On the codimension three submanifolds of Euclidean Space with nonnegative sectional curvatures

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Abstract

We show that a complete submanifold M in codimension three with nonnegative sectional curvature which contains no lines and is covered by $\bar{M} \times \mathbb{R}$ has nonnegative curvature operator. In addition, we give a classification of codimension three compact manifolds of nonnegative sectional curvature and infinite fundamental group.

1 Introduction

The study of the topology of Riemannian manifolds of nonnegative sectional curvature is a classical problem in differential geometry. In the case that the curvature operator is nonnegative (see definition in Section 2), the results of several authors ([5], [7], [8], [11]) lead to a topological classification of such manifolds. They are covered by Riemannian products of manifolds of the following types: homeomorphic to spheres, diffeomorphic to Euclidean spaces, biholomorphic to complex projective spaces or symmetric spaces of compact type. This classification can be found in [6]. If the dimension of the manifold is three, the nonnegativity of the sectional curvatures implies the nonnegativity of the curvature operator (see Lemma 2.1). This implication is also true in the case of hypersurfaces of Space Forms or manifolds that are immersed in Space Forms with zero normal curvature ($R^{\perp} = 0$). A result of Weinstein, [12], states that codimension two nonnegatively curved submanifolds of the Euclidean space also have nonnegative curvature operator. Using this fact, Baldin and Mercuri classified compact codimension two submanifolds in [1] and [2], and in [9], Noronha did the complete, non-compact case. They are homeomorphic to spheres, or a Riemannian product of two manifolds, each of which homeomorphic to spheres or a product of manifold homeomorphic to a sphere with a manifold diffeomorphic to the Euclidean space, or diffeomorphic to the total space of a bundle over S^1 or over the projective plane $\mathbb{R}\mathrm{P}^2$, or cylinders over the two-dimensional Klein bottle.

Codimension three submanifolds of nonnegative seccional curvature have not yet been studied. The aim of this article is to start such a study by investigating conditions on codimension three submanifolds that imply the nonnegativity of the curvature operator. Our first result is the following:

Theorem 1.1 *Let $f : M^n \to \mathbb{R}^{n+3}$ be an isometric immersion of a complete, Riemannian manifold M with nonnegative sectional curvatures. If M contains no lines and is covered by $\bar{M} \times \mathbb{R}$, then the curvature operator of $\bar{M}$ is nonnegative. In particular, M has nonnegative curvature operator.*

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It follows from a result of Cheeger and Gromoll (Theorem 9.1 in [4]) that if the fundamental group of a compact manifold of nonnegative curvature is infinite, then its universal covering splits isometrically as $\bar{M} \times \mathbb{R}^k$, for $k > 0$. We use this result and Theorem 1.1 to prove the following:

Theorem 1.2 *Let $f : M^n \to \mathbb{R}^{n+3}$ be an isometric immersion of a compact manifold with nonnegative sectional curvature. If the fundamental group $\pi_1(M)$ is infinite, then M is covered by one of the following:*
(i) $\bar{M} \times \mathbb{R}^k$, where $\bar{M}$ is homeomorphic to a sphere or the Riemannian product of two manifolds, each of which is homeomorphic to a sphere, and $k \leq 2$.
(ii) $\bar{M} \times \mathbb{R}$, where $\bar{M}$ is biholomorphic to the Complex Projective Plane $\mathbb{C}P^2$.

2 Algebraic Preliminaries

Let ∇ and R denote the Riemannian connection and the Riemann curvature tensor of M respectively. Let $\Lambda^2(T_pM)$ denote the exterior product of the tangent space T_pM. The tensor R defines a symmetric operator $\mathcal{R} : \Lambda^2(T_pM) \to \Lambda^2(T_pM)$ given by

$$\mathcal{R}(e_{ij}) = \frac{1}{2}\sum_{k,l} R_{ijlk} e_{kl}$$

where $\{e_i\}$ is an orthonormal basis of T_pM, e_{ij} denotes the 2-form $e_i \wedge e_j$ and $R_{ijlk} = \langle R(e_i, e_j)e_l, e_k\rangle$.We endow $\Lambda^2(T_pM)$ with its natural inner product, that is,

$$\langle X \wedge Y, Z \wedge W\rangle = \langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle,$$

which implies that $\{e_{ij}\}_{i<j}$ is an orthonormal basis of $\Lambda^2(T_pM)$.

