Generalized Pisot Numbers *†‡

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1 Introduction

In this paper, we discuss certain sets of algebraic integers related to Pisot and Salem numbers, which are defined below, and their properties. Much is known about Pisot numbers, and there are many known ways to construct them. In comparison, little is known about Salem numbers, and their construction is difficult. There are still many open questions about Salem numbers, including determining the infemum of the set.

In this section, we define new sets of generalized Pisot numbers. Sections 2 and 3 are concerned with the arithmetic properties and limit points of one of these sets, Pisot-2 pairs. For this set we obtain some results analogous to those known about Pisot numbers. We then, in section 4, discuss another set, $o(k)$-Pisot numbers, and its connections to Salem numbers, including a relationship with the infemum of Salem numbers. Finally, section 5 gives arithmetic properties of these $o(k)$-Pisot numbers.

Definition 1.1. *Let U be the set of real algebraic integers $\theta > 1$ whose remaining conjugates have modulus at most equal to 1.*

The following two sets partition U.

Definition 1.2. *A number $\alpha \in U$ is called a Pisot number if all remaining conjugates have modulus strictly less than 1. We denote the set of Pisot numbers by S.*

A great deal is known about this set. For example, if the integer coefficients of the minimal polynomial of α, $P(z) = z^d + q_{d-1}z^{d-1} + \cdots + q_0$, satisfy $1 + \sum_{i=0}^{d-1} q_i < 0$ and $|q_{n-1}| > 1 + \sum_{i=0}^{d-2} |q_i|$ then α is a Pisot number. Also, for $\alpha \in S$ it is known that $\alpha^n \in S$. One important theorem about Pisot numbers is that the set S is closed.

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Definition 1.3. *A number $\tau \in U$ is called a Salem number if it has at least one conjugate having modulus equal to* 1*. We denote the set of Salem numbers by T.*

Definition 1.4. *The reciprocal polynomial of a polynomial P is defined by $P^*(z) = z^{degP}P(z^{-1})$. A polynomial is called reciprocal if $P = P^*$.*

In general, we will denote the reciprocal polynomial of P by Q. The minimal polynomial of a Salem number will always be reciprocal, and thus, there are no Salem numbers with odd degree or degree less than 4. Constructing Salem numbers is much more difficult than Pisot numbers. For instance, graph theory can be used to construct some but not nearly all. The smallest Salem number is still not known, though it is conjectured to be largest root of $1 + z - z^3 - z^4 - z^6 - z^7 + z^9 + z^{10}$.

The remaining sections of our paper are concerned with the following definitions. We adjust the definition of Pisot numbers to consider slightly different sets of algebraic integers.

Definition 1.5. *A pair of real distinct algebraic conjugates (α_1, α_2) is called a Pisot-*2 *pair if α_1 and α_2 are greater than* 1 *and all remaining conjugates have modulus strictly less than* 1*. We denote the set of Pisot-*2 *pairs by S_2.*

Definition 1.6. *Let β be a real algebraic integer greater then* 1 *with degree at least* 2*, and $\beta_2, \dots, \beta_d$ its conjugates. Then if there exist exactly k conjugates of β that lie outside the unit disc, we say that β is an $o(k)$-Pisot number.*

Definition 1.7. *Let β be an $o(k)$-Pisot number. We say that β is a strong $o(k)$-Pisot number if $(\beta + 1) \in T$ or $(\beta^{-1} + 1) \in T$. We denote the set of strong $o(k)$-Pisot numbers by $O(k)$. Also, we let $O = \bigcup_{k \in N^*} O(k)$.*

The definition of strong $o(k)$-Pisot numbers was motivated by the fact that, if $\tau \in T$ with $\tau - 1$ a unit then $(\tau - 1)^{-1}$ or $\tau - 1$ is a strong $o(k)$-Pisot number for some $k \in \mathbb{Z}$. So we will relate, in section 4, the $\inf T$ and the boundedness of a particular subset of O.

2 Arithmetic Properties of Pisot-2 Pairs

In this section we discuss some basic arithmetic properties of S_2 that are similar to those of S. It is known that if $\alpha \in S$ then $\alpha^n \in S$, $\forall n \in \mathbb{N}$. Also, we have that $\{\alpha^k\} \longrightarrow 0$, where $\{x\}$ denotes the fractional part of x. Our last proposition relates a specific subset of S_2 to S.

