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OVER-DETERMINED ELLIPTIC BOUNDARY-VALUED PROBLEMS AND HOMOGENIZATION ISSUES

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Abstract. In these paper we derive several simple conditions for the existence of a limit of solutions to a regularized boundary-value problem, in particular if the equation is of the form $F(D^2u) = 0$ where F is convex. In addition, we analyze the behavior of the free-boundary problem $F(D^2u) = 0$ with prescribed boundary values in a region. We determine that if the equation has a special divergence structure and satisfies a multiplier condition for the inner-product of $\partial_k \nabla u$ with the vector potential, then u satisfies a special free-boundary condition.

1. Introduction

An important class of free-boundary problems are the following over-determined elliptic problems. Given $\Omega \subseteq \mathbb{R}^n$, $\partial\Omega$ a smooth compact hypersurface in $\mathbb{R}^n$, a nonnegative function $f : \mathbb{R}^n \to \mathbb{R}$ and a postive function $g : \bar{\Omega} \to \mathbb{R}$, an important question is whether one can find another compact hypersurfce $\Gamma = \partial\Omega' \subseteq \Omega$, such that it is possible to solve the overdetermied elliptic boundary value problem

$$\begin{cases} F(D^2u) = 0 & \text{in } \Omega\backslash\Omega' \\ u = f & \text{on } \partial\Omega \\ u = 0 & \text{and } u_\nu = g \text{ on } \Gamma \end{cases}$$

where u_ν is the normal derivative of u related to Γ and F is an elliptic operator, either in divergence form, or not. For example, systems of the above form appear in flame propagation problems.

In this paper we consider problems relating to a perturbation technique to construct possible solutions of such a problem. A possible solution of it will be obtained as the limit of the approximating regularized problem

$$\begin{cases} F(D^2u_\epsilon) = \beta_\epsilon(u_\epsilon) \text{ in } \Omega \\ u_\epsilon = f \text{ on } \partial\Omega, \end{cases}$$

where $\beta_\epsilon(x) = \epsilon\beta(x/\epsilon)$ for some non-negative $\beta \in C_c^\infty([0,1])$ extended to be zero outside of $[0,1]$. Sequences of approximating solutions as $\epsilon \to 0^+$ to the above

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system have been considered previously in the work of Professors Teixeira and Moreira at Universidade Federal do Ceara. The connection of the first section to this problem is highly technical and hence will be omitted, but the relation of the latter material is much more direct.

2. Homogenization Issues

We begin by letting $F : \mathrm{Sym}(n) \to \mathbb{R}$ be some operator, where $\mathrm{Sym}(n)$ denotes the $n \times n$ real symmetric matrices. Additionally, throughout this paper we assume $F(0) = 0$. Let P be the diagonalized form of any symmetric matrix P', and let $P = P^+ + P^-$, where the only nonzero entries of P^+ and P^- are the positive and negative eigenvalues of P, respectively. We say that F is elliptic if there exist $\Lambda \geqslant \lambda > 0$ such that

$$F(N+P) \leqslant F(N) + \Lambda \left\|P^+\right\| - \lambda \left\|P^-\right\|,$$

for all $N, P \in \mathrm{Sym}(n)$. Note the norms can be any fixed norm on $\mathrm{Sym}(n)$ (e.g. the Frobenius norm or the Euclidian operator norm). We call Λ, λ the ellipticity constants. It is an immediate consequence of the definition that $F(N+P) > F(N)$ if P is positive.

We now prove several elementary properties of elliptic operators.

Proposition 2.1. *If $F \in \mathrm{Sym}(n)$ is linear and positive for every non-negative matrix, then F is elliptic.*

Proof. Since F is linear, we have $F(N+P) = F(N) + F(P^+) + F(P^-)$. Thus, proving the ellipticity of F reduces to proving $F(P^+) + F(P^-) \leqslant \Lambda \|P^+\| - \lambda \|P^-\|$. So it suffices to prove there exist constants $\Lambda \geqslant \lambda > 0$ such that $F(P^+) \leqslant \Lambda \|P^+\|$ and $F(-P^-) \geqslant \lambda \|P^-\|$. Since P^+ and $-P^-$ are both non-negative and F is linear, proving these two inequalities is equivalent to showing that there exist constants $\Lambda \geqslant \lambda > 0$ such that $0 < \lambda \leqslant F(P) \leqslant \Lambda$ on the set $A = \{P : P \in \mathrm{Sym}(n), P$ is non-negative, and $\|P\| = 1\}$.

