Gauss Words and Planar Curves

Allen Matthew Jarboe*
Martin Ames Harrison†
Michael Marcondes de Freitas‡

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Acknowledgments

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1 Preliminaires

We define a **normal closed curve (n.c.c.)** with n-double points in a compact oriented surface to be a closed curve in this surface with n self-intersections which are all tranverse double points. Whenever it is possible to realize a n.c.c. on a sphere, it will be said to be **planar** and might be drawn in a plane, since the plane and a punctured sphere are homeomorphic. Figure 1 shows a typical example for $n = 2$.

To each n.c.c. we associate a string of signed characters called a **Gauss word**. This string is obtained by walking once through the whole curve and recording the double points in a way we describe now using the curve on Figure 1 as an example. First label the double points and choose both a starting place and a direction for the walk. The green point indicates the starting place and the arrow the choosen direction. Each double point will be visited twice and one of these visits will be labeled positive and the other negative. When you arrive at a double point, x in the case of Figure 2, record x if the crossing curve

*Purdue University, USA: jarboe@purdue.edu .
†University of California, Santa Barbara, USA: martinharrison@umail.ucsb.edu .
‡Universidade de Brasília, Brazil: http://www.mat.unb.br/∼michael .

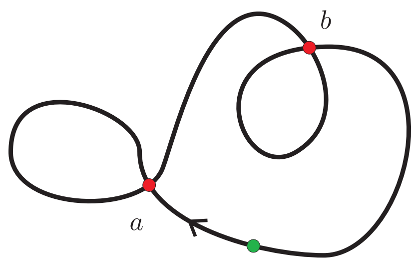

Figure 1: A n.c.c. with 2 double points.

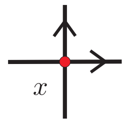

Figure 2: The neighborhood of a double point.

goes from left to right and x^- if the crossing curve goes from right to left. The word for the curve in Figure 1 considering all these conventions is then aa^-b^-b.

In general, given any n letters $a_1, \cdots, a_n$, we call an **(abstract) signed Gauss word** any permutation of $a_1 \cdots a_n a_1^- \dots a_n^-$. A more formal definition of an abstract Gauss word will be introduced when it is necessary.

2 Equivalence Classes of Gauss Words

Given a normal closed curve on an compact oriented surface there are several ways of reading this curve each of them leading to a different Gauss word. You can relabel the crossings, you can change your starting point, you can change your position from the outside of the surface to the inside or vice versa and you can reverse the direction in which you walk through the curve. Each of these has a specific effect on the Gauss word first considered for the curve.

A relabeling of the crossings will translate into a permutation of the letters ignoring the signs. The change of starting point translates into a cyclic permutation of the Gauss word. If you are changing your point of view from the outside to the inside of the surface or vice versa then the letters that were signed positive will now be signed negative and conversely. And to reverse the direction in which you walk through the curve is the same as to rewrite the Gauss word backwards. We describe these operations in more detail in the next example using the n.c.c. in Figure 3.

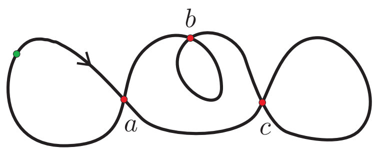

Figure 3: The n.c.c. $ac^-cbb^-a^-$.

First observe that it does not matter what symbol (choosen from the list previously fixed, that is, a, b, c) is used to label each of the double points. So you can permute the letters a, b and c, relabel the double points accordingly, and then reread the word, obtaining a different Gauss word for the given curve. (Figure 4.)

Next, given a particular labeling, direction and sign convention, we may choose any of the $2n$ arcs of the curve to begin the walk. Two words are equivalent by this relation if they result from the same curve but are read from different starting points. (See Figure 5.)

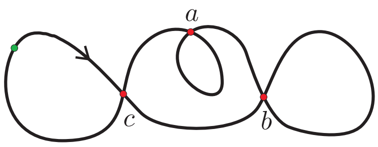

Figure 4: $ac^{-}cbb^{-}a^{-} \equiv cb^{-}baa^{-}c^{-}$.