Proposition 2.1. *If $(\alpha_1, \alpha_2) \in S_2$ then $\forall n \in \mathbb{N}$, $(\alpha_1^n, \alpha_2^n) \in S_2$.*

Proof. Let $(\alpha_1, \alpha_2) \in S_2$, $n \in \mathbb{N}$, and $M \in \mathbb{Z}[x]$ the minimal polynomial of α_1 and α_2. Let $\alpha_3, \alpha_4, ..., \alpha_k$ denote the other roots of M. We know that the product of any 2 algebraic integers is, itself, an algebraic integer. Since α_1

is an algebraic integer, α_1^n is also an algebraic integer (by a simple inductive argument). Let $P \in \mathbb{Z}[x]$ be the minimal polynomial of α_1^n. Now consider σ_i, the embedding of $\mathbb{Q}(\alpha_1)$ into $\mathbb{C}$, which fixes $\mathbb{Q}$ and maps α_1 to α_i.

$$P(\alpha_i^n) = P((\sigma_i(\alpha_1))^n) = P(\sigma_i(\alpha_1^n)) = \sigma_i(P(\alpha_1^n)) = \sigma_i(0) = 0$$

So, $\forall i \leq k, \alpha_i^n$ satisfies P. By elementary Galois theory, $[\mathbb{Q}(\alpha_1^n) : \mathbb{Q}] \leq [\mathbb{Q}(\alpha_1) : \mathbb{Q}]$. This shows that $deg(P) \leq deg(M)$. So $\alpha_1^n, \alpha_2^n, ..., \alpha_k^n$ are all the roots of P. If $3 \leq i \leq k$ then $|\alpha_i^n| = |\alpha_i|^n < 1^n = 1$. If $i = 1, 2$ then, $|\alpha_i^n| = |\alpha_i|^n > 1^n = 1$. Therefore, $(\alpha_1^n, \alpha_2^n) \in S_2$.

□

Proposition 2.2. *If* $(\alpha_1, \alpha_2) \in S_2$ *then* $\{\alpha_1^n + \alpha_2^n\} \longrightarrow 0$.

Proof. Let (α_1, α_2) be a Pisot-2 pair. Let $\alpha_3, \alpha_4, ..., \alpha_k$ denote the conjugates of α_1 and α_2. By the proof of Proposition 2.1, $\forall n \in \mathbb{N}$ $\alpha_1^n, \alpha_2^n, ..., \alpha_k^n$ are the roots of some degree k irreducible polynomial, P_n, in $\mathbb{Z}[x]$. Also,

$$tr(P_n) = \sum_{i=1}^{k} \alpha_i^n \in \mathbb{Z}.$$

So $\{tr(P_n)\} = 0$. The above can be rewritten as

$$\left\{ \sum_{i=1}^{k} \alpha_i^n \right\} = \left\{ \alpha_1^n + \alpha_2^n + \sum_{i=3}^{k} \alpha_i^n \right\} = 0.$$

Since, $\forall i \in \mathbb{Z}$ with $3 \leq i \leq k$, by definition $|\alpha_i| < 1$, therefore $\alpha_i^n \longrightarrow 0$. Thus we have $\{\sum_{i=3}^{k} \alpha_i^n\} \longrightarrow 0$. Therefore $\{\alpha_1^n + \alpha_2^n\} \longrightarrow 0$.

□

In some cases we can use Pisot-2 pairs to generate Pisot numbers.

Proposition 2.3. *If* $(\alpha_1, \alpha_2) \in S_2$ *with minimal polynomial,* $P \in \mathbb{Q}[x]$*, of degree* 3 *and* $P(0) = -1$*, then* $\alpha_1\alpha_2 \in S$.

Proof. Let $(\alpha_1, \alpha_2) \in S_2$ *s.t.* their minimal polynomial, P, has degree 3 and $P(0) = -1$. Let α_3 be the third root of P. Since $P(x) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$, $-\alpha_1\alpha_2\alpha_3 = P(0) = -1$. Therefore $\alpha_3 = \frac{1}{\alpha_1\alpha_2}$. Consider $Q(x) = -x^3 P(\frac{1}{x})$. Clearly Q is monic, irreducible over $\mathbb{Z}[x]$, and $\frac{1}{\alpha_3} = \alpha_1\alpha_2 > 1$, $\frac{1}{\alpha_1} < 1$ and $\frac{1}{\alpha_2} < 1$ are its roots. Therefore $\alpha_1\alpha_2$ is a Pisot number.

□

3 Limit Points of Pisot-2 Pairs

In this section, we prove, using methods of complex analysis, that $\overline{S_2} \subset S_2 \cup (S \times \mathbb{R}) \cup (\mathbb{R} \times S)$. The following results will be used in the proof of Theorem 3.6. Definition 3.1 and Theorem 3.2 are taken directly from [4].