The existence of the desired constants now follows because the set A is compact. This is seen most easily using the Frobenius norm on $\mathrm{Sym}(n)$ (i.e. thinking of $\mathrm{Sym}(n)$ as a subset of $\mathbb{R}^{n^2}$ and giving it the induced metric). Then A is bounded by definition and closed because it is the intersection of the two closed sets of the unit sphere in $\mathbb{R}^{n^2}$ and the non-negative matrices. Now take λ and Λ to be the infimum and supremum, respectively, of F on A. It follows immediately from their definition that $\Lambda \geqslant \lambda$. Furthermore, $\lambda > 0$ because $F > 0$ on A by assumption. In greater detail, $\lambda \geqslant 0$ because $F > 0$ on A, and if $\lambda = 0$ then by compactness there would exist $P \in A$ such that $F(P) = 0$, a contradiction. This completes the proof that F is elliptic.

□

We will be interested in taking the limit of elliptic operators in a manner similar to approximations to a delta function. To do this, we define the following notation.

Definition 2.2. : Let $F : \mathrm{Sym}(n) \to \mathbb{R}$ be elliptic, $A \in \mathrm{Sym}(n)$, and define for each $\epsilon > 0$,

$$F_\epsilon(A) := \epsilon F\left(\frac{1}{\epsilon} A\right).$$

We further define $F^*(A) = \lim_{\epsilon \to 0^+} F_\epsilon(A)$ if the limit exists.

The next several propositions prove some elementary properties of $\{F_\epsilon\}_{\epsilon>0}$ and F^*. In particular, we give proofs (which are all simple computations) that $\{F_\epsilon\}_{\epsilon>0}$ shares the same ellipticity constants as F, that if F is homogeneous that $F^* = F$, and the partial converse that if F^* exists then it is homogenous of degree 1 for positive multipliers.

Proposition 2.3. *F_ϵ is elliptic with the same ellipticity constants as F, namely Λ, λ.*

Proof. For two matrices N, P as defined above, we have:

$$\begin{aligned} F_\epsilon(N+P) - F_\epsilon(N) &= \epsilon F\left(\frac{1}{\epsilon}(N+P)\right) - \epsilon F\left(\frac{1}{\epsilon}N\right) \\ &= \epsilon\left[F\left(\frac{1}{\epsilon}(N+P)\right) - F\left(\frac{1}{\epsilon}N\right)\right] \\ &\leqslant \epsilon[\Lambda\left\|\frac{1}{\epsilon}P^+\right\| - \lambda\left\|\frac{1}{\epsilon}P^-\right\|] \\ &= \Lambda\left\|P^+\right\| - \lambda\left\|P^-\right\|. \end{aligned}$$

□

Proposition 2.4. *Assume F is homogeneous. Then F^* exists, and in fact, $F^* = F$.*

Proof. Consider F_ϵ for any $\epsilon > 0$:

$$F_\epsilon = \epsilon F\left(\frac{A}{\epsilon}\right) = F(A),$$

where the final equality follows since F is homogeneous. Thus, since this is true for all ϵ, then

$$\lim_{\epsilon\to 0} F_\epsilon(A) = F(A) = F^*(A).$$

□

Theorem 2.5. *If F^* exists, then F^* is homogeneous of degree one for positive multipliers, i.e. $F^*(cA) = cF^*(A)$ for $c > 0$.*

Proof. First, let $A \in \mathrm{Sym}(n)$, and fix $\alpha \geqslant 0$. Since $F_\epsilon(A) \to F^*(A)$ as $\epsilon \to 0$, then it follows that

$$F_{\alpha\epsilon}(A) \to F^*(A) \text{ as } \alpha\epsilon \to 0. \tag{2.1}$$

Next, applying F_ϵ to $\frac{A}{\alpha}$, we have

$$F_\epsilon\left(\frac{A}{\alpha}\right) \to F^*\left(\frac{A}{\alpha}\right) \text{ as } \epsilon \to 0.$$

