., a_{j-k+1})$.Then, ψ_j is the number of admissible vectors in the form

$$(\underbrace{a_s, a_{s-1}, ..., a_{j+1}}, \underbrace{0, b_2, ..., b_{k-1}, 0}, \underbrace{a_{j-k}, a_{j-k-1}, ..., a_1}),$$

where $(a_s, a_{s-1}, ..., a_{j+1})$ and $(a_{j-k}, a_{j-k-1}, ..., a_1)$ are admissible vectors.

By previous Lemma, we have that

$$\psi_j = (\phi_{s-j})(\phi_{j-k}).$$

Then,

$$\sum_{j=k}^{s} \psi_j = \sum_{j=k}^{s} (\phi_{s-j})(\phi_{j-k}) = \sum_{j=0}^{s-k} (\phi_{s-k-j})(\phi_j) = S_{s-k}.$$

Remark 1.17. By the previous Lemmas, we must now evaluate:

$$\lim_{n\to\infty} \frac{S_{n-k}}{n\phi_n}.$$

We first determine:

$$\lim_{n\to\infty} \frac{S_{n-1}}{n\phi_n},$$

or equivalently

$$\lim_{n\to\infty} \frac{S_n}{(n+1)\phi_{n+1}}.$$

$\square$

Lemma 1.18. *Given* $\varepsilon > 0$, *there is a sequence* (N_k) *such that for each* k *we have:*

((*)) $$n \geq N_k \Rightarrow \phi_n \phi^k (1-\varepsilon) \leq \phi_{n+k} \leq \phi_n \phi^k (1+\varepsilon).$$

We can take (N_k) *a increasing sequence.*

Proof. The ϕ_n is the Fibonacci‘s n-th number and satisfies the following identity:

$$\frac{\phi_{n+k}}{\phi_n} \to \phi^k,$$

where $\phi = \frac{\sqrt{5}+1}{2}$ is the golden ratio. Then, we can suppose that for,given $\varepsilon > 0$, there are N_k such that (*) is true.

We can suppose that (N_k) is increasing because (N_k) can be taken increasing in the induction process. $\square$

Lemma 1.19. -

$$\lim_{n\to\infty} \frac{\phi_{\lfloor \frac{n}{2} \rfloor} \phi_{\lceil \frac{n}{2} \rceil}}{S_{n-1}} = 0.$$

Proof. Take $\varepsilon = \frac{1}{2}$.

We must prove that

$$\lim_{n\to\infty} \frac{S_{n-1}}{\phi_{\lfloor \frac{n}{2} \rfloor}\phi_{\lceil \frac{n}{2} \rceil}} = +\infty.$$

Likewise,we must prove that

$$\lim_{n\to\infty} \frac{S_{2n-1}}{\phi_n\phi_n} = \lim_{n\to\infty} \frac{S_{2n}}{\phi_{n+1}\phi_n} = +\infty,$$

where

$$\frac{S_{2n-1}}{\phi_n\phi_n} \quad and \quad \frac{S_{2n}}{\phi_{n+1}\phi_n}$$

represents the cases n even or odd.

First case: -

$$\frac{S_{2n-1}}{\phi_n\phi_n}.$$

For each $k \in \mathbb{N}$, using the previous lemma with $\varepsilon = \frac{1}{2}$, we have that for $n \geq N_k \geq N_{k-1} \geq ... \geq N_1$:

$$\begin{aligned}
\frac{S_{2n-1}}{\phi_n\phi_n} &\geq \frac{\phi_{n+k}\phi_{n-k-k} + ... + \phi_{n+1}\phi_{n-2}}{\phi_n\phi_n} = \sum_{j=1}^{k}\left(\frac{\phi_{n+j}}{\phi_n}\frac{\phi_{n-j-1}}{\phi_n}\right) \\
&\geq \sum_{j=1}^{k}\left(\left(\phi^j(1-\frac{1}{2})\right)\left(\phi^{-(j+1)}(1+\frac{1}{2})^{-1}\right)\right) \\
&= \sum_{j=1}^{k}\frac{1}{3\phi} = \frac{k}{3\phi}.
\end{aligned}$$

Since for $n \geq N_k$, $\frac{S_{2n-1}}{\phi_n\phi_n} \geq \frac{k}{3\phi}$, we have that $\frac{S_{2n-1}}{\phi_n\phi_n} \to \infty$.