Definition 3.1. *If $q \in \mathbb{N}^*$, $k \in \mathbb{N}$, $\delta \in \mathbb{R}^{+*}$, we denote by $\mathcal{F}(q,k,\delta)$ the set of rational functions f that can be written in the form $f = \frac{A(z)}{Q(z)}$ where A and Q are polynomials with integer coefficients such that*

(i) $Q(0) = q, A(0) \neq 0$

(ii) *Q has no more than k zeros in $D(0,1)$ and is non-zero in $D(0,\delta) \cup \{z \in \mathbb{C}, |z| = 1\}$*

(iii) $\left|\frac{A(z)}{Q(z)}\right| \leq 1$ *if* $|z| = 1$.

Theorem 3.2. *The family $\mathcal{F}(q,k,\delta)$ is compact for the uniform convergence topology on the compacts of $D(0,\delta)$.*

The following lemma is analogous to Rouche's Theorem.

Lemma 3.3. *If f and g are two analytic functions on $D(0,r)$ with $r > 1$ such that*

(i) $|f(z)| \leq |g(z)|$ *if* $|z| = 1$

(ii) $f(z) - g(z) = \gamma_n z^n + (\textit{higher order terms}) \qquad \textit{where } \gamma_n \neq 0,$

then g has at least n zeros in $D(0,1)$.

Lemma 3.4. *If $f = \frac{P}{Q}$ is a meromorphic function with no pole at $z = 0$, $Q(0) = 1$, and taylor series $f(x) = \sum_{i=0}^{\infty} u_i x^i$ then for all $i \in \mathbb{N}$, $u_i \in \mathbb{Z}$.*

Proof. We will prove this by induction on the index of the coefficient of the taylor expansion.

Base case: $i = 0$

We know $\sum_{i=0}^{\infty} u_i x^i = f(x) = \frac{P(x)}{Q(x)}$. So $P(x) = Q(x)\sum_{i=0}^{\infty} u_i x^i$. If we expand out $P(x)$ and $Q(x)$ we get

$$(p_0 + p_1 x + \cdots + p_{n_1} x^{n_1}) = (q_0 + q_1 x + \cdots + q_{n_2} x^{n_2}) \sum_{i=0}^{\infty} u_i x^i.$$

Comparing coefficients gives us $p_0 = q_0 u_0$. Since $q_0 = Q(0) = 1$, we have $p_0 = u_0 \in \mathbb{Z}$.

Inductive Case: Assume that for $i \leq k$ $u_i \in \mathbb{Z}$.

Either $k \geq n_1$ or $k < n_1$. If $k \geq n_1$ then the coefficient of x^{k+1} in P is 0. So comparing coefficients of x^{k+1} we get

$$0 = \sum_{i=1}^{k+1} u_i q_{k+1-i} = u_{k+1} q_0 + \sum_{i=1}^{k+1} u_i q_{k+1-i}.$$

By the inductive hypothesis, we have

$$u_{k+1} = u_{k+1}q_0 = -\sum_{i=1}^{k+1} u_i q_{k+1-i} \qquad \in \mathbb{Z}.$$

If instead $k < n$ then, by a similar argument,

$$u_{k+1} = p_{k+1} - \sum_{i=1}^{k+1} u_i q_{k+1-i} \qquad \in \mathbb{Z}.$$

□

Lemma 3.5. *Let $(\alpha_1, \alpha_2) \in S_2$. Then there exists at least one polynomial A with integer coefficients, different from Q, and such that*

(i) $A(0) \geq 1$

(ii) $|A(z)| \leq |Q(z)|$ *if* $|z| = 1$.

Proof. If P is not reciprocal, take $A(z) = \pm P(z)$ (such that $A(0)$ is positive). Otherwise, we know that the conjugates of α_1 and α_2 are contained in $\mathbb{R}^+$. So we have

$$\begin{aligned} |Q(z)| &= |(z-\alpha)||(z-\alpha_2)|\ldots|(z-\alpha_d)| \\ &\geq \big||z|-|\alpha|\big|\big||z|-|\alpha_2|\big|\ldots\big||z|-|\alpha_d|\big| \\ &= \big|(1-\alpha)(1-\alpha_2)\ldots(1-\alpha_d)\big| \\ &= |Q(1)| \qquad \forall z \in C(0,1) \end{aligned}$$

Therefore, since $Q \in \mathbb{Z}[z]$ and $Q(1)$ is the sum of the coefficients of Q, we know $Q(1) \in \mathbb{Z}$. Also, since Q is non-zero on $C(0,1)$ we have $Q(1) \geq 1$, so we let $A = 1$.