We will consider immersions into the Euclidean space. In this case, denote by α the second fundamental form given by

$$\alpha(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$$

and the Gauss equation is as follows

$$\langle R(X, Y)Z, W\rangle = \langle \alpha(X, W), \alpha(Y, Z)\rangle - \langle \alpha(X, Z), \alpha(Y, W)\rangle.$$

Denoting by A_ξ the symmetric Weingarten operator given by

$$\langle A_\xi X, Y\rangle = \langle \alpha(X, Y), \xi\rangle ,$$

where ξ is a normal direction, the Gauss equation can be written as follows

$$\mathcal{R}(X \wedge Y) = \sum_i A_{\xi_i} X \wedge A_{\xi_i} Y.$$

In the following lemmas, U(p) will denote the orthogonal complement of the relative nullity subspace

$$\mathrm{N(p)} = \{\mathrm{X} \in \mathrm{T_xM} \mid \alpha(\mathrm{X}, \mathrm{Y}) = 0, \forall\ \mathrm{Y} \in \mathrm{T_xM}\}.$$

Lemma 2.1 *If $dim(U(p)) \leq 3$ and M has nonnegative sectional curvature k, then the curvature operator is nonnegative at p.*

Proof: If $dim(U(p)) < 2$, then $\Lambda^2(U(p)) = 0$. If $dim(U(p)) = 2$, then there exist orthonormal vectors X_1 and X_2 such that $X_1 \wedge X_2$ spans the one-dimensional space $\Lambda^2(U(p))$. Therefore

$$\langle \mathcal{R}(X_1 \wedge X_2), X_1 \wedge X_2 \rangle = k(X_1, X_2) \geq 0.$$

If $dim(U(p)) = 3$, consider any 2-form $\omega \in \Lambda^2(U(p))$. It is well-known that ω is decomposable, that is, there exists a pair of vectors $X, Y \in U(p)$ such that $\omega = X \wedge Y$. In fact, write

$$\omega = a_1 e_2 \wedge e_3 + a_2 e_1 \wedge e_3 + a_3 e_1 \wedge e_2.$$

If $a_1 = 0$, then $X = e_1, Y = a_2 e_3 + a_3 e_2$ will work. If $a_1 \neq 0$, we let $X = a_2 e_1 + a_1 e_2$ and $Y = -\frac{a_3}{a_1} e_1 + e_3$. Therefore,

$$\langle \mathcal{R}(\omega), \omega \rangle = \langle \mathcal{R}(X \wedge Y), X \wedge Y \rangle = ||X \wedge Y||^2 k(X, Y) \geq 0.$$

■

In order to show that the hypotheses of Theorem 1.1 imply nonnegative curvature operator in the case of $dim(U(p)) \geq 4$, we need first the lemma below.

Lemma 2.2 *Let $f : \bar{M} \times \mathbb{R} \to \mathbb{R}^{n+3}$ be an isometric immersion. Assume that $M \times \mathbb{R}$ has nonnegative sectional curvature and suppose that for X_1 tangent to $\mathbb{R}$ in $T_p\bar{M} \times \mathbb{R}$ we have $\alpha(X_1, X_1)(p) \neq 0$. If $dim(U(p)) \geq 4$, then exists an orthonormal frame, $\{\xi_1, \xi_2, \xi_3\}$ of $T_p^\perp M$ such that $rank(A_{\xi_1}) \leq 1$.*

Proof: Consider X_1 as described above and define $\xi_1 = \alpha(X_1, X_1)/\left\|\alpha(X_1, X_1)\right\|$. Let $\xi_2, \xi_3 \in T_p^\perp M$ such that $\{\xi_1, \xi_2, \xi_3\}$ is an orthonormal basis for $T_p^\perp M$. Consider the space

$$V = \{X \in U(p) \mid \alpha(X_1, X) = 0\}.$$

Since $dim(U(p)) \geq 4$, we have that $dim(V) \geq dim\ U(p) - 3 \geq 1$ and hence there exists a non-null vector $X \in V$. For $X \in V$, the Gauss equation implies

$$\mathcal{R}(X_1 \wedge X) = \sum_{i=1}^{3} A_{\xi_i} X_1 \wedge A_{\xi_i} X = 0,$$

where the last equality follows from the fact that X_1 is tangent to $\mathbb{R}$. Note that our choice of $\{\xi_1, \xi_2, \xi_3\}$ gives us the following:

$$\langle \alpha(X_1, X), \xi_i \rangle = \langle A_{\xi_i} X, X_1 \rangle = 0, \qquad i = 1, 2, 3.$$

$$\langle \alpha(X_1, X_1), \xi_i \rangle = \langle A_{\xi_i} X_1, X_1 \rangle = 0, \qquad i = 2, 3.$$