Second case:

$$\frac{S_{2n}}{\phi_{n+1}\phi_n}.$$

For each $k \in \mathbb{N}$, using the previous lemma with $\varepsilon = \frac{1}{2}$, we have that for $n \geq N_k \geq N_{k-1} \geq ... \geq N_1$:

$$\begin{aligned}
\frac{S_{2n}}{\phi_{n+1}\phi_n} &\geq \frac{\phi_{n+k+1}\phi_{n-k-1} + ... + \phi_{n+2}\phi_{n-2}}{\phi_{n+1}\phi_n} = \sum_{j=1}^{k}\left(\frac{\phi_{n+j+1}}{\phi_{n+1}}\frac{\phi_{n-j-1}}{\phi_n}\right) \\
&\geq \sum_{j=1}^{k}\left(\left(\phi^j(1-\frac{1}{2})\right)\left(\phi^{-(j+1)}(1+\frac{1}{2})^{-1}\right)\right) \\
&= \sum_{j=1}^{k}\frac{1}{3\phi} = \frac{k}{3\phi}.
\end{aligned}$$

Since for $n \geq N_k$, $\frac{S_{2n}}{\phi_{n+1}\phi_n} \geq \frac{k}{3\phi}$, we have that $\frac{S_{2n}}{\phi_{n+1}\phi_n} \to \infty$. □

2. Working towards the proof that C_ϕ is ϕ-normal

Definition 2.1. A sequence $\{x_n\}$ in $\mathbb{R}$ is *finite Cauchy* if $\lim_{n\to\infty}|x_n - x_{n-1}| = 0$.

Lemma 2.2. *The sequence $x_n = \frac{S_n}{(n+1)\phi_{n+1}}$ is finite Cauchy.*

Proof. Observe that $x_n \in [0,1)$ because S_n is the number of zeroes in the admissible vectors of length $n+1$ and $(n+1)\phi_{n+1}$ is the number of digits in the admissible vectors of length $n+1$.

Since x_n is bounded, we know that $|x_n - x_{n-1}| \to 0$ if $\frac{x_n}{x_{n-1}} \to 1$. So to prove that $\{x_n\}$ is finite Cauchy it suffices to prove that $\frac{x_{n1}}{x_{n-1}} \to 1$. Now

$$\frac{x_n}{x_{n-1}} = \frac{S_n}{(n+1)\phi_{n+1}}\frac{n\phi_n}{S_{n-1}} = \frac{S_n n\phi_n}{S_{n-1}(n+1)\phi_{n+1}},$$

where $\frac{n}{n+1} \to 1$ and $\frac{\phi_n}{\phi n+1} \to \phi^{-1}$, so we just need to prove that $\frac{S_n}{S_{n-1}} \to \phi$. □

Let $\epsilon > 0$, by properties of the Fibonacci sequence we know that there exists N such that for$n \geq N$ we have that

$$\phi_n(1-\epsilon)\phi \leq \phi_{n+1} \leq \phi_n(1+\epsilon)\phi.$$

Using the second inequality we get that for $n > 2N$ we have that

$$\begin{aligned}
S_n &= \sum_{j=0}^{n}\phi_{n-j}\phi_j = (\sum_{j=0}^{\lceil n/2\rceil-1}\phi_{n-j}\phi_j) + \phi_{\lfloor n/2\rfloor}\phi_{\lceil n/2\rceil} + (\sum_{j=\lceil n/2\rceil+1}^{n}\phi_{n-j}\phi_j) \\
&\leq \phi(1+\epsilon)\left(\sum_{j=0}^{\lceil n/2\ \rceil-\ 1}\phi_{n-1-j}\phi_j\right) + \phi_{\lfloor n/2\rfloor}\phi_{\lceil n/2\rceil} + \phi(1+\epsilon)(\sum_{j=\ \lceil n/2\ \rceil+\ 1}^{n}\phi_{n-j}\phi_{j-1}) \\
&= \phi(1+\epsilon)(\sum_{j=0}^{n-1}\phi_{n-j}\phi_j) + \phi_{\lfloor n/2\rfloor}\phi_{\lceil n/2\rceil} \\
&= \phi(1+\epsilon)S_{n-1} + \phi_{\lfloor n/2\rfloor}\phi_{\lceil n/2\rceil}.
\end{aligned}$$