□

This lemma allows us to associate to each Pisot-2 pair an element of $\mathcal{F}(1,2,\delta)$. This will be useful when looking at the limit points of S_2 as we will be able to obtain a sequence of rational functions which, by Theorem 3.2, will converge to a function in the family $\mathcal{F}(1,2,\delta)$. For simplicity, we will denote this family by $\mathcal{F}(\delta)$.

Theorem 3.6. *The limit points of S_2 lie either in S_2, $S \times \mathbb{R}$, or $\mathbb{R} \times S$.*

Proof. Let $(\alpha, \beta) \in S_2'$. There exists a sequence $(\alpha_\nu, \beta_\nu) \in S_2$ such that $(\alpha_\nu, \beta_\nu) \longrightarrow (\alpha, \beta)$. Note that $(\alpha_\nu, \beta_\nu) \not\to (1,1)$.

It follows from Lemma 3.5 that for each pair (α_ν, β_ν) we can associate a rational function $f_\nu = A_\nu / Q_\nu \in \mathcal{F}(\delta)$ with $\delta < \inf_{\nu \in N}\{\frac{1}{\alpha_\nu}, \frac{1}{\beta_\nu}\}$. Since $\mathcal{F}(\delta)$ is compact, there is a subsequence, which we will also denote (f_ν), that converges to $f \in \mathcal{F}(\delta)$. Let

$$f_\nu(z) = \sum_{n=0}^{\infty} u_{\nu,n} z^n \qquad \text{and} \qquad f(z) = \sum_{n=0}^{\infty} u_n z^n$$

be the Taylor expansions of the functions f_ν and f with $u_{\nu,n}, u_n \in \mathbb{Z}$. Since f_ν converges uniformly to f on the compacts of $D(0,\delta)$, we have that $u_{\nu,n} \longrightarrow u_n$ in $\mathbb{Z}$. So for all n there exists $\nu(n)$ such that $u_{\nu,n} = u_n$ for all $\nu > \nu(n)$. Therefore, since $1 \le A_\nu(0)/Q_\nu(0) = f_\nu(0) = u_{\nu,0}$ we have $u_0 \ge 1$.

Now for every $\nu \in N$ the function f_ν has exactly two poles inside $D(0,1)$. We wish to show that f has at least one pole in $D(0,1)$. If we consider the function

$$\phi(z) = (f_\nu(z) - u_{\nu,0})Q_\nu(z),$$

we find that

$$|f_\nu(z)Q_\nu(z)| = |A_\nu(z)| \le |Q_\nu(z)| \le |u_{\nu,0}Q_\nu(z)| \qquad \forall z \in C(0,1).$$

So we may conclude that there does not exist $\nu \in \mathbb{N}$ with $u_{\nu,1} = u_{\nu,2} = 0$. Otherwise, by Lemma 3.3, we would have that $u_{\nu,0}Q_\nu$ has three zeros in $D(0,1)$, a contradiction. If f is analytic on $D(0,1)$, then, by the maximum modulus theorem, it attains its maximum on $C(0,1)$. Since $|f(z)| \le 1$ on $C(0,1)$ and $f(0) = 1$, we know that f must be the constant function $f(z) = 1$. Therefore $u_1 = u_2 = 0$ contradicting our previous conclusion. Therefore, f has at least one pole in $D(0,1)$ and at most two.

If f has two poles in $D(0,1)$, then they are at $\frac{1}{\alpha}$ and $\frac{1}{\beta}$. We may write $f = A/Q$ where $A, Q \in \mathbb{Z}[z]$, are relatively prime, and satisfy $|A(z)| \le |Q(z)|$ on $C(0,1)$. Since $Q(\frac{1}{\alpha}) = Q(\frac{1}{\beta}) = 0$ and $Q(0) = 1$, we have that $P(z) = Q^*(z)$ is monic and $P(\alpha) = P(\beta) = 0$. It is possible that one of these roots can be factored out of the polynomial, but not both. Therefore, either $(\alpha,\beta) \in S_2$, $(\alpha,\beta) \in S \times \mathbb{R}$, or $(\alpha,\beta) \in \mathbb{R} \times S$.

If f has only one pole in $D(0,1)$, then we have one of the following cases:

(1) $\frac{1}{\alpha} = \frac{1}{\beta}$

(2) $\frac{1}{\alpha} \ne \frac{1}{\beta}$ and either $\frac{1}{\alpha}$ or $\frac{1}{\beta}$ is a pole.