But this remains true if we multiply by α:

$$\alpha F_\epsilon\left(\frac{A}{\alpha}\right) \to \alpha F^*\left(\frac{A}{\alpha}\right) \text{ as } \epsilon \to 0.$$

At the same time

$$\begin{aligned} \alpha F_\epsilon\left(\frac{A}{\alpha}\right) &= \alpha\epsilon F\left(\frac{A}{\alpha\epsilon}\right) \\ &= F_{\alpha\epsilon}(A) \to \alpha F^*\left(\frac{A}{\alpha}\right) \text{ as } \alpha\epsilon \to 0. \end{aligned} \tag{2.2}$$

Comparing (2.1) and (2.2), we have, by uniqueness of the limit, that $F^*(A) = \alpha F^*(\frac{A}{\alpha})$. But the proof would not change for any real number in place of α, and therefore F^* is homogeneous. □

Proposition 2.6. *F and F_ϵ are Lipschitz continuous with Lipschitz constant Λ.*

Proof. By Proposition 2.3, we know F_ϵ has the same ellipticity constants as F. This implies, for any two symmetric matrices A, B, $||F_\epsilon(A) - F_\epsilon(B)|| \leqslant \Lambda||A - B||$. To see this, let $A, B, C \in \mathrm{Sym}(n)$ such that $A = B + C$, so $||A - B|| = ||C||$. Then,

$$\begin{aligned} F(B+C) - F(B) &\leqslant \Lambda||C^+|| - \lambda||C^-|| \\ &\leqslant \Lambda||C^+|| \\ &\leqslant \Lambda||C||. \end{aligned} \tag{2.3}$$

Similarly,

$$\begin{aligned} F(B) - F(B+C) &= F((B+C) - C) - F(B+C) \\ &\leqslant \Lambda||C^-|| - \lambda||C^+|| \\ &\leqslant \Lambda||C^-|| \\ &\leqslant \Lambda||C||. \end{aligned} \tag{2.4}$$

By (2.3) and (2.4), we have $||F(A) - F(B)|| \leqslant \Lambda||A - B||$. □

Proposition 2.7. *If F is convex, then $F_\epsilon \to F$ as $\epsilon \to 0$.*

Proof. Fix $A \in \mathrm{Sym}(n)$. Since F is convex, $F(\lambda A) \leqslant \lambda F(A)$ for all $\lambda \in [0, 1]$. If $\epsilon_1 \leqslant \epsilon_2$, then

$$F_{\epsilon_2}(A) = \epsilon_2 F\left(\frac{1}{\epsilon_2}(A)\right) = \epsilon_2 F\left(\frac{\epsilon_1}{\epsilon_2} \cdot \frac{1}{\epsilon_1} A\right) \leqslant \epsilon_1 F\left(\frac{1}{\epsilon_1}(A)\right) = F_{\epsilon_1}(A).$$

Thus, the sequence $F_\epsilon(A)\}$ is nondecreasing as $\epsilon \to 0$. Also, the set $\{F_\epsilon(A)\}$ is bounded, because F is Lipschitz continuous:

$$||F_\epsilon(A)|| = ||F_\epsilon(A) - F_\epsilon(0)|| \leqslant K||A||.$$

We conclude that $F_\epsilon(A)$ is a bounded monotonous sequence, and so it converges. □

Theorem 2.8. *If $F_\epsilon(A) \to F^*(A)$ pointwise as $\epsilon \to 0$, then $F_\epsilon \to F^*$ uniformly on compact subsets.*

Proof. Let $K \subseteq \mathrm{Sym}(n)$ be compact and $S = \{F_{\epsilon_n}\}$. Then Proposition 2.3 implies S is equicontinuous and pointwise bounded in K. To see this, for any $A \in \mathrm{Sym}(n)$ and a given $\eta > 0$, consider the ball $B(A, \eta)$. Then for every $B \in B(A, \eta)$,

$$||F_\epsilon(A) - F_\epsilon(B)|| \leqslant \Lambda||A - B||$$

for every $\epsilon > 0$. This proves equicontinuity, and the same inequality with $B = 0$ shows that $\{F_\epsilon\}$ is pointwise bounded. Next, by the Ascoli-Arzelá Theorem, every sequence $\{F_n\} \subseteq S$ has a subsequence that converges uniformly in K. But uniform convergence guarantees pointwise convergence, and since $F_n \to F^*$, the limit of the uniformily convergent subsequence must be F^*.

