Calabi Conjecture for minimal hypersurfaces with Ricci curvature with strong quadratic decay

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Abstract

We show that Calabi Conjecture on minimal hypersurfaces of $\mathbb{R}^n$, $n \geq 4$ holds in the class of minimal hypersurfaces with Ricci curvature with strong quadratic decay.

1 Introduction

Eugenio Calabi [2], in mid 1960s made two conjectures about complete minimal hypersurfaces in $\mathbb{R}^n$. He conjectured that *the complete minimal hypersurfaces of* $\mathbb{R}^n$ *are unbounded* and in a second and stronger statement he conjectured that *the complete minimal non-flat hypersurfaces of* $\mathbb{R}^n$ *have unbounded projection on every* $(n-2)$*-dimensional subspace.* See also [**?**], [3]. It is well known that these two conjectures are false, after the examples of Nadirashvili [8] and Jorge-Xavier [6]. The purpose of this paper is to show that the Calabi conjectures hold in the class of complete minimal hypersurfaces with Ricci curvature with strong quadratic decay in $\mathbb{R}^n$, $n \geq 4$. Recall that a complete Riemannian manifold M is said to have *Ricci curvature with strong quadratic decay* if

$$Ric_M(x) \geq -c^2[1 + \rho_M^2(x) log^2(\rho_M(x) + 2)]$$

where ρ_M is the distance function on M to a fixed point x_0 and $c = c(x_0) > 0$ is a constant depending on x_0. We prove the following theorem.

Theorem 1. *Let* $\varphi : M \hookrightarrow \mathbb{R}^n$ *be a complete hypersurface with Ricci curvature with strong quadratic decay and mean curvature* H*. If* $\varphi(M) \subset B_{\mathbb{R}^2}(R) \times \mathbb{R}^{n-2}$ *then*

$$\sup_M |H| \geq \frac{1}{(n-1)R} > 0. \tag{1}$$

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Remark 1.1. *Theorem (1) does not imply Calabi's second conjecture in the class of complete minimal hypersurfaces with Ricci curvature with strong quadratic decay for* $n = 3$ *although implies Calabi first conjecture, (that was proved by Chen-Xin in [7]). However, in dimension* $n \geq 5$*, it proves a stronger statement than Calabi's second conjecture.*

Notice that Theorem 1 is a consequence of the following more general theorem which generalizes the main theorem proved in [1].

Theorem 2. *Let* N *be a complete* n*-dimensional Riemannian manifold with radial sectional curvature bounded* $K_N \leq k$ *above along the geodesics issuing from a point* x_0*. Let* $\varphi : M \hookrightarrow N \times \mathbb{R}^l$ *be a complete* m*-dimensional immersed submanifold such that* $\varphi(M) \subset B_N(R) \times \mathbb{R}^l$, $l \leq m-1$*, where* $B_N(R)$ *is the geodesic ball in* N *centered at* x_0 *and radius* R, $R < \min\{\mathrm{inj}_N(x_0), \pi/2\sqrt{k}\}$*. Assume that* $\pi/2\sqrt{k} = \infty$ *if* $k \leq 0$*. Then*

i. If M *has Ricci curvature with strong quadratic decay then*

$$\sup_M |H| \geq \frac{(m-l)}{m}\mu(R); \tag{2}$$

ii. If $|H| \leq \dfrac{(m-l-1)}{m}\mu(R)$ *then* M *is non-parabolic;*

where μ *is given in (8).*

The following result is a corollary of the proof of Theorem (2).

Theorem 3. *Let* N *be a complete* n*-dimensional Riemannian manifold with sectional curvature bounded* $K_N \leq k$*. Let* $\varphi : M \hookrightarrow N \times \mathbb{R}^l$ *be a complete* m*-dimensional immersed submanifold with bounded mean curvature* $|H| \leq (m-l-1)$*. Let* K *be a compact subset of* $\varphi(M)$ *with piecewise smooth boundary (possibly empty) such that* $R_K < \min\{\mathrm{inj}_N(x_K), \pi/2\sqrt{k}\}$*. Assume that* $\pi/2\sqrt{k} = \infty$ *if* $k \leq 0$ *then we have that*

$$\frac{vol_{m-1}(\partial K)}{vol_m(K)} \geq (m-l-1)\mu(R_K) - m\sup_M |H| \tag{3}$$

Where R_K *is the radius of the compact set* $p_1(K)$*,* x_K *is its barycenter and* $p_1 : N \times \mathbb{R}^l \to N$ *is the projection on the first factor.*