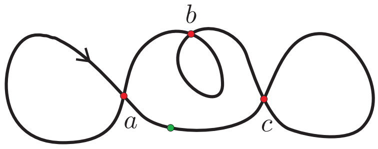

Figure 5: $ac^{-}cbb^{-}a^{-} \equiv c^{-}cbb^{-}a^{-}a$.

If you change your perspective from the outside to the inside of the surface (or vice versa) you will have the same Gauss word except for the fact that the signs of the letters will be opposite. This defines another equivalence relation. (Figure 6 illustrates this.)

Finally, if you reverse the direction in which you walk along the curve, the resulting word will be a mirror image of the one read at first. This gives the fourth equivalence relation. (Look at Figure 7.)

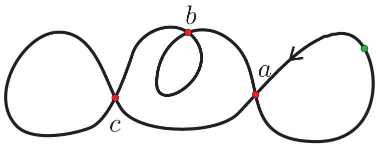

Figure 6: $ac^{-}cbb^{-}a^{-} \equiv a^{-}cc^{-}b^{-}ba$.

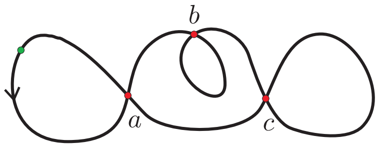

Figure 7: $ac^{-}cbb^{-}a^{-} \equiv a^{-}b^{-}bcc^{-}a$.

As another example, the Gauss words $acb^{-}a^{-}bc^{-}$, $bc^{-}acb^{-}a^{-}$, $c^{-}ba^{-}b^{-}ca$, $a^{-}c^{-}bab^{-}c$ and $bac^{-}b^{-}ca^{-}$ all reffer to the same curve.

A full algebraic description in a very formal setup of these equivalence relations on the set of all Gauss words of a prescribed size n is given by Daniel Frohardt in [4].

3 The Minimal Genus Problem

The first problem considered is the problem of finding a compact orientable surface of the smallest possible genus in which a normal closed curve realizing a given Gauss word fits. This will be referred to as the **minimal genus problem**.

J. Scott Carter presents in [3, pages 87–88] or [2, Section 3] what he claims to be an algorithm for finding this surface once a Gauss word has been given. But he gives neither a formal justification of the algorithm nor an indication of where one could be found.

We would be convinced that this algorithm always works if Conjectures 1 and 2 turned out to be true.

Conjecture 1. *Given any Gauss word, there exists a compact orientable surface in which a normal closed curve realizing this Gauss word fits.*

Conjecture 2. *Consider a normal closed curve in a compact orientable surface. The turning left procedure in Carter's algorithm applied to this setup leads us to a collection of disks if, and only if this surface is a surface of smallest possible genus for a normal closed curve realizing the Gauss word of the given curve.*

Assuming these two facts to be true, we would get a very visual interpretation of the algorithm. Given any Gauss word, Conjecture 1 would give us the existence of a compact oriented surface in which a normal closed curve realizing this Gauss word could be drawn; without any loss of generality we could suppose that this surface was of minimal genus. Conjecture 2 would then give us several disks which, when glued together by matching the twin edges, would result again in this minimal genus surface; the appropriate concatenation of these gluing edges would be the curve for the given Gauss word.

Grant Cairns and Daniel M. Elton call a surface for a Gauss word obtained through this algorithm the **Carter's surface** for the given Gauss word. And so shall we do.

Conjecture 2 becomes rather intuitive after running the algorithm in some particular examples. We also considered what would happen if we tried to apply the turning left procedure to a normal closed curve not in its minimal genus surface; in all particular cases considered the surface was not split into disks. But no formal proof for Conjecture 2 was achieved.

Conjecture 1 is even more intuitive. To convince oneself of this one should pick any Gauss word and start drawing a normal closed curve for this Gauss word on the sphere. Whenever you are left in a situation where you would not be able to get to the next crossing without adding an extra crossing, use the third dimension of the space by adding a handle to the surface that takes you to the next crossing. But we were not satisfied with this rather algorithmic argument. Although we were not able to find a formal proof, we at least have a very geometrical idea for a non-algorithmic justification of Conjecture 1.