Lemma 2.3. *For $n \geq 2$ we have that*

$$S_n = (n+1)\phi_{n+1} - S_{n-2} - 2\phi_{n-1}.$$

Proof. From the recurrent relation

$$\phi_{n+1} = \phi_n + \phi_{n-1}$$

and

$$\phi_{-1} = \phi_0 = 1$$

it follows that for $0 \leq j \leq n$,

$$\phi_{n+1} = \phi_{n-j}\phi_j + \phi_{n-1-j}\phi_{j-1}.$$

Using this identity we get the following:

$$\begin{aligned}(n+1)\phi_{n+1} &= \sum_{j=0}^{n}\phi_{n+1} = \sum_{j=0}^{n}\phi_{n-j}\phi_j + \sum_{j=0}^{n}\phi_{n-1-j}\phi_{j-1} \\ &= S_n + \sum_{j=1}^{n-1}\phi_{n-1-j}\phi_{j-1} + 2\phi_{n-1} \\ &= S_n + \sum_{j=0}^{n-2}\phi_{n-2-j}\phi_j + 2\phi_{n-1} \\ &= S_n + S_{n-2} + 2\phi_{n-1}.\end{aligned}$$

Therefore for $n \geq 2$ we have that $S_n = (n+1)\phi_{n+1} - S_{n-2} - 2\phi_{n-1}$. □

Proposition 2.4. *The sequence*

$$x_n = \frac{S_n}{(n+1)\phi_{n+1}}$$

converges to $\nu(0) = \frac{5+\sqrt{5}}{10}$.

Proof. As we observed in the proof of Lemma (?) we know that $x_n \in [0,1)$, and therefore there exists a convergent subsequence of $\{x_n\}$. Therefore to prove that $x_n \to \nu(0)$ it suffices to prove that x is a limit point of $\{x_n\}$ only if $x = \nu(0)$.

Let x a limit point of $\{x_n\}$. Then there are $X_{n_j} \to x$. Since x_n is finite cauchy we have that

$$\left|x_{n_j-2} - x_{n_j}\right| \to 0.$$

Then $x_{n_j-2} \to x$. Using the previous lemma we have that

$$\frac{S_{n_j}}{(n_j+1)\left(\phi_{n_j+1}\right)} = 1 - \frac{S_{n_j-2}}{(n_j+1)\left(\phi_{n_j+1}\right)} - \frac{2\phi_{n_j-2}}{(n_j+1)\left(\phi_{n_j+1}\right)}.$$

Then

$$\begin{aligned}x &= \lim_{n_j\to\infty}\frac{S_{n_j}}{(n_j+1)\left(\phi_{n_j+1}\right)} = 1 - \lim_{n_j\to\infty}\frac{S_{n_j-2}}{(n_j+1)\left(\phi_{n_j+1}\right)} \\ &= 1 - \lim_{n_j\to\infty}\frac{S_{n_j-2}}{(n_j-1)\left(\phi_{n_j-1}\right)}\frac{\phi_{n_j-1}}{\phi_{n_j+1}}\frac{(n_j-1)}{(n_j+1)} \\ &= 1 - x\frac{1}{\phi}.\end{aligned}$$

The unique value that soluble this equation is $v_0 = \frac{5+\sqrt{5}}{10}$. □

Notation. Given an admissible vector $M_s = (a_s, ..., a_1, a_0)$ and $0_k = (0, b_2, ..., b_{k-1}, 0)$. we denote by

- F_s the number of occurrences of 0_k in all admissible vectors of lenght $\leq s$.
- G_s the number of digits writed in all admissibles vectors of lenght $\leq s$.
- f_{M_s} the number of (occurrences) of 0_k in all vectors from $(\underbrace{0,0,...,0,0}_{s+1})$ to $(a_s, ..., a_0) = M_s$.
- g_{M_s} the number of digits writed in all admissible vectors from $(\underbrace{0,0,...,0,0}_{s+1})$ to $(a_s, ..., a_0) = M_s$.