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2 Preliminaries

Let $\varphi : M \hookrightarrow N$ be an isometric immersion of a m-dimensional Riemannian manifold M into a n-dimensional complete Riemannian manifold N. Consider a smooth function $g : N \to \mathbb{R}$ and $f = g \circ \varphi : M \to \mathbb{R}$. Let $q \in M$ and X,$Y \in T_qM$. Identifying X with $d\varphi(X)$, we have

$$\langle \text{grad}\, f, X\rangle = df(X) = dg(X) = \langle \text{grad}\, g, X\rangle. \tag{4}$$

Therefore,

$$\text{grad}\, g = \text{grad}\, f + (\text{grad}\, g)^{\perp} \tag{5}$$

where $(\text{grad}\, g)^{\perp}$ is perpendicular to T_qM.

Let ∇ and $\overline{\nabla}$ be the Riemannian connection on M and N respectively, and let α be the second fundamental form. Using the equations (4) and (5) we have:

$$\text{Hess}\, f(q)(X,Y) = \text{Hess}\, g(\varphi(q))(X,Y) + \langle \text{grad}\, g, \alpha(X,Y)\rangle_{\varphi(q)}. \tag{6}$$

Taking the trace in (6) with respect to an orthonormal basis $\{e_i\}_{i=1,\dots m}$ for T_qM, we have that

$$\triangle f(q) = \sum_{i=1}^{m} \text{Hess}\, g(\varphi(q))(e_i, e_i) + \langle \text{grad}\, g, \vec{H}\rangle, \tag{7}$$

where $\vec{H}$ is the mean curvature vector, with $\|\vec{H}\| = m|H|$. Both of the above two formulas are proved in [5].

Now we are going to remind two well known results. The first one is the Hessian Comparison Theorem.

Theorem 4. *Let M be a complete Riemannian manifold and $\rho(x) = dist_M(x, x_0)$ be the distance function to $x_0 \in M$. Let γ be a minimizing geodesic joining x_0 and x. Let K_γ be the radial sectional curvatures of M along γ, let $k = \sup K_\gamma$ and let $\mu(\rho)$ be the function defined below.*

$$\mu(\rho) = \begin{cases} \sqrt{-k}\cdot \coth(\sqrt{-k}\cdot \rho(x)), & if \quad k < 0 \\ 1/\rho(x), & if \quad k = 0 \\ \sqrt{k}\cdot \cot(\sqrt{k}\cdot \rho(x)), & if \quad k > 0 \ and\ \rho < \pi/2\sqrt{k}. \end{cases} \tag{8}$$

Then the Hessian of ρ satisfies

$$Hess\, \rho(x)(X,X) \geq \mu(\rho(x)).\|X\|^2, Hess\, \rho(x)(\gamma', \gamma') = 0,$$

where X is any vector in T_xM perpendicular to $\gamma'(\rho(x))$.

Last, the proof of our results uses the Omori-Yau Maximum Principle, proved in [9], [10] generalized in [4] and [?].

Theorem 5. *(Omori-Yau Maximum Principle) Let M be a complete Riemannian manifold with Ricci Curvature with strong quadratic decay. Let f be a C^2 function bounded above on M. Then for any sequence $\varepsilon_k \to 0$ of positive numbers there exists a sequence of points $x_k \in M$ such that:*

i. $\lim_{k\to\infty} f(x_k) = \sup_M f$

ii. $|grad\, f(x_k)| < \varepsilon_k$

iii. $\triangle f(x_k) < \varepsilon_k$

Remark 2.1. *It is clear from the proof of the Omori-Yau Maximum Principle that f is allowed to be C^0 in a set of measure zero. For instance, f can be the distance function to a point $x_0 \in M$, which is C^∞ in $M \setminus (cut\, locus(x_0) \cup \{x_0\})$ and C^0 in $cut\, locus(x_0) \cup \{x_0\}$.*

3 Proof of the Results

Let us begin with M an immersion as in the Theorem (2) and let $g : N \times \mathbb{R}^l \to \mathbb{R}$ be defined by $g(x, y) = \rho_N(x)$, where $\rho_N : N \to \mathbb{R}$ is given by $\rho_N(x) = \mathrm{dist}_N(x_0, x)$, let $f = g \circ \varphi : M \to \mathbb{R}$, q a point of M and $p = \pi_1(\varphi(q)) \in N$, where $p_1 : N \times \mathbb{R}^l \to N$ is the projection on the first factor. Likewise, $p_2 : N \times \mathbb{R}^l \to \mathbb{R}^l$ is the projection on the second factor.