We illustrate this idea with an example. Consider the Gauss word aba^-b^-. Start drawing the two crossings (a and b) on a sphere as shown in Figure 8. Now choose a disk neighborhood for each of the end points of the crossings in such a way that they are pairwise disjoint. (See Figure 9.) We will now add some handles to the sphere in order to assure that it will be possible to finish drawing the curve. The handles will be glued to the boundaries of two of these disk neighborhoods; the Gauss word given will tell us which of these disks should be paired. For instance, b (positive) follows a (positive), so we should add a handle to the sphere joining neighborhoods O_1 and O_2. Next, there must be a handle joining O_3 and O_4, since the Gauss word aba^-b^- tells us a^- should follow b, and so on. When this process ends, we will have a surface of genus four in which a normal closed curve realizing the Gauss word aba^-b^- certainly fits. Of course this proccess will not always lead to the minimal genus surface for the given Gauss word. For instance, the minimal genus surface for this specific Gauss word is a torus. (See [3, page 88].) The point of this argument is to obtain not a good upper bound for the genus of the minimal genus surface, but a non-algorithmic proof of its existence.

We believe that with a good notation and clear definitions of the objects in question it would not be complicated to obtain a formalization of this argument.

4 The Planarity Problem for Signed Gauss Words

An abstract signed Gauss word w on n letters is a bijection between the sets $\{1, \ldots, 2n\}$ and $\{a_1, \ldots, a_n\} \times \{+, -\}$. One can think of it as a string of length $2n$ in which each of the letters a_i occurs exactly once with a positive sign and

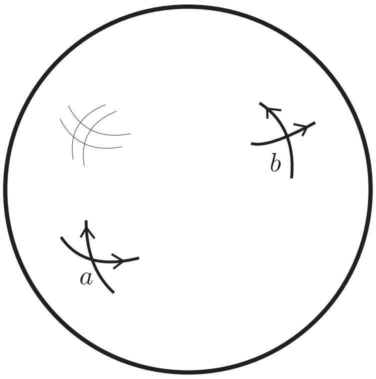

Figure 8: A surface for the Gauss word $aba^{-}b^{-}$: step 1

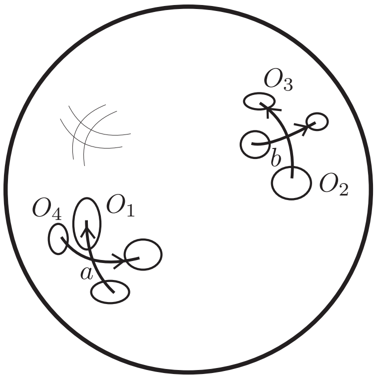

Figure 9: A surface for the Gauss word $aba^{-}b^{-}$: step 2

exactly once with a negative sign. We then actually write $(a_i, +) = a_i$ and $(a_i, -) = a_i^{-}$ for all i.

We have seen already that an arbitrary Gauss word can always be realized on some surface of sufficiently large genus. But when can a signed Gauss word be realized as a curve on a sphere? This question has been answered by several people, but most answers have been algorithmic. Cairns and Elton have given purely combinatorial criteria for planarity (realiziability in the sphere) in [1], which were further simplified by Tanio in [5]. We'll describe these criteria and give a heuristic for one of them. First, we need some notation.

Definition 1. *Fix any signed Gauss word w with letters $a_1, \ldots, a_n, a_1^{-}, \ldots, a_n^{-}$. For $i = 1, \ldots, n$, let S_i be the collection of letters appearing strictly between a_i and a_i^{-} in w, reading from left to right. (For this purpose you can cyclically permute the letters in the word in order to make a_i appear before a_i^{-} on w.) Denote $S_i \cup \{a_i, a_i^{-}\}$ by $\overline{S}_i$. Let S^{+} be the set of letters in w with $+$ signs, and S^{-} be the set of letters in w with $-$ signs. For a subset A of the letters of w, define $\sigma(A)$ to be $\#(A \cap S^{+}) - \#(A \cap S^{-})$. Define $\alpha_i(w)$ to be $\sigma(S_i)$ and $\beta_{ij}(w)$ to be $\sigma(\overline{S}_i \cap S_j)$.*

Fact 1. *An abstract signed Gauss word w is a Gauss word of a curve on the sphere if, an only if $\beta_{ij}(w) = \alpha_i(w) = 0$ for all $i, j = 1, \ldots, n$.*

Cairns and Elton proved this theorem by showing that an abstract signed Gauss w word satisfies the above conditions exactly when the Carter surface of w has genus zero. The proof is not trivial, and we will not reproduce it here. We will, however, give a heuristic argument for the necessity of the condition on the α's.