In both cases, we claim that α or β is Pisot.

The proof for both cases is the same. Without loss of generality we can assume that $1/\alpha$ is a root of Q and α is a root of the monic polynomial $P = Q^*$. If P is not irreducible then we can reduce it to an irreducible polynomial $P^{'}$. It remains to show that α is a root of $P^{'}$. We know that the roots of $P^{'}$ are roots of P, and that the product of the roots is an integer. Since all of the other roots of $P^{'}$, $\alpha^{(2)}, \cdots, \alpha^{(d)}$, have modulus less than 1, we know that α must be a root of $P^{'}$. Thus, α is a Pisot number.

□

4 Properties of o(k)-Pisot Numbers

It is known that for all $\tau \in T$ the minimal polynomial is reciprocal and of even degree. Also, we know that for all integers a, greater than 1, there exists a sequence of Salem numbers (τ_n) converging to a, with $\tau_n - 1$ a unit. We will show that this property remains true for the set O, even for $a = 1$ and we find a relation between the boundedness of a particular subset of O and $\inf T$.

Definition 4.1. *Let $\beta \in O(k)$ for some $k \in \mathbb{Z}$. Either $\beta+1 \in T$ or $\beta^{-1}+1 \in T$. Define α by*

$$\alpha = \begin{cases} \beta + 1, & \text{if } \beta + 1 \in T \\ \beta^{-1} + 1, & \text{if } \beta^{-1} + 1 \in T \end{cases}$$

We call α the Salem conjugate of β.

Clearly $\alpha - 1$ is a unit. Also it is easy to verify that if $\beta \in O$ and its Salem conjugate α is less than 2 then, we get that $(\alpha - 1)^{-1}$ belongs to O. So for $\beta \in O$ we will say that $\beta \in O_1$ if its Salem conjugate is less than 2, and $\beta \in O_2$ otherwise. Clearly $O = O_1 \cup O_2$. We will show that O numbers have very similar arithmetic properties to Salem numbers.

Lemma 4.2. *Let β be a strong o(k)-Pisot number and α the Salem conjugate of β. Let $\alpha = \alpha_1$, $\alpha^{-1} = \alpha_2$, and $\alpha_3, \ldots, \alpha_d$ be the conjugates of α. Then the conjugates of β are either,*

$$\begin{cases} \beta = \beta_1,\ \frac{-\beta}{\beta-1} = \beta_2, \text{ and } \beta_3 = \alpha_3 - 1, \ldots, \beta_d = \alpha_d - 1 & \text{if } \alpha = \beta + 1 \\ \\ \beta = \beta_1,\ -1-\beta = \beta_2, \text{ and } \beta_3 = \alpha_3^{-1} - 1, \ldots, \beta_d = \alpha_d^{-1} - 1 & \text{if } \alpha = \beta^{-1} + 1 \end{cases}$$

and the minimal polynomial of β has even degree. Also we have that if the Salem conjugate α of β is less than 2 then $\Re(\beta_j) = -\frac{1}{2}$ for all $3 \le j \le d$.

Proof. Consider σ_i, the embedding of $\mathbb{Q}(\alpha_1, \ldots, \alpha_d)$ into $\mathbb{C}$, which fixes $\mathbb{Q}$ and maps α_1 to α_i. Let Q be the minimal polynomial of β. Now we have 2 cases:

<u>Case 1:</u> $\alpha = \beta + 1$

$$0 = Q(\beta) = Q(\alpha - 1) = \sigma_i(Q(\alpha - 1)) = Q(\sigma_i(\alpha - 1)) = Q(\alpha_i - 1)$$

Giving us that the roots of Q are exactly $\alpha_i - 1, \forall 1 \le i \le d$.

<u>Case 2:</u> $\alpha = \beta^{-1} + 1$

In this case we use the same arguments and get that the roots of Q are exactly $\alpha_i^{-1} - 1, \forall 1 \leq i \leq d$.

We know that Q is irreducible over $\mathbb{Q}[X]$ and that Q has d roots. d is even because the degree of α is even, so we get that the minimal polynomial of β is reciprocal and of even degree.

Knowing the conjugates of β, and recalling that $|\alpha_j| = 1$ for all $3 \leq j \leq d$, a simple computation shows that $\Re(\beta_j) = -\frac{1}{2}$.