Consider an orthonormal basis $\{\mathrm{grad}\rho_N, \frac{\partial}{\partial\theta_2}, ..., \frac{\partial}{\partial\theta_n}\}$ for T_pN and an orthonormal basis $\{\frac{\partial}{\partial t_1}, ..., \frac{\partial}{\partial t_l}\}$ for $T_{p_2(\varphi(q))}\mathbb{R}^l \simeq \mathbb{R}^l$. Then we have $\{\mathrm{grad}\rho_N, \frac{\partial}{\partial\theta_2}, ..., \frac{\partial}{\partial\theta_n}, \frac{\partial}{\partial t_1}, ..., \frac{\partial}{\partial t_l}\}$ an orthonormal basis for $T_{\varphi(q)}(N \times \mathbb{R}^l)$. Now taking $\{e_1, e_2, ..., e_m\}$ an orthonormal basis for T_qM we can write, for each $i \in \{1, ..., m\}$,

$$e_i = \alpha_i \text{ grad } \rho_N + \sum_{j=2}^{n} a_{ij} \frac{\partial}{\partial\theta_j} + \sum_{j=1}^{l} b_{ij} \frac{\partial}{\partial t_j}.$$

The Levi-Civita connection $\overline{\nabla}$ of $N \times \mathbb{R}^l$ decomposes as $\overline{\nabla} = \nabla^N + \nabla^{\mathbb{R}^l}$. We have that $\mathrm{Hess}_{N\times\mathbb{R}^l}\, \rho_N(e_i, e_i) = \langle \nabla^N_{e_i} \mathrm{grad}\rho_N,\, e_i\rangle + \langle \nabla^{\mathbb{R}^l}_{e_i} \mathrm{grad}\rho_N,\, e_i\rangle = \mathrm{Hess}_N\, \rho_N(e_i, e_i)$. To simplify the notation let us call $\nabla^N = \nabla$.

$$\begin{aligned}\mathrm{Hess}_{N\times\mathbb{R}^l}\, \rho_N(e_i, e_i) \ &= \alpha_i \langle \nabla_{\mathrm{grad}\ \rho_N} \mathrm{grad}\ \rho_N, e_i\rangle + \sum_{j=2}^{n} a_{ij} \langle \nabla_{\frac{\partial}{\partial\theta_j}} \mathrm{grad}\ \rho_N, e_i\rangle \\ &\quad + \sum_{j=1}^{l} b_{ij} \langle \nabla_{\frac{\partial}{\partial t_j}} \mathrm{grad}\ \rho_N, e_i\rangle,\end{aligned} \tag{9}$$

Notice that

$$\nabla_{\mathrm{grad}\ \rho_N} \mathrm{grad}\ \rho_N = 0 \text{ and } \langle \nabla_{\frac{\partial}{\partial t_j}} \mathrm{grad}\ \rho_N, e_i\rangle = 0.$$

So the equality in (9) becomes

$$\begin{aligned}\operatorname{Hess}_{N\times\mathbb{R}^l}\rho_N(e_i,e_i) &= \sum_{j=2}^{n} a_{ij}\Big[\alpha_i\langle\nabla_{\frac{\partial}{\partial\theta_j}}\operatorname{grad}\rho_N,\operatorname{grad}\rho_N\rangle\\ &+ \sum_{k=2}^{n} a_{ik}\langle\nabla_{\frac{\partial}{\partial\theta_j}}\operatorname{grad}\rho_N,\frac{\partial}{\partial\theta_k}\rangle\\ &+ \sum_{k=1}^{l} b_{ik}\langle\nabla_{\frac{\partial}{\partial\theta_j}}\operatorname{grad}\rho_N,\frac{\partial}{\partial t_k}\rangle\Big].\end{aligned}$$

However have that $\langle\nabla_{\frac{\partial}{\partial\theta_j}}\operatorname{grad}\rho_N,\operatorname{grad}\rho_N\rangle = 0 = \langle\nabla_{\frac{\partial}{\partial\theta_j}}\operatorname{grad}\rho_N,\frac{\partial}{\partial t_k}\rangle$ therefore,

$$\operatorname{Hess}_{N\times\mathbb{R}^l}\rho_N(e_i,e_i) = \sum_{j=2}^{n} a_{ij}\sum_{k=2}^{n} a_{ik}\langle\nabla_{\frac{\partial}{\partial\theta_j}}\operatorname{grad}\rho_N,\frac{\partial}{\partial\theta_k}\rangle.$$