To conclude the proof we must now show that every sequence $\{F_n\} \subseteq S$ converges uniformly to F^* in K. Let F_n be a sequence and suppose, for the sake of contradiction, that it does not converge unformly to F^* in K. Then there exists $\eta > 0$, such that

$$||F_n - F^*|| > \eta$$

for all n, where the above norm is $||F|| := \sup\{||F(A)|| \,|\, A \in K\}$. But we know that $\{F_n\}$ has a subsequence that converges to F^*, which implies that there exists N such that for all $n \geqslant N$

$$||F_n - F^*|| < \eta.$$

But this is a contradiction, so we have uniform convergence. $\square$

3. A Generalization of the Free Boundary Condition

We now compute a fairly general free boundary condition for the free boundary problem stated in the introduction. We recall that free boundary problem was that we were given $F : \mathrm{Sym}(n) \to \mathbb{R}$ satisfying $F(0) = 0$, $\beta \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\beta \subset [0,1]$, β non-negative, and $\beta_\epsilon(x) = \epsilon\beta(x/\epsilon)$. Then the Free-boundary problems we investigate are solutions u_ϵ to

$$F(D^2u_\epsilon) = \beta_\epsilon(u_\epsilon).$$

In this particular case we assume that F has a hidden divergence structure in the sense that there exists some function E of x, u, and ∇u such that $\operatorname{div} E = F(D^2u) + G(\nabla u, x)$, where the $G(\nabla u, x)$ encapsulates lower order terms.

To be specific, let $E(x, u, \nabla u)$ be a vector valued function of x, $u : \mathbb{R}^n \to \mathbb{R}$, and ∇u. We further assume that E is differentiable, say C^1, and that it is uniformly Lipshitz continuous in ∇u. We further suppose that for solutions u_ϵ of the singularly perturbed equation $F(D^2u_\epsilon) = \beta_\epsilon(u_\epsilon)$ that the following condition is satisfied:

$$\langle \nabla \partial_k u_\epsilon, E(x, u, \nabla u)\rangle = \gamma(x, u)\partial_k \langle \nabla u_\epsilon, E(x, u, \nabla u)\rangle$$

for some $\gamma(x, u) > 0$ which is C^1. This result is more general than is usually required. We give an example of a class of functions it applies to after the proof.

Theorem 3.1. *If u_0 is the limit of solutions u_ϵ to the Free Boundary problems $F(D^2u) = \beta_\epsilon(u)$, then*

$$\langle \nabla u_0, E(x, u_0, \nabla u_0)\rangle = \frac{1}{1 - \gamma(x, 0)}$$

along the free boundary $\partial\{u_0 = 0\}$.

Proof. The computation consists of using the divergence theorem to obtain two identities governing the behavior of u_0 along $\partial\{u_0 = 0\}$ and then substituting one equation into the other. The first equation is obtained by applying the divergence theorem to u_ϵ and taking the limit as $\epsilon \to 0^+$ and the second from applying the divergence theorem to u_0 directly.

For the remained of the proof, let B be a ball contained in the domain and centered at a point of $\{u_0 = 0\}$. Further, let $\varphi \in C_c^\infty(B)$ and ν denote the outward normal vector to the boundary of any region of integration. (They are all sufficiently regular for ν to be well-defined.) Finally, we define $C = B \cap \{u_0 > 0\}$.

By the divergence theorem, since $\varphi|_{\partial B} \equiv 0$ we have

$$\begin{aligned} 0 &= \int_{\partial B} \varphi \partial_k u_\epsilon \langle E, \nu \rangle \\ &= \int_B \operatorname{div}(\varphi \partial_k u_\epsilon E) \\ &= \int_B (\partial_k u_\epsilon \langle \nabla\varphi, E\rangle + \varphi \langle \partial_k \nabla u_\epsilon, E\rangle) + \varphi \partial_k u_\epsilon \operatorname{div}(E). \end{aligned}$$

By the computation above, we have $\operatorname{div} E = F + G = F(D^2 u, \nabla u, x) + G(\nabla u, x)$. Additionally, we recall that we have also assumed

$$\langle \nabla \partial_k u_\epsilon, E(x, u, \nabla u)\rangle = \gamma(x, u)\partial_k \langle \nabla u_\epsilon, E(x, u, \nabla u)\rangle .$$