□

Theorem 4.3. [1] *Any integer greater than* 1 *is a limit point of a sequence of Salem numbers* (α_n)*, where the algebraic integers* $(\alpha_n - 1)$ *are units for all* n*.*

Corollary 4.4. *For* $a \in \mathbb{N}^*$*, there exists a sequence* $(\beta_n)_{n \in N}$ *such that* $\beta_n \in O$ *for every* n*, and* $\beta_n \longrightarrow a$*.*

Proof. By Theorem 4.4 we know that there is a sequence $(\alpha_n)_{n\in\mathbb{N}} \subset T$ such that $\alpha_n \longrightarrow a + 1$ and $\alpha_n - 1$ is a unit $\forall n \in \mathbb{N}$. Now, for each $i \in \mathbb{N}$, if $\alpha_i > 2$, define $\beta_i = \alpha_i - 1$, and otherwise define $\beta_i = (\alpha_i - 1)^{-1}$. So $\beta_n \in O$ for every $n \in \mathbb{N}$, and $\beta_n \longrightarrow a$.

□

Corollary 4.5. *The set* O *is not bounded above.*

Proof. By corollary 4.5 we know that for any positive integer a, there exists a sequence of O-numbers converging to a. So it follows immediately that O is not bounded above.

□

The importance of strong $o(k)$-Pisot numbers is given by the following lemma, which shows a relationship between O_1 and the $\inf T$.

Theorem 4.6. *There exists* $(\alpha_n)_{n\in\mathbb{N}} \subset T$ *with* $\alpha_n - 1$ *a unit* $\forall n \in \mathbb{N}$*, and* $\alpha_n \longrightarrow 1$ *if and only if* O_1 *is not bounded above.*

Proof. Assume that there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$, of Salem numbers, converging to 1, with $\alpha_n - 1$ a unit for all $n \in \mathbb{N}$. We can find a subsequence, which we will also denote$(\alpha_n)_{n\in\mathbb{N}}$, such that $\alpha_n \leq 2$, $\forall n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we let $\beta_n = (\alpha_n - 1)^{-1}$. A simple computation shows that $\beta_n \in O$, so it follows that $\beta_n \in O_1$ since $\alpha_n < 2$. Now, as $n \longrightarrow \infty$, we know $\alpha_n \longrightarrow 1$, so $\beta_n \longrightarrow \infty$. Therefore O_1 is unbounded.

Now suppose that O_1 is not bounded above. We can find a sequence $(\beta_n) \subset O_1$ which diverges to ∞ as $n \longrightarrow \infty$. We know that $\alpha_n = \beta_n^{-1} + 1$, and that $\alpha_n - 1$ is a unit, so as β_n goes to ∞, α_n goes to 1. Thus there exists a sequence of Salem numbers (α_n) converging to 1 with $\alpha_n - 1$ a unit.

□

As two immediate consequences of this theorem we have that: if $\inf T > 1$ then O_1 is bounded above, and if O_1 is bounded above and $\inf T = 1$, then every sequence of Salem numbers (α_n) converging to 1 has at most finitely many elements such that $(\alpha_n - 1)$ is a unit.

It is known that if $\tau \in T$ then $\tau^n \in T$. However, for $\beta \in O$, if $\beta^n \in O_1$ for infinitely many n, then it follows that O_1 is not bounded.

5 Arithmetical properties of O-numbers

If τ is a Salem number and $\frac{1}{\tau}, e^{2\pi i\theta_3}, \ldots, e^{2\pi i\theta_d}$ are it's conjugates, then its known that $1, \theta_3, \theta_4, \ldots, \theta_d$ are $\mathbb{Q}$-linearly independent. Also A. Dubickas has shown, in [3], that if τ is either a Pisot or a Salem number, then $\lfloor \tau^n \rfloor$ is composite for infinitely many values of n. We will show in this section that O-numbers are closely related to Salem numbers with respect to arithmetical properties. By using these results we will show that the above properties of Salem numbers remain true for O numbers.