Corollary 2.5. -

$$\frac{S_{n-k}}{n\phi_n} \to P_0 (\pi_{00})^{k-1} = P_0 \left(\frac{1}{\phi}\right)^{k-1}.$$

Proof.

$$\frac{S_{n-k}}{n\phi_n} = \frac{S_{n-k}}{(n-k+1)\phi_{n-k+1}} \frac{n-k+1}{n} \frac{\phi_{n-k+1}}{\phi_n} \to P_0 \left(\frac{1}{\phi}\right)^{k-1}.$$

□

Remark 2.6. Remember that

$$F_s = \sum_{j=k}^{s} S_{j-k} \quad \textit{and} \quad G_s = \sum_{j=1}^{s} j\phi_j,$$

then by previous corolary and Lemma (?) we have that $\frac{F_s}{G_s} \to P_0 \left(\frac{1}{\phi}\right)^{k-1}$.

Using an argument due to Champernowne we obtain

Lemma 2.7. *If*

$$\frac{P_0 \left(\frac{1}{\phi}\right)^{k-1} g_{M_s} - f_{M_s}}{G_s} \overset{M_s \to \infty}{\to} 0$$

then C_ϕ is normal.

Proof. Since $\frac{F_s}{G_s} \to P_0 \left(\frac{1}{\phi}\right)^{k-1}$, we have that

$$\frac{F_s - P_0 \left(\frac{1}{\phi}\right)^{k-1} G_s}{G_s} \to 0.$$

Then

$$\frac{F_s - P_0 \left(\frac{1}{\phi}\right)^{k-1} G_s + f_{M_s} - P_0 \left(\frac{1}{\phi}\right)^{k-1} g_{M_s}}{G_s} \to 0.$$

Then

$$\frac{F_s + f_{M_s} - P_0 \left(\frac{1}{\phi}\right)^{k-1} (G_s + g_{M_s})}{G_s + g_{M_s}} \to 0.$$

Then

$$\frac{F_s + f_{M_s}}{G_s + g_{M_s}} - P_0 \left(\frac{1}{\phi}\right)^{k-1} \to 0.$$

Therefore C_ϕ is normal. □

Lemma 2.8. -

i)

$$g_{Ms} = (s+1) \left(\sum_{j:a_j \neq 0} \phi_j \right) + (s+1).$$

ii)

$$f_{M_s} = \left(\sum_{j:a_j \neq 0} \Psi(j)\phi_j + S_{j-k} \right) + \Psi(0),$$

where $0 \leq \Psi(j) \leq s+1-j$ *and* $S_n = 0$ *if* $n < 0$.

Proof. Let $j_1 = \max\{j \in \{0,...,s\} : a_j \neq 0\}$. From $(\underbrace{0,0,...,0,0}_{s+1})$ to $(\underbrace{0,...,0}_{s-j_1},1,\underbrace{0,...,0,0}_{j_1})$ so, there are ϕ_{j_1} admissible vectors, all in the form $(\underbrace{0,...,0}_{s-j_1+1},c_{j_1-1},...,c_0)$, where $(c_{j_1-1},...,c_0)$ is admissible.

Let $j_2 = \max\{j \in \{0,...,j_1-1\} : a_j \neq 0\}$. Using the same arguments,there are ϕ_{j_2} admissible vectors from $(\underbrace{0,...,0}_{s-j_1},1,\underbrace{0,...,0,0}_{j_1})$ to $(\underbrace{0,...,0}_{s-j_1},1,0,...,0,1,\underbrace{0,...,0,0}_{j_2})$.