This implies that

$$\langle \mathcal{R}(X_1 \wedge X), X_1 \wedge Y \rangle = \langle A_{\xi_1} X_1, X_1 \rangle \langle A_{\xi_1} X, Y \rangle = 0, \qquad \forall\ Y \in T_pM$$

and, since $\langle A_{\xi_1} X_1, X_1 \rangle \neq 0$, we conclude that

$$X \in Ker(A_{\xi_1}), \qquad \forall\ X \in V.$$

We claim that $\alpha(X, X) \neq 0, \forall\ X \in V$. In fact, supposing $\alpha(X, X) = 0$, from the Gauss equation, we would have that

$$0 \leq \langle \alpha(X, X), \alpha(Y, Y) \rangle - ||\alpha(X, Y)||^2 = ||\alpha(X, Y)||^2 \leq 0,$$

hence $\alpha(X,Y)=0, \ \ \forall\, Y\in T_pM$, that is, $X\in N(p)$, contradicting the fact that $X\in U(p)$.

We fix $X_2\in V$ and since we have $\langle\alpha(X_1,X_1),\alpha(X_2,X_2)\rangle=0$, we define $\xi_2=\alpha(X_2,X_2)/||\alpha(X_2,X_2)||$. Since $X_2\in Ker(A_{\xi_1})$, with an argument similar to the one used above, that is, considering the inner product $\langle\mathcal{R}(X_1\wedge X_2),X_2\wedge Y\rangle=0,\forall\, Y\in T_pM$, we obtain

$$X_1\in Ker(A_{\xi_2}).$$

We now have the following expression

$$\mathcal{R}(X_1\wedge X)=A_{\xi_3}X_1\wedge A_{\xi_3}X=0,\qquad \forall\, X\in V,$$

which gives us that either $X_1\in Ker(A_{\xi_3})$ or $A_{\xi_3}X=\lambda_X A_{\xi_3}X_1, \ \forall\, X\in V$. The former case implies that

$$\mathcal{R}(X_1\wedge Y)=A_{\xi_1}X_1\wedge A_{\xi_1}Y=0,\qquad \forall\, Y\in T_pM,$$

thus,

$$rank(A_{\xi_1})\leq 1.$$

For the other case, we have $A_{\xi_3}X_1\neq 0$. Consider

$$V'=\{Z\in T_pM/\,\langle\alpha(X_1,Z),\xi_3\rangle=0\}.$$

Clearly, $dim(V')\geq n-1$, and $X_1\in V'$. Let $Z\in V'$ and $Y\perp A_{\xi_1}X_1$. It follows that

$$\begin{aligned}0=\langle\mathcal{R}(X_1\wedge Z),X_1\wedge Y\rangle &=\langle A_{\xi_1}X_1,X_1\rangle\langle A_{\xi_1}Z,Y\rangle-\langle A_{\xi_1}X_1,Y\rangle\langle A_{\xi_1}Z,X_1\rangle\\ &+\langle A_{\xi_3}X_1,X_1\rangle\langle A_{\xi_3}Z,Y\rangle-\langle A_{\xi_3}X_1,Y\rangle\langle A_{\xi_3}Z,X_1\rangle\\ &=\langle A_{\xi_1}X_1,X_1\rangle\langle A_{\xi_1}Z,Y\rangle,\end{aligned}$$

and hence $\langle A_{\xi_1}Z,Y\rangle=0, \ \ \forall\, Y\perp A_{\xi_1}X_1$. This implies that $A_{\xi_1}Z=\lambda A_{\xi_1}X_1$, thus, $rank(A_{\xi_1}|_{V'})=1$, that is, $rank(A_{\xi_1})\leq 2$.