Thus the final integral above equals

$$\begin{aligned} &\int_B \partial_k u_\epsilon \langle \nabla\varphi, E\rangle + \varphi\gamma(x, u_\epsilon)\partial_k \langle \nabla u_\epsilon, E\rangle + \varphi\partial_k u_\epsilon(F + G) \\ &= \int_B \partial_k u_\epsilon \langle \nabla\varphi, E\rangle - \partial_k(\varphi\gamma(x, u_\epsilon)) \langle \nabla u_\epsilon, E\rangle + \varphi\partial_k u_\epsilon(F + G). \end{aligned}$$

Note that, as before, $\int_B \varphi\partial_k u_\epsilon F = -\int_B \partial_k\varphi B_\epsilon(u_\epsilon)$. Since $B_\epsilon(u_\epsilon) \to \chi_{\{u_0>0\}}$ in $\mathscr{D}'(\Omega)$ and $u_\epsilon \to u_0$ in $L^2(\Omega)$, as $\epsilon \to 0^+$ the above expression tends to

$$\int_B \partial_k u_0 \langle \nabla\varphi, E\rangle - \partial_k(\varphi\gamma(x, u_0)) \langle \nabla u_0, E\rangle - \partial_k\varphi\chi_{\{u_0>0\}} + \varphi\partial_k u_0 G,$$

where E, F, and G are evaluated at u_0 and ∇u_0 instead of u_ϵ and ∇u_ϵ. A simple integration by parts shows $\int_B \partial_k\varphi\chi_{\{u_0>0\}}\, dx = \int_C \partial_k\varphi = -\int_{\partial C} \varphi\nu_k$. Since the above integral is equal to 0 and $u_0 \equiv 0$ on $B \cap C^C$, we rearrange to obtain

$$\int_C \partial_k u_0 \langle \nabla\varphi, E\rangle = \int_{\partial C} \varphi\nu_k + \int_C \partial_k(\varphi\gamma(x, u_0)) \langle \nabla u_0, E\rangle - \varphi\partial_k u_0 G.$$

We now go through a similar computation using u_0. Noting that $F(D^2u, \nabla u, x) \equiv 0$ on C, we have

$$\begin{aligned} \int_{\partial C} \varphi\partial_k u_0 \langle E, \nu\rangle &= \int_C \operatorname{div}(\varphi\partial_k u_0 E) \\ &= \int_C \partial_k u_0 \langle \nabla\varphi, E\rangle + \varphi \langle \nabla\partial_k u_0, E\rangle + \varphi\partial_k u_0(F + G) \\ &= \int_C \partial_k u_0 \langle \nabla\varphi, E\rangle + \varphi\gamma\partial_k \langle \nabla u_0, E\rangle + \varphi\partial_k u_0 G \\ &= \int_C \partial_k u_0 \langle \nabla\varphi, E\rangle - \partial_k(\varphi\gamma) \langle \nabla u_0, E\rangle + \varphi\partial_k u_0 G + \int_{\partial C} \varphi\gamma \langle \nabla u_0, E\rangle . \end{aligned}$$

Substituting the expression for $\int_C \partial_k u_0 \langle \nabla\varphi, E\rangle$ from the previous paragraph into the above expression, we obtain the much simpler equality

$$\int_{\partial C} \varphi\partial_k u_0 \langle E, \nu\rangle = \int_{\partial C} \varphi\nu_k + \int_{\partial C} \varphi\gamma \langle \nabla u, E\rangle \nu_k.$$

As before, $\partial_k u_0 \nu = \nabla u \nu_k$, so the above may be rearranged to

$$0 = \int_{\partial C} (1 - (1 - \gamma) \langle \nabla u, E\rangle) \varphi\nu_k.$$

Since this holds for all $\varphi \in C_c^\infty(B)$ and $k = 1, \dots, n$, it follows that

$$0 = 1 - (1-\gamma)\langle \nabla u, E\rangle$$

along the free boundary, or equivalently

$$\langle \nabla u, E\rangle = \frac{1}{1-\gamma}$$

along the free boundary.