Lemma 5.1. *Let β be an O number and $\beta = \beta_1, \beta_2, e^{2\pi i\phi_3}, \ldots, e^{2\pi i\phi_d}$ be its conjugates. Then the numbers $1, \phi_3, \ldots, \phi_d$ are $\mathbb{Q}$-linearly independent.*

Proof. Suppose there exist integers $l_2, l_3, \ldots, l_d$ such that $l_2 + \sum_{j=3}^{d} l_j\phi_j = 0$. It follows that $\exp(2\pi i \sum_{j=3}^{d} \beta_j^{\phi_j}) = 1$. In the Galois extension $\mathbb{Q}(\beta, \beta_2, \ldots, \beta_d)$ there exists an automorphism σ such that $\sigma(\beta_2) = \beta$. Hence we have

$$\beta^{l_3} \prod_{j=4}^{d} (\sigma(\beta_j))^{l_j} = 1.$$

However, this equation cannot hold if $l_2 \neq 0$. Using the same argument we get that $l_3, \ldots, l_d$ are zero and the reals $1, \phi_3, \ldots, \phi_d$ are $\mathbb{Q}$-Linearly independent. $\square$

Corollary 5.2. *Let β be an algebraic integer with conjugates $e^{2\pi\phi_2}, \ldots, e^{2\pi\phi_d}$ with $1, \phi_2, \ldots, \phi_d$ $\mathbb{Q}$-linearly independent. Then $\sum_{j=3}^{d} \cos 2n\pi\phi_j$ is dense in $[-2m, 2m]$, where $d = 2m + 2$.*

This corollary is a well known result that we get by using Kronecker's theorem (see [4]).

A well known and important result for Salem numbers is that if $\tau \in T$, then $\{\tau^n\}$ is dense but not uniformly distributed in $[0, 1]$. The next theorem shows that this result is still true for O_2-numbers.

Theorem 5.3. *Let $\beta \in O$ such that the Salem conjugate of β is greater than 2. Then, the sequence $\{\beta^n\}$ is dense but not uniformly distributed in $[0, 1]$.*

Proof. Let $\beta \in O$. Let d be the degree β and $m = \frac{d-2}{2} \in \mathbb{Z}$. The number $S_n = \beta^n + (-1)^n(\frac{\beta}{\beta+1})^n + \sum_{j=3}^{d} \beta_j^n$ is a rational integer and the distribution of the sequence $\{\beta^n\}$ is the same as that of the sequence $\{(-1)^n(\frac{\beta}{\beta+1})^n + \sum_{j=3}^{d} \beta_j^n\}$. The sequence $(-1)^n(\frac{\beta}{\beta+1})^n$ converges to 0, so we will only consider the distribution of the sequence $\{\sum_{j=3}^{d} \beta_j^n\}$.

- 1) The sequence β^n is dense in $[0,1]$. By corollary 4.8 we know that the sequence $\{\sum_{j=3}^{d} \beta_j^n\}$ is dense in $[0,1]$. Therefore, it follows that the sequence $\{\beta^n\}$ is dense in $[0,1]$.

- 2) The sequence $\{\beta^n\}$ is not uniformly distributed in $[0,1]$.

Because the reals $1, \phi_3, \phi_4, \ldots, \phi_d$ are linearly independent, it follows (from theorem 4.6.3 of [5]) that the sequence $(n\phi_j)_{j=2,\ldots,d}$ is uniformly distributed in $\mathbb{R}^{m-1}$. Consider the function $x \longmapsto 2\cos 2\pi x$. The integral

$$\int_0^1 exp(4i\pi h \cos 2\pi t)dt = J_0(4h\pi),$$

where J_0 denotes the Bessel function of order 0, is not zero for all $h \in \mathbb{Z}^*$. We complete the proof by using theorem 4.6.5 (of [4]).

□

Lemma 5.4. *[3] Suppose F is a polynomial with integer coefficients in d variables, and let (b_k) be the sequence given by:*

$$b_k = F(b_{k-1}, b_{k-2}, \ldots, b_{k-d})$$

where $k \geq d+1$. Then, for all $q \in \mathbb{N}$, $\exists k(q) \in \mathbb{Z}$, such that $(b_k)_{k \geq k(q)}$ is periodic modulo q.

Proof. Consider the vector $v_k = (b_k, b_{k+1}, \ldots, b_{k+d-1})$ modulo q. By the pigeonhole principle, among the (q^d+1) vectors $v_1, v_{d+1}, v_{2d+1}, \ldots, v_{Dd+1}$, where $D = q^d$, at least two are equal because there are just q^d distinct vectors with d entries modulo q. Suppose $v_{kd+1} = v_{ld+1}$, with $k < l$. By the definition of the sequence (b_k), we have that $b_{kd+j} = b_{ld+j}$ modulo q for $j = 1, \ldots, (l-k)d$. Therefore, b_{kd+j}, $j = 1, 2, \ldots$ modulo q is periodic with period less than or equal to $(l-k)d$.