Now we claim that $Im(A_{\xi_1})=Im(A_{\xi_3})$. First, let us show that $rank(A_{\xi_1})=rank(A_{\xi_3})$. In fact, if $Y\in Ker(A_{\xi_1})$, we have

$$\mathcal{R}(X_1\wedge Y)=A_{\xi_3}X_1\wedge A_{\xi_3}Y=0,$$

which gives

$$A_{\xi_3}Y=cA_{\xi_3}X_1.$$

Denoting by

$$\Delta=Ker(A_{\xi_1})+span\{X_1\},$$

we have $dimKer(A_{\xi_3}|_{\Delta})=dim\Delta-1=dimKer(A_{\xi_1})$ and hence $rank(A_{\xi_1})\geq rank(A_{\xi_3})$. Similarly, considering $Y'\in Ker(A_{\xi_3})$, we get that $rank(A_{\xi_3})\geq rank(A_{\xi_1})$, and thus

$$rank(A_{\xi_1})=rank(A_{\xi_3}).$$

Now, if $rank(A_{\xi_1})=2$, we consider W such that $\{A_{\xi_1}X_1,A_{\xi_1}W\}$ is a basis for $Im(A_{\xi_1})$. Since $\mathcal{R}(X_1\wedge W)=0$, we have

$$0\neq A_{\xi_1}X_1\wedge A_{\xi_1}W=A_{\xi_3}W\wedge A_{\xi_3}X_1.$$

Thus

$$Im(A_{\xi_1})=Im(A_{\xi_3}).$$

Denoting $\tilde{A}_{\xi_1}=A_{\xi_1}|_{Im(A_{\xi_1})}$ and $\tilde{A}_{\xi_3}=A_{\xi_3}|_{Im(A_{\xi_3})}$, it follows from the Gauss equation that $\det(\tilde{A}_{\xi_1})=-\det(\tilde{A}_{\xi_3})$. Let $\nu_p=span\{\xi_1,\xi_3\}$ and consider the determinant mapping

$$\det:\nu_p\subset T_p^{\perp}M\to\mathbb{R}$$

$$\xi \mapsto \det \tilde{A}_\xi.$$

From the continuity of the determinant mapping, we conclude that there exists a direction η_1 where $\det(\tilde{A}_{\eta_1}) = 0$. Fixing that η_1, choose η_2 orthogonal to η_1, we will have that $rank(A_{\eta_1}) \leq 1$ and $rank(A_{\eta_2}) \leq 1$, completing the proof.

■

Lemma 2.3 *Let $f : \bar{M}\times\mathbb{R} \to \mathbb{R}^{n+3}$ be an isometric immersion. Assume that $M\times\mathbb{R}$ has nonnegative sectional curvature. If $dim(U(p)) \geq 4$ then the curvature operator of $\bar{M}\times\mathbb{R}$ is nonnegative at p.*

Proof: Lemma 2.2 above implies that there exists an orthonormal frame, $\{\xi_1, \xi_2, \xi_3\}$ of $T_p^\perp M$ such that $rank(A_{\xi_1}) \leq 1$. Consider the mapping $\bar{\alpha} : T_pM \times T_pM \to W = span\{\xi_2, \xi_3\}$ given by

$$\bar{\alpha}(X, Y) = \langle A_{\xi_2}X, Y\rangle\xi_2 + \langle A_{\xi_3}X, Y\rangle\xi_3.$$

We define the associated curvature operator $R_{\bar{\alpha}} : \Lambda^2(T_pM) \times \Lambda^2(T_pM) \to \Lambda^2(T_pM)$ by

$$\langle R_{\bar{\alpha}}(X \wedge Y), Z \wedge W\rangle = \langle\bar{\alpha}(X, Z), \bar{\alpha}(Y, W)\rangle - \langle\bar{\alpha}(X, W), \bar{\alpha}(Z, Y)\rangle.$$