Consider the crossing a_1 of a normal closed planar curve with Gauss word w. To see why $\alpha_1(w)$ must be zero, look at Figure 10. Since a_1 is a double point of a planar curve, we can picture it as the vertex of a "loop" which intersects other arcs of the curve. The points where this loop intersects other arcs of the curve are precisely the elements of S_1. Suppose you are a Massey bug entering the inside of the loop at the point x. Then $x^{-} \in S_1$. Since this curve is in the sphere, you (the Massey bug) cannot escape the inside of the loop via a handle; you must cross the loop once again, this time from the inside to the outside.

The point at which you exit, y, will have a positive orientation if the point at which you entered had a negative orientation. And it is not hard to convince oneself that the entry points and exit points are always opposite in orientation, so that signs always cancel. Thus, α_1 must be zero.

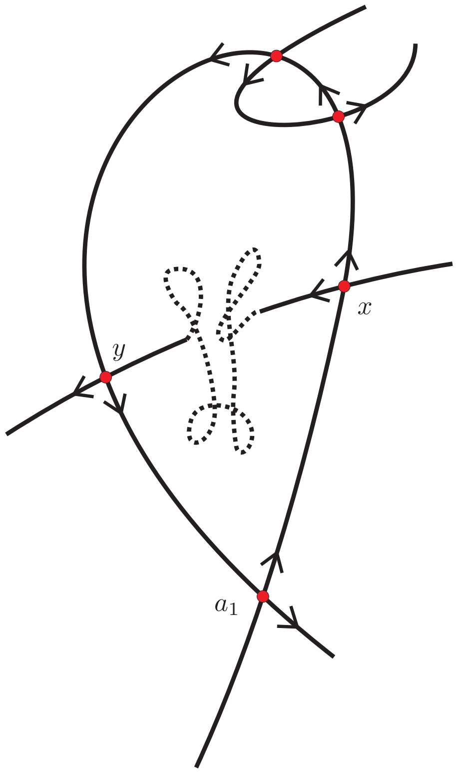

Figure 10: Why the α's must vanish.

Hitoe Tanio points out in [5, page 2] that this condition on the α's is superfluous. The condition on the β's alone is neccessary and sufficient for a signed Gauss word to be planar.

5 Creating Planar Curves

In this section we will examine the ways planar curves with n double points are created from closed curves with $n-1$ double points. If a pattern can be found in how the planar curves with n double points are made then there might be some way to recursively count them.

There is only one curve with one double point, denoted simply aa^-, and it is shown here.

All curves in this section will be denoted in a similar fashion. In the figures at the end of the section, the orange dot represents the starting place on the curve and the purple arrows indicate the orientation of the curve. The orange letter near the curve is the name of the curve. Since we are considering only normal curves, the only way to add a single double point to a planar curve is

Figure 11: aa^-

to add a simple loop xx^- or x^-x. There are two planar curves that can be constructed from x and they are y and z.

Figure 12: ab^-ba^-

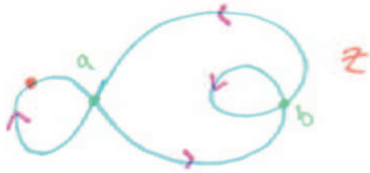

Figure 13: abb^-a^-

Now we will add simple loops to y and z to see which curves result. According to [Carter] there are six unique planar curves with three double points. Adding simple loops to y produces four unique curves. The curves are not unique in the sense that they are porduced soley by y, but in the sense that they are the only curves obtainable from y by adding loops. The curves produced by y in this manner are A, B, D and E.