□

Lemma 5.5. *Let $N \in \mathbb{N}$. Then if $\beta \in O$, there exists a constant $k = k(N)$ such that the sequence $(S_n)_{n \in \mathbb{N}}$, where $S_n = Tr(\beta^n)$, is periodic modulo N with period $p = p(N, \alpha)$, satisfying $p \leq N^d$.*

Proof. Let $Q(x) = a_0 + a_1x + \cdots + a_{d-1}x^{d-1} + x^d$ be the minimal polynomial of β. Then since $Q(\alpha_j) = 0$ for all $j = 1, 2, \ldots, d$. We get d equalities:

$$a_0\alpha_j^n + a_1\alpha_j^{n+1} + \cdots + a_{d-1}\alpha_j^{n+d-1} + \alpha_j^{n+d} = 0 \qquad \forall j = 1, \ldots, d.$$

So by taking the sum we have that

$$a_0S_n + a_1S_{n+1} + \cdots + a_{d-1}S_{n+d-1} + S_{n+d} = 0$$

which implies

$$a_0S_n + a_1S_{n+1} + \cdots + a_{d-1}S_{n+d-1} = -S_{n+d}. \tag{1}$$

Now in (2) we express S_{n+d} as a polynomial with integer coefficients and variables $S_n, \ldots, S_{n+d-1}$, so by Lemma 5.2 we have that $(t_n)_{n\geq k}$ is periodic for some $k \in N$.

□

In [3] A.Dubickas has shown that if τ is either a Piost or a Salem number then there are infinitely many composite numbers in the sequence $\lfloor \tau^n \rfloor$. We will show in the next theorem that this result remains true for O_2-numbers.

Theorem 5.6. *Let β be an O_2-number. Then, there exist infinitely many composite numbers in the sequence $\lfloor \beta^n \rfloor$.*

Proof. Let β be an O_2-number. So we know that

$$\frac{S_n}{2} = \frac{\beta^n}{2} + \frac{\beta_2^n}{2} + \frac{1}{2}\sum_{j=3}^{d} \cos n\phi_j$$

Now, by lemma 5.3, there exists a constant $k(2)$ such that $(S_n)_{n\geq k(2)}$ is periodic mod 2. Now choose $M \geq k(2)$, with M divisible by the period of $(S_n \mod 2)$. We will consider now the sequence $(S_{tM})_{t\in\mathbb{Z}}$. By Lemma 5.3 we get that S_{tM} are all even, or all odd.

<u>Case 1:</u> S_{tM} are all even.

If they are all even we have that

$$\left\{\frac{\beta^{tM}}{2}\right\} = \left\{-\frac{\beta_2^{tM}}{2} - \frac{1}{2}\sum_{j=3}^{d} \cos tM\phi_j\right\}$$

Now, by lemma 5.1 we know that $1, \phi_3, \ldots, \phi_d$ are $\mathbb{Q}$-linearly independent, so $M, M\phi_3, \ldots, M\phi_d$ are $\mathbb{Q}$-linearly independent.Thus it follows by corollary 4.8 that the sum $\sum_{j=3}^{d} \cos tM\phi_j$ takes on infinitely many values in the interval $(\frac{-1}{3}, \frac{-1}{4})$. Since $\beta > 1$, and $|\beta_2| = |\frac{\beta}{\beta+1}| < 1$, we have, for t sufficiently large, that $\{\beta_2^{tM}\} < \frac{1}{6}$. Thus $\{\frac{\beta^{tM}}{2}\} < \frac{1}{2}$ for infinitely many values of t. Thus $\lfloor \beta^{tM} \rfloor = 2\lfloor \frac{\beta^{tM}}{2} \rfloor$ is even, and consequently composite, for infinitely many values of t.

Case 2: S_{tM} are all odd.

If they are all odd we have that

$$\left\{\frac{\beta^{tM}}{2}\right\} = \left\{\frac{1}{2} - \frac{\beta_2^{tM}}{2} - \frac{1}{2}\sum_{j=3}^{d} \cos tM\phi_j\right\}$$

Now, by the same argument as case 1 we have that the sum $\sum_{j=3}^{d} \cos tM\phi_j$ takes infinitely many values in $(\frac{-1}{5}, \frac{1}{4})$. By taking t suficiently large we get that $\{\frac{\beta_2^{tM}}{2}\} < \frac{1}{6}$. So it follows that $\{\frac{\beta^{tM}}{2}\} < \frac{1}{2}$ for infinitely many $t \in \mathbb{Z}$. Thus the numbers $\lfloor \beta^{tM} \rfloor = 2\lfloor \frac{\beta^{tM}}{2} \rfloor$ are even, and consequently composites, for infinitely many values of t.

□

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