□

4. Further Generalizations and Examples

In this section several generalizations of the theorem in the previous section and several examples of its use are given.

4.1. **Example 1:** We can weaken the hypotheses of the theorem in the previous section by considering the case,

$$\langle \nabla \partial_k u, E(x,u,\nabla u)\rangle = \gamma(x,u)\partial_k w(x,u,\nabla u, E),$$

where w is a C^1 function which depends of $x, u, \nabla u, E$. It is more general, though the computations are the same as in the proof of the previous theorem and we obtain the following similar free-boundary condition

$$\langle \nabla u_0, E\rangle = 1 + \gamma w.$$

This is more general than the previous theorem, and as an application we can consider $w = \langle \nabla u, E(x,u,\nabla u)\rangle$. In this case, $\langle \nabla\partial_k u, E\rangle = \gamma(x,u)\partial_k \langle \nabla u, E\rangle$, then $\langle \nabla u_0, E\rangle = 1 + \gamma w = 1 + \gamma\langle \nabla u_0, E\rangle$, that is, $\langle \nabla u_0, E\rangle = 1/(1-\gamma)$

4.2. **Example 2:** An analogous calculation to the one done on the proof of theorem 3.1 can been done to generalize it in another direction. Suppose the following weaker compatibility condition is satisfied:

$$\partial_k(\langle \nabla u, \varphi\rangle) = \frac{1}{\gamma(x,u)}(\langle \nabla u_k, \varphi\rangle + h(x,u,\nabla u)),$$

where h has the same continuity hypothesis as G and vanishes in the region $\{u_0 = 0\}$. In this case, the same free boundary condition holds.

As an application of this generalization, suppose $\varphi(x,u,\nabla u) = A(x,u)\nabla u$, where A is a symmetric $n \times n$ matrix. So the equality $\varphi_k(x,u,\nabla u) = A_k(x,u)\nabla u + A(x,u)\nabla u_k$ and the symmetry $A = A^T$ imply

$$\partial_k(\langle \nabla u, \varphi\rangle) = 2\langle \nabla u_k, \varphi\rangle + \langle \nabla u, A_k \nabla u\rangle.$$

Taking $h(x,u,\nabla u) = \langle \nabla u, A_k\nabla u\rangle$, we conclude from the last computation that the free boundary condition is

$$\langle \nabla u, \varphi\rangle = \langle \nabla u, A(x,u)\nabla u\rangle = 2.$$

And computing the divergence of φ

$$\operatorname{div}\varphi = \sum_{i,j}(a_{ji})_j u_i + Tr(AD^2u).$$

So this generalization applies to equations of the above form.

4.3. **Example 3.** The most important example where this computation can be applied is the Laplacian operator, where

$$\varphi(\nabla u) = \nabla u$$

so that $\operatorname{div} \varphi(\nabla u) = \Delta u$.

This example can be generalized to the p-Laplacian, which has applications in several fields, including glaciology, non-Newtonian fluid flow, and flow through porous media. The p-Laplacian is the divergence of the following function:

$$\varphi(\nabla u) = |\nabla u|^{p-2}\nabla u$$

Taking the divergence, we see the associated PDE operator is

$$\operatorname{div} \varphi = F(\nabla u, D^2 u) = (p-2)|\nabla u|^{p-4}\sum_{i=1}^{n}\langle \nabla u, \nabla u_i\rangle u_i + |\nabla u|^{p-2}\Delta u.$$

So we can apply theorem 3.1 to a non-linear operator. As we have $\partial_k\langle \nabla u, \varphi(\nabla u)\rangle = p\langle \varphi, \nabla u_k\rangle$, the p-laplacian satisfies the compatibility condition with $\gamma = \frac{1}{p}$. So in this example the free boundary condition is

$$\langle \varphi, \nabla u\rangle = |\nabla u|^p = \frac{p}{p-1}.\text{along the free boudary}$$

Notice that taking A=Id in the first example and p=2 in the second, we have the Laplacian.

Therefore we can now solve the over-determined system:

$$\begin{cases} \Delta_p u = 0 \text{ in } \Omega\backslash\Omega' \\ u = f \text{ on } \partial\Omega \\ u = 0 \text{ and } |\nabla u|^p = \frac{p}{p-1} \text{ on } \partial\Omega' \end{cases}$$