Figure 14: $ab^-cc^-ba^-$

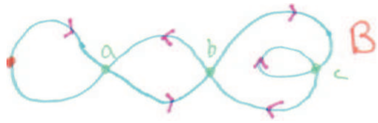

Figure 15: $ab^-c^-cba^-$

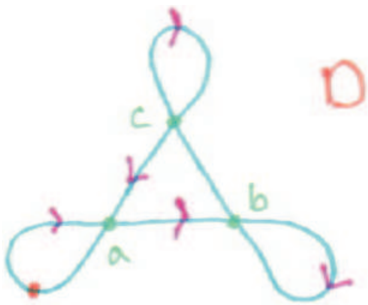

Figure 16: $ab^-bc^-ca^-$

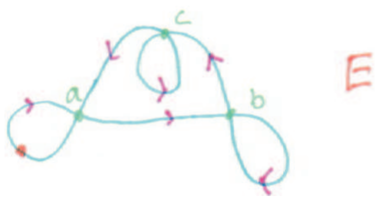

Figure 17: $ab^-bc^-ca^-$

Now doing the same procedure with z we see that it produces three unique curves. However we will notice that B and E are produced by both y and z. The curves produced by adding simple loops to z are B, E and C.

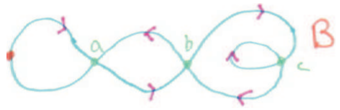

Figure 18: $ab^-c^-cba^-$

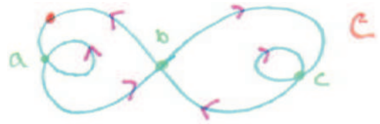

Figure 19: $aa^-b^-bcc^-$

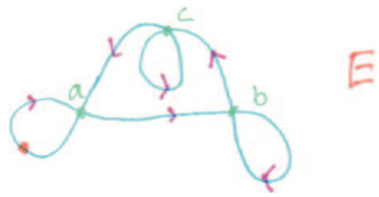

Figure 20: $ab^-bc^-ca^-$

There is however a planar curve with three crossings that is not produced by adding simple loops to either x or y. It is F and shown also at the end of the section.

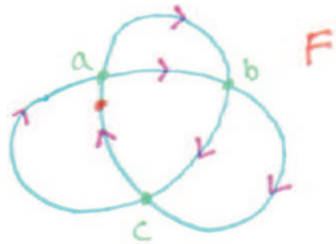

Figure 21: $ab^-ca^-bc^-$

However this curve can be formed from the one curve with two double points on the torus, ab^-a^-b. Normally when drawing ab^-a^-b you come to a point where you are stuck in a closed loop and forced to add a handle to complete the curve. This time instead of adding a handle, force a new double point to complete the closed curve. This will construct the curve F, the missing planar curve with three double points. Forcing a new double point is analagous to squishing the handle of the torus ab^-a^-b is on and making it a planar curve with three double points. So by two different manipulations of curves with two double points we are able to produce all the planar curves with three double points.

Now taking the six planar curves with three double points we add simple loops to them to make planar curves with four double points. There are 19 planar curves with four double points [Carter] and 18 of these can be made from adding simple loops to the planar curves with three double points. By adding simple loops to A we get the four curves 1, 6, 10 and 14 and they are as follows.

Adding loops to B we get the eight curves 6, 8, 10, 11, 12, 13, 14 and 15.

The curve 6 is the same by equivalence relations to the curves produced by A and D. Also 15 is equivalent to the curve produced by E. The curve C with

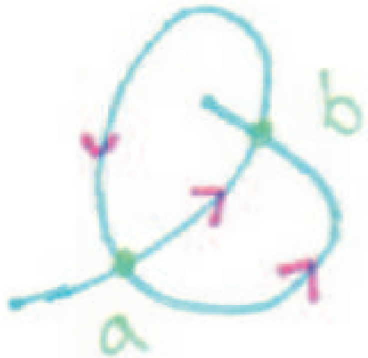

Figure 22: Squishing a Torus.