Then by a simple argument, we can see that there are $\sum_{j:a_j\neq 0}\phi_j$ admissible vectors from $(\underbrace{0,0,...,0,0}_{s+1})$ to $(a_s,...,a_0) = M_s$. Notice that the lenght of these vectors is $(s+1)$. This finishes the proof of i).

ii) Let $j_1 = \max\{j \in \{0,...,s\} : a_j \neq 0\}$. From $(\underbrace{0,0,...,0,0}_{s+1})$ to $(\underbrace{0,...,0}_{s-j_1},1,\underbrace{0,...,0,0}_{j_1})$ there are $(\Psi(j_1)\phi_{j_1} + S_{j_1-k})$ occurrences of 0_k, where $\Psi(j_1)$ is the number of occurrences of 0_k in $(\underbrace{0,...,0}_{s-j_1+1})$.

Let $j_2 = \max\{j \in \{0,...,j_1-1\} : a_j \neq 0\}$. From $(\underbrace{0,...,0}_{s-j_1},1,\underbrace{0,...,0,0}_{j_1})$ to $(\underbrace{0,...,0}_{s-j_1},1,0,...,0,1,\underbrace{0,...,0,0}_{j_2})$ there are $(\Psi(j_2)\phi_{j_2} + S_{j_2-k})$ occurrences of 0_k, where $\Psi(j_2)$ is the number of occurrences of 0_k in $(\underbrace{0,...,0}_{s-j_1},1,\underbrace{0,...,0}_{j_1-j_2})$. Using the same arguments of part i) we can conclude the prove. □

3. Proof that C_ϕ is ϕ-normal

We are now prepared to prove Theorem 1.11, that C_ϕ is ϕ-normal.

Proof. Remember that, by Lemma 2.7, is sufficient to prove that

$$\frac{P_0\left(\frac{1}{\phi}\right)^{k-1} g_{M_s} - f_{M_s}}{G_s} \overset{M_s\to\infty}{\to} 0.$$

We need to evaluate

$$\lim_{M_s\to\infty} \frac{P_0\left(\frac{1}{\phi}\right)^{k-1}\left[(s+1)\left(\sum_{j:a_j\neq 0}\phi_j\right) + (s+1)\right] - \left(\sum_{j:a_j\neq 0}\Psi(j)\phi_j + S_{j-k}\right) + \Psi(0)}{\sum_{j=1}^{s} j\phi_j}.$$

Since

$$\frac{\sum_{j:a_j\neq 0}\Psi(j)\phi_j}{\sum_{j=1}^{s} j\phi_j} \leq \frac{\sum_{j=1}^{s}(s+1-j)\phi_j}{\sum_{j=1}^{s} j\phi_j},$$

and using ([Vorobiev], pg. 93, 98), we have that

$$\lim_{s\to\infty}\frac{\sum_{j=1}^{s}(s+1-j)\phi_j}{\sum_{j=1}^{s} j\phi_j}=\lim_{s\to\infty}\frac{\sum_{j=1}^{s}(s-j)\phi_j}{\sum_{j=1}^{s} j\phi_j}=\lim_{s\to\infty}\frac{s\sum_{j=1}^{s}\phi_j}{\sum_{j=1}^{s} j\phi_j}-1$$

$$=\lim_{s\to\infty}\frac{s(\phi_{s+2}-\phi_2)}{s\phi_{s+2}-\phi_{s+3}+2}-1=0. \qquad ((*))$$

Since

$$\lim_{s\to\infty}\frac{P_0\left(\frac{1}{\phi}\right)^{k-1}\left[(s+1)+\sum_{j:a_j\neq 0}\phi_j\right]+\Psi(0)}{\sum_{j=1}^{s} j\phi_j}=0,$$

it's enough to show that

$$\lim_{M_s\to\infty}\frac{P_0\left(\frac{1}{\phi}\right)^{k-1} s\left(\sum_{j:a_j\neq 0}\phi_j\right)-\left(\sum_{j:a_j\neq 0}S_{j-k}\right)}{\sum_{j=1}^{s} j\phi_j}=0.$$