It is easy to see that $R_{\bar{\alpha}}(X \wedge Y, X \wedge Y) = \langle\mathcal{R}(X \wedge Y), X \wedge Y\rangle$:

$$\begin{aligned} R_{\bar{\alpha}}(X \wedge Y, X \wedge Y) &= \langle\bar{\alpha}(X, X), \bar{\alpha}(Y, Y)\rangle - ||\bar{\alpha}(X, Y)||^2 \\ &= \langle A_{\xi_2}X, X\rangle\langle A_{\xi_2}Y, Y\rangle + \langle A_{\xi_3}X, X\rangle\langle A_{\xi_3}Y, Y\rangle \\ &\quad - \langle A_{\xi_2}X, Y\rangle\langle A_{\xi_2}Y, X\rangle - \langle A_{\xi_3}X, Y\rangle\langle A_{\xi_3}Y, X\rangle \\ &= \langle A_{\xi_2}X \wedge A_{\xi_2}Y + A_{\xi_3}X \wedge A_{\xi_3}Y, X \wedge Y\rangle \\ &= \langle\mathcal{R}(X \wedge Y), X \wedge Y\rangle \end{aligned}$$

Thus,

$$\langle\bar{\alpha}(X, X), \bar{\alpha}(Y, Y)\rangle \geq ||\bar{\alpha}(X, Y)||^2 \geq 0.$$

Using arguments similar to those in [12], it can be shown that there exists an orthonormal frame $\{\zeta_1, \zeta_2\}$ of ν such that $\langle\bar{\alpha}(X, X), \zeta_i\rangle \geq 0$. Thus the forms $\bar{\alpha}_i : (X, Y) \mapsto \langle\bar{\alpha}(X, Y), \zeta_i\rangle$ are both nonnegative. This implies that the operators $\mathcal{R}_i$ given by

$$\mathcal{R}_i(X \wedge Y) = A_{\zeta_i}(X) \wedge A_{\zeta_i}(Y)$$

are both nonnegative, which in turns, implies that $\mathcal{R}$ is nonnegative, since

$$\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2.$$

■

3 Proof of the Theorems

3.1 Proof of Theorem 1.1

Consider $\bar{M} \times \mathbb{R}$ with the covering metric and the isometric immersion $\tilde{f} = f \circ P$, where P is the covering map. We claim that for $x \in \bar{M}$, there exists a point $p = (x, t) \in \bar{M} \times \mathbb{R}$ such that $\alpha(X_1, X_1)(p) \neq 0$, for $X_1 \in \mathbb{R}$ and tangent to $\bar{M} \times \mathbb{R}$. The claim follows from an argument used to prove Theorem 2.1 in [10], which we repeat here for completeness. Given x in $\bar{M}$, consider the geodesic $\gamma(t) = exp_x t\, X_1$. Suppose that for all points on the geodesic $\gamma(t)$, $\alpha(X_1, X_1) = 0$. Then $\tilde{f}$ would take a complete geodesic onto a straight line in $\mathbb{R}^{n+3}$. Therefore f would take $P(\gamma)$ on the same line. Since M contains no lines, given two points a and b on $P(\gamma)$ there exists a minimal geodesic σ connecting a to b. Hence, the isometric immersion f would take $f(\sigma)$ into a curve in $\mathbb{R}^{n+3}$ that would connect $f(a)$ to $f(b)$ with a length shorter than the straight line, which is a contradiction. Therefore, there exists $p \in \gamma$ with the property that $\alpha(X_1, X_1)(p) \neq 0$.

Now we use Lemmas 2.1, 2.2, and 2.3 to conclude that the curvature operator of $\bar{M} \times \mathbb{R}$ is nonnegative at p and hence the curvature operator of $\bar{M}$ is nonnonnegative at x. This implies that $\bar{M}$ has nonnegative curvature operator, which in turn implies that $\bar{M} \times \mathbb{R}$ and M have nonnegative curvature operator.

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3.2 Proof of Theorem 1.2

It is clear that if $n \leq 3$ that the curvature operator is nonnegative. Thus for the rest of this proof we assume that $n \geq 4$. From the compactness of M, it follows that M contains no lines. Furthermore, there exists $x_0 \in M$ such that

$$\alpha(X, X)(x_0) \neq 0, \quad \forall\, X \in T_{x_0}M.$$

From the Cheeger-Gromoll splitting theorem, we have that the universal covering $\tilde{M}$ of M is a Riemannian product $\bar{M} \times \mathbb{R}^k$, where $\bar{M}$ is compact . Since $\pi_1(M)$ is infinite we have that $k \geq 1$. Let $N = \bar{M} \times \mathbb{R}^{k-1}$. Then M is covered by $N \times \mathbb{R}$. Thus, by Theorem 1.1, M has $\mathcal{R} \geq 0$. The classification of such manifolds is known and depends on the holonomy algebra h of M. From the above, we know that h is a subset of the orthogonal algebra $o(n-k)$. The following proposition will complete the proof of Theorem 1.2:

Proposition 3.1 *If M satisfies the conditions of Theorem 1.2, then the holonomy algebra h of M is one of the following:* $h = \begin{cases} o(n-1) \\ o(n-\ell-1) + o(\ell) \\ u(2) \quad \text{and} \quad dim\ M = 5 \end{cases}$

Proof: Consider x_0 as above and let $r(x_0)$ denote the algebra generated $Im(\mathcal{R})$. Consider a point $p_0 \in P^{-1}(x_0)$, P being the covering map, $\tilde{M}$ with the covering metric, and similarly define the algebra $r(p_0)$. We then have that $\alpha(X, X)(p_0) \neq 0$, for all $X \in T_{p_0}\tilde{M}$. It is well known that $r(p_0) = r(x_0) \subset h$. We will then study the possible cases for $r(p_0)$. Let $\{\xi_1, \xi_2, \xi_3\}$ be the orthonormal frame given by Lemma 2.2. Here we have two cases to consider:
Case 1: There exists a vector $X_1 \in Ker(A_{\xi_2}) \cap Ker(A_{\xi_3})$.
Let $V = Im(A_{\xi_2})$, $W = Im(A_{\xi_3})$, $U = V + W$. Then $r(p_0) \subset o(U)$. Since $X_1 \notin U$, we have that

$$dim(U) \leq n-1.$$

Suppose $dim(U) \leq n-2$. Then there exists $X \in T_{p_0}M$ linearly independent with X_1 such that $X \in Ker(A_{\xi_2}) \cap Ker(A_{\xi_3})$. Since $rank(A_{\xi_1}) \leq 1$, we have that $A_{\xi_1}(X) = cA_{\xi_1}(X_1)$. Thus,

$$\bar{X} = X - cX_1 \in Ker(A_{\xi_1}) \cap Ker(A_{\xi_2}) \cap Ker(A_{\xi_3}) \Rightarrow \alpha(\bar{X}, \bar{X}) = 0,$$

which contradicts the fact that $\alpha(X, X)(p_0) \neq 0$. Therefore,

$$dim(U) = n-1.$$

Moreover, $Im(\mathcal{R}) = Im(R_{\bar{\alpha}})$, where $R_{\bar{\alpha}}$ is as in the proof of Lemma 2.2. Therefore we can use the results of Bishop in [3], since they depend only on the fact that U is the sum of the images of only two Weingarten operators.

It follows that $r(p_0)$ and hence the holonomy algebra of $\tilde{M}$ and of M is one of the following:

$$h = \begin{cases} o(U) = o(n-1) \\ o(V) + o(W) = o(n-\ell-1) + o(\ell) \\ u(2), \text{ if dim(U)=4.} \end{cases}$$

The classification of compact manifolds with nonnegative curvature operator shows that in the first case, M is covered by $S^{n-1} \times \mathbb{R}$. In the second case, if $\ell = 1$, then M is covered by $S^{n-2} \times \mathbb{R}^2$. If $\ell > 2$, then M is covered by $S^{n-\ell-1} \times S^\ell \times \mathbb{R}$. The last case implies that M is covered by $\mathbb{C}P^2 \times \mathbb{R}$.

Case 2: We consider $U = Im(A_{\xi_2})$, $rank(A_{\xi_1}) \leq 1$ and $rank(A_{\xi_3}) \leq 1$ Again we have that $dim(U) \leq n-1$. We claim now that $dim(U) \geq n-2$. In fact, if $dimKer(A_{\xi_2}) \geq 3$, the proof of Lemma 2.2 shows that there exist two orthonormal vectors in $span\{\xi_1, \xi_3\}$ whose corresponding Weingarten operators have a common kernel of dimension at least $n-2$. Therefore we would find again a vector

$$X \in Ker(A_{\xi_1}) \cap Ker(A_{\xi_2}) \cap Ker(A_{\xi_3}),$$

which implies that $\alpha(X, X) = 0$, contradicting the choice of p_0.

This case implies that $r(p_0)$ and hence the holonomy algebra of $\tilde{M}$ is $o(n-k)$, $0 < k \leq 2$ and M is covered by $S^{n-k} \times \mathbb{R}^k$, $0 < k \leq 2$. ∎

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