Figure 23: $ab^-cdd^-c^-ba^-$

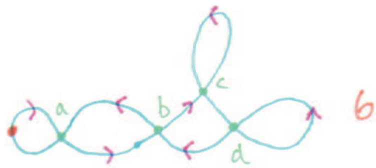

Figure 24: $ab^-cc^-dd^-ba^-$

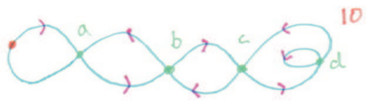

Figure 25: $ab^-cdd^-c^-ba^-$

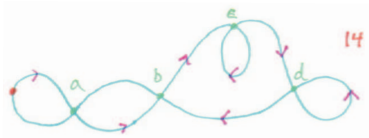

Figure 26: $ab^-cc^-dd^-ba^-$

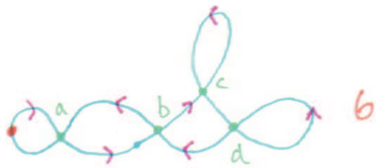

Figure 27: $ab^-cc^-dd^-ba^-$

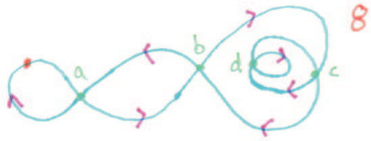

Figure 28: $ab^-c^-d^-dcba^-$

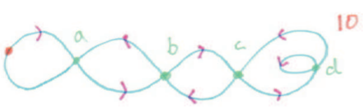

Figure 29: $ab^-cdd^-c^-ba^-$

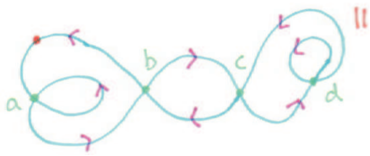

Figure 30: $aa^-b^-cdd^-c^-b$

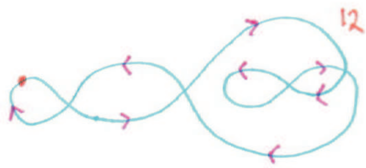

Figure 31: $ab^-c^-dd^-cba^-$

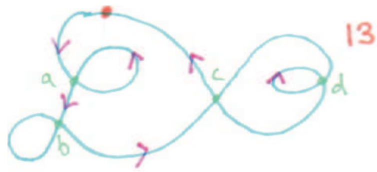

Figure 32: $aa^-b^-bc^-d^-dc$

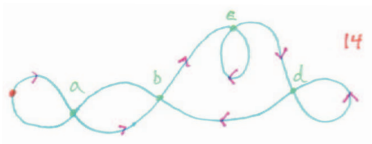

Figure 33: $ab^-cc^-dd^-ba^-$

Figure 34: $ab^-bcd^-dc^-a^-$

three double points produces four unique curves by adding simple loops: 7, 8, 9, and 13.

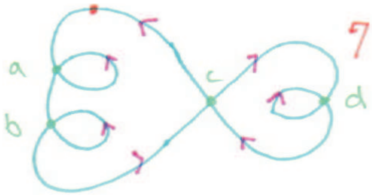

Figure 35: $ab^-bcdd^-c^-a^-$

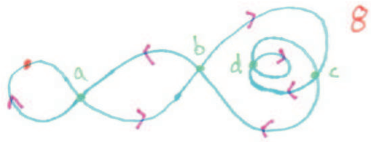

Figure 36: $ab^-c^-d^-dcba^-$

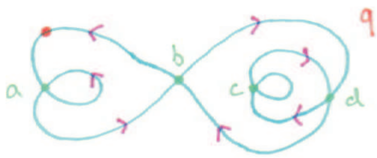

Figure 37: $aa^-b^-d^-c^-cd$

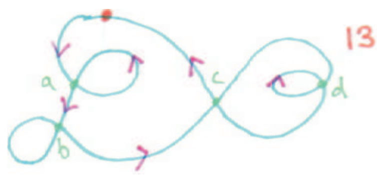

Figure 38: $aa^-b^-bc^-d^-dc$

The curve D also produces four unique curves by adding simple loops: 2, 3, 6, 16.

The curve E produces six unique curves: 4, 5, 7, 13, 14 and 15

Finally the curve F produces two planar curves with four double points by adding simple loops, 17 and 19. They are as follows.

18 of the 19 planar curves with four double points are produced by adding simple loops to the six planar curves with three double points. To create the final curve we will again look at a curve on the torus, $bdc^-b^-cd^-$, this time with three double points. Once more we will force a double point when we would normally be required to add a handle. This creates the final planar curve with four crossings. It seems, then, that all planar curves can be made from curves with one less double point by either adding a simple loop or squishing a handle