But $\langle\nabla_{\frac{\partial}{\partial\theta_j}}\operatorname{grad}\rho_N,\frac{\partial}{\partial\theta_k}\rangle$ is zero if $j\neq k$ and it is equal to $\operatorname{Hess}_N\rho_N(\frac{\partial}{\partial\theta_j},\frac{\partial}{\partial\theta_j})$ if $j=k$. Thus

$$\operatorname{Hess}_{N\times\mathbb{R}^l}\rho_N(e_i,e_i) = \sum_{j=2}^{n}(a_{ij})^2\operatorname{Hess}_N\rho_N(\frac{\partial}{\partial\theta_j},\frac{\partial}{\partial\theta_j}). \tag{10}$$

From the equations (7), (10) we can compute the Laplacian of f as follows

$$\triangle f(q) = \sum_{i=1}^{m}\sum_{j=2}^{n}(a_{ij})^2\operatorname{Hess}_N\rho_N(p)(\frac{\partial}{\partial\theta_j},\frac{\partial}{\partial\theta_j}) + \langle\operatorname{grad} g,\vec{H}\rangle.$$

For the Theorem (4) we have

$$\sum_{i=1}^{m}\sum_{j=2}^{n}(a_{ij})^2\operatorname{Hess}_{N\times\mathbb{R}^l}\rho_N(p)(\frac{\partial}{\partial\theta_j},\frac{\partial}{\partial\theta_j}) \geq \sum_{i=1}^{m}\sum_{j=2}^{n}(a_{ij})^2\mu(\rho_N(p)).$$

Therefore, using that $\|e_i\|^2 = 1$ and $\langle\operatorname{grad} g,\vec{H}\rangle \geq -m|H|$, we find the following inequality

$$\triangle f(q) \geq \mu(\rho_N(p))\left(m - \sum_{i=1}^{m}(\alpha_i)^2 - \sum_{i=1}^{m}\sum_{j=1}^{l}(b_{ij})^2\right) - m|H|. \tag{11}$$

Since that M has Ricci curvature with strong quadratic decay and that $\varphi(M)$ is cylindrically bounded, we can apply the Omori-Yau Maximum Principle to f. Hence, given $\{\varepsilon_k\}_{k\in\mathbb{N}}$ a sequence of positive numbers with $\varepsilon_k\to 0$ there exists $\{x_k\}_{k\in\mathbb{N}}\subset M$ such that

$$\lim_{k\to\infty} f(x_k) = \sup_M f, \quad \triangle f(x_k) < \varepsilon_k \quad \text{and} \quad |\operatorname{grad} f(x_k)| < \varepsilon_k. \tag{12}$$

Notice that

$$\alpha_i = \langle e_i, \text{grad } \rho_N \rangle = \langle e_i, \text{grad } f \rangle + \langle e_i, (\text{grad } g)^{\perp} \rangle < \varepsilon_k,$$

so

$$\sum_{i=1}^{m} (\alpha_i)^2 < m(\varepsilon_k)^2. \tag{13}$$

On the other hand,

$$\sum_{i=1}^{m} \sum_{j=1}^{l} (b_{ij})^2 = \sum_{j=1}^{l} \sum_{i=1}^{m} (b_{ij})^2 \leq l, \tag{14}$$

since $\sum_{i=1}^{m} (b_{ij})^2 \leq |\frac{\partial}{\partial t_j}|$.

From (11), (12), (13) and (14), we have

$$\varepsilon_k > (m - m(\varepsilon_k)^2 - l)\mu(\rho_N(y_k)) - m|H|,$$

where $y_k = p_1(\varphi(x_k)) \in N$. Making $k \to \infty$ and using (12), the previous equation becomes

$$sup_M |H| \geq \frac{(m-l)}{m} \mu(R).$$

What proves item i. of Theorem (2).

Using the equations (11) and (14), in addiction to $\sum_{i=1}^{m} (\alpha_i)^2 < 1$ and the fact that μ is decreasing, we have the following inequality:

$$\triangle f(q) \geq \mu(R)(m - l - 1) - \|\vec{H}\|.$$

If M has bounded mean curvature $|H| \leq (m - l - 1)\, \mu(R)/m$ then f is a bounded non-constant subharmonic function. And this says that M is non-parabolic. This proves ii. of the Theorem (2).

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