Take $\varepsilon>0$. by Corolary (?) there are N_0 such that

$$j\geq N_0\Rightarrow(1-\varepsilon)j\phi_j P_0\left(\frac{1}{\phi}\right)^{k-1}\leq S_{j-k}\leq(1+\varepsilon)j\phi_j P_0\left(\frac{1}{\phi}\right)^{k-1}.$$

Then,

$$\sum_{j:\ a_j\neq 0}S_{j-k}=\sum_{j\leq N_0:\ a_j\neq 0}S_{j-k}+\sum_{\substack{j>N_0\\ a_j\neq 0}}S_{j-k}\geq\left(\sum_{j\leq N_0:\ a_j\neq 0}S_{j-k}\right)+\left((1-\varepsilon)P_0\left(\frac{1}{\phi}\right)^{k-1}\sum_{j>N_0:\ a_j\neq 0}j\phi_j\right).$$

Since

$$\lim_{M_s\to\infty}\frac{P_0\left(\frac{1}{\phi}\right)^{k-1} s\left(\sum_{j\leq N_0:\ a_j\neq 0}\phi_j\right)-\left(\sum_{j\leq N_0:\ a_j\neq 0}S_{j-k}\right)}{\sum_{j=1}^{s} j\phi_j}=0,$$

we have that:

$$\limsup_{M_s\to\infty}\frac{P_0\left(\frac{1}{\phi}\right)^{k-1} s\left(\sum_{j:a_j\neq 0}\phi_j\right)-\left(\sum_{j:a_j\neq 0}S_{j-k}\right)}{\sum_{j=1}^{s} j\phi_j}$$

$$\leq\limsup_{M_s\to\infty}\frac{P_0\left(\frac{1}{\phi}\right)^{k-1} s\left(\sum_{j:a_j\neq 0}\phi_j\right)-\left((1-\varepsilon)P_0\left(\frac{1}{\phi}\right)^{k-1}\sum_{j>N_0:\ a_j\neq 0}j\phi_j\right)-\left(\sum_{j\leq N_0:\ a_j\neq 0}S_{j-k}\right)}{\sum_{j=1}^{s} j\phi_j}$$

$$=\limsup_{M_s\to\infty}\frac{P_0\left(\frac{1}{\phi}\right)^{k-1} s\left(\sum_{j>N_0:a_j\neq 0}\phi_j\right)-\left((1-\varepsilon)P_0\left(\frac{1}{\phi}\right)^{k-1}\sum_{j>N_0:\ a_j\neq 0}j\phi_j\right)}{\sum_{j=1}^{s} j\phi_j}$$

$$=P_0\left(\frac{1}{\phi}\right)^{k-1}\limsup_{M_s\to\infty}\frac{\left(\left(\sum_{j>N_0:a_j\neq 0}(s-j)\phi_j\right)+\varepsilon\left(\sum_{j>N_0:a_j\neq 0}j\phi_j\right)\right)}{\sum_{j=1}^{s} j\phi_j}$$

$$\overset{(*)}{=}P_0\left(\frac{1}{\phi}\right)^{k-1}\limsup_{M_s\to\infty}\frac{\varepsilon\sum_{j>N_0:a_j\neq 0}j\phi_j}{\sum_{j=1}^{s} j\phi_j}$$

$$\leq P_0\left(\frac{1}{\phi}\right)^{k-1}\varepsilon.$$

Using the same argument, we have that

$$\liminf_{M_s\to\infty} \frac{P_0\left(\frac{1}{\phi}\right)^{k-1} s\left(\sum_{j:a_j\neq 0}\phi_j\right) - \left(\sum_{j:a_j\neq 0} S_{j-k}\right)}{\sum_{j=1}^{s} j\phi_j} \geq -P_0\left(\frac{1}{\phi}\right)^{k-1}\varepsilon.$$

Since ε is arbitrary, we conclude that C_ϕ is normal. □

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