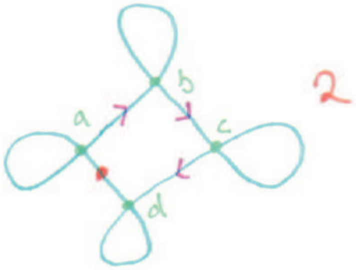

Figure 39: $aa^{-}bb^{-}cc^{-}dd^{-}$

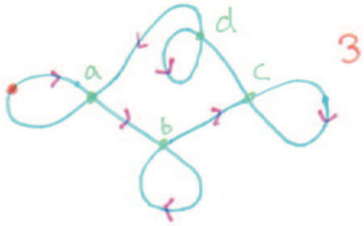

Figure 40: $ab^{-}bc^{-}cdd^{-}a^{-}$

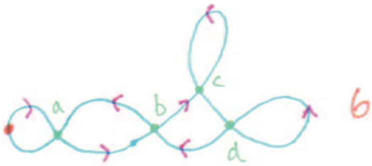

Figure 41: $ab^{-}cc^{-}dd^{-}ba^{-}$

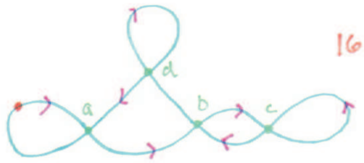

Figure 42: $ab^{-}cc^{-}bd^{-}da^{-}$

Figure 43: $ab^{-}bcc^{-}dd^{-}a^{-}$

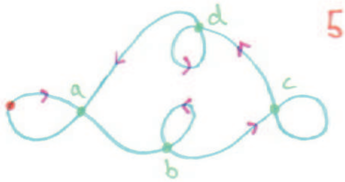

Figure 44: $abb^{-}c^{-}cdd^{-}a^{-}$

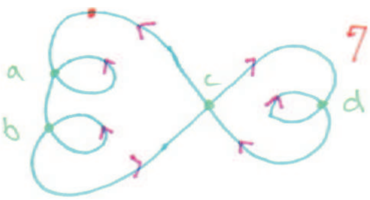

Figure 45: $ab^{-}bcdd^{-}c^{-}a^{-}$

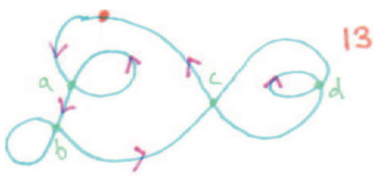

Figure 46: $aa^{-}b^{-}bc^{-}d^{-}dc$

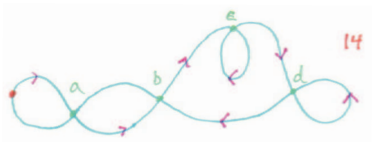

Figure 47: $ab^{-}cc^{-}dd^{-}ba^{-}$

Figure 48: $ab^{-}bcd^{-}dc^{-}a^{-}$

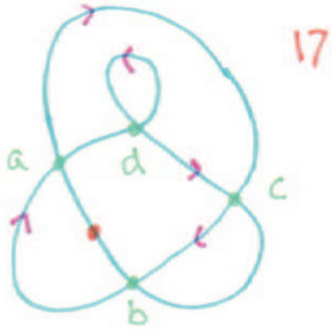

Figure 49: $ac^{-}ba^{-}dd^{-}cb^{-}$

Figure 50: $a^{-}bc^{-}ab^{-}cdd^{-}$

for a curve on a torus. However since a formal proof is not obvious we will pose this idea as a conjecture.

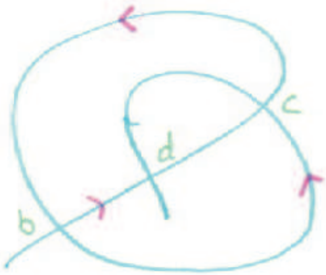

Figure 51: Creating curve 18.

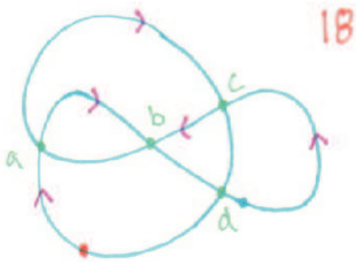

Figure 52: $a^{-}bdc^{-}b^{-}acd^{-}$

Conjecture 3. *Every planar curve with n double points can be constructed by manipulating curves with $n-1$ double points.*

Now we will look at how the curves with different numbers of double points relate to each other. We will begin by looking at the simplest case, going from one double point to two double points. The following diagrams show how the different curves are related. (All of the following diagrams could be very easily formalized into graphs) The vertices of the first diagram represent the different planar curves with their respective names and the different edges represent all the possible ways to add loops to x and produce planar curves with two double points. In general the edges on the diagrams connect curves that can be made equivalent by adding or removing a single loop.

The second diagram shows the relationships between planar curves with two double points and planar curves with three double points.

The vertex F is not connected to any other vertices because it was produced by squishing a handle and not by adding a loop to some planar curve with two double points. In all the subsequent diagrams, the set of curves produced by adding loops to F will be disjoint from the set of curves produced by the other planar curves with two double points because there is no way to construct the

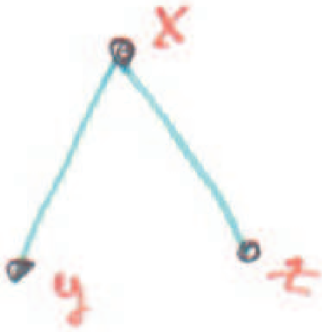

Figure 53: From 1 to 2 double points.

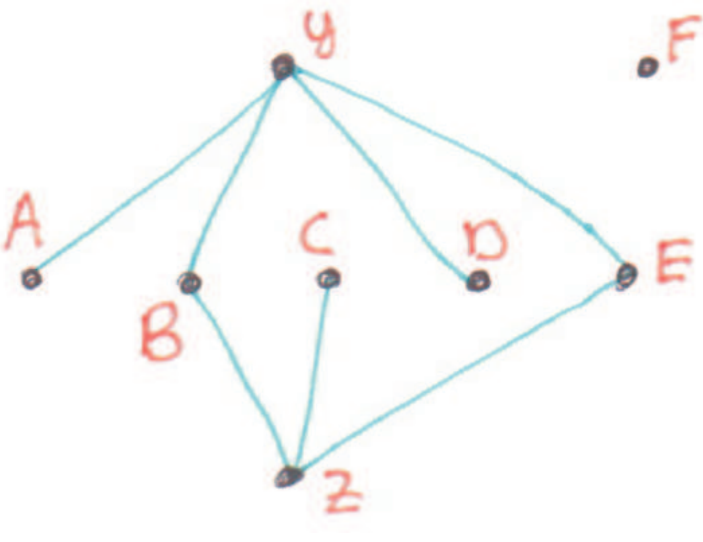

Figure 54: From 2 to 3 double points.

crossings in *F* by just adding loops. This seems to create different classes of loops based on how complicated the orignal curve is. As seen in the following diagram, a new vertex, 18, is isolated and it again has no simple loops in it. However in the examples given, it is not obvious how the number of planar curves increases with an increase in the number of double points. Designing a computer program to check the relation in cases with greater numbers of double points might reveal a pattern.

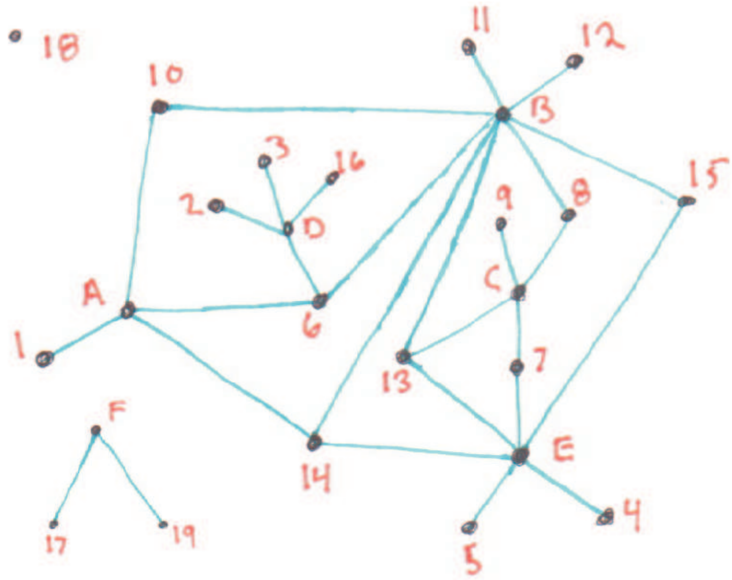

Figure 55: From 3 to 4 double points.

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