FUNDAMENTAL EQUATIONS FOR SURFACES IN ANTI DE SITTER SPACES

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ABSTRACT. We prove an analogue of Bonnet's theorem in Anti de Sitter spaces endowed with an exotic metric tensor. We prove that necessary and sufficient conditions to characterize a surface in AdS space, up to left translation, are given by Gauss and Codazzi equations in the first and second fundamental forms as well as information about lapse and shift functions.

1. INTRODUCTION

Three-dimensional versions of Gravity theory constitute a beautiful and complex research theme which combines theoretical elements from General Relativity, Riemann surface theory - including Teichmüller spaces apparatus, and analytical techniques involving harmonic maps. Witten's work (see [16]) laid down the basis for an impressive amount of work about physics and geometry of $(2+1)$-gravity modeled upon Anti de Sitter (AdS) spaces. For instance, we should mention the extremely geometric in nature construction in [2] of the BTZ black-holes solutions for three-dimensional gravity with a negative cosmological constant. The initial value formulation for this theory in terms of an ADM formalism is nicely presented in references as [1], [8], [11], [13] and [14]. Quite naturally, this formalism is based on the existence of an initial Riemann surface where are defined a set of data as the induced metric, its Lie derivative in the form of the extrinsic curvature and lapse and shift data satisfying constraint and dynamical equations which are deduced using the Gauss and Codazzi equations.

Our aim in this paper is to establish the fundamental equations for surface theory in exotic AdS spaces. Considering the same background manifold, we define an one-parameter family of metrics g_τ, $\tau > 0$, in AdS space, one of these corresponding to the usual metric induced by embedding in $\mathbb{R}^{2,2}$. These metrics are solutions for a $(2+1)$-dimensional Einstein's equation with negative cosmological constant $-\tau^2$ and an energy-momentum tensor which is non-zero for $\tau \neq 1$. The fundamental equations we refer to are integrability conditions for the existence of a space-like isometrically immersed surface in AdS space with prescribed extrinsic curvature.

These exotic AdS spaces are all endowed with a time-like Killing vector field whose flow lines form a congruence of geodesics. The Hopf fibration over the Poincaré disc may be used to make explicit the fact that the metrics g_τ are stationary with non-integrable rest spaces. Tangential and normal components of this Killing vector field over an space-like Cauchy surface determine the lapse and shift data.

We are able to prove (Theorems 5.1 and 5.2 in Section 5) that the integrability conditions we alluded to are Gauss and Codazzi equations and other two additional conditions relating the induced metric, the extrinsic curvature and the prescribed lapse and shift data. In this sense, we are motivated by the versions of Bonnet's theorem which recently appeared in the literature as may be seen in references [3], [10], [9], [7], [6] and [4], among others.

Further investigation topics which have been currently investigated by the authors are solving for analytic data the initial value problem for $(2+1)$-gravity in exotic AdS spaces and construct the analogues of BTZ black-hole modeled in these spaces. The integrability conditions for minimal and mean curvature surfaces have been recently closely examined in [5], where they are reduced to the solving a non-linear sigma model.

This paper has the following presentation: in Sections 2 and 3 we establish some basic geometric facts for surfaces in AdS spaces and its physical counterparts. In Section 4, we prove a sequence of necessary conditions relating the intrinsic and ambient geometries all of them satisfied by an immersed surface in an AdS space. The equations in this section are presented in such a way that in Section 5 are restated as integrability conditions. This section is devoted to the proof of the existence theorems 5.1 and 5.2. The paper ends with an appendix containing a technical proposition used in the core of the proof of the theorems.

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2. AdS space

Anti de Sitter (AdS) space is modeled as the universal cover of the hyperquadric $\bar{M}$ in $\mathbb{R}^4$ defined in terms of linear coordinates by $q(x,y,u,v)=1$, where

$$q(x,y,u,v)=x^2+y^2-u^2-v^2.$$

It is easily seen that $\bar{M}$ is a three-dimensional differentiable manifold which may be identified to the Lie group

$$SU_{1,1}=\left\{Z=\begin{pmatrix} z & w\\ \bar{w} & \bar{z}\end{pmatrix}\in GL(2,\mathbb{C}) : |z|^2-|w|^2=1\right\}$$

after identifying points $(x,y,u,v)\in\bar{M}$ with the pair of complex numbers $z=x+iy, w=u+iv$. The Lie algebra $\mathfrak{su}_{1,1}$ of $SU_{1,1}$ is spanned by the tangent vectors

$$\varsigma_1=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix},\quad \varsigma_2=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\quad \varsigma_3=\begin{pmatrix} 0 & i\\ -i & 0\end{pmatrix}, \tag{2.1}$$

which generate the left-invariant vector fields

$$Z\in SU_{(1,1)}\mapsto Z\varsigma_i\in T_Z SU_{1,1},\quad i=1,2,3. \tag{2.2}$$

Given a positive constant $\tau\in\mathbb{R}$, one defines the *Lorentzian* metric in $\mathfrak{su}_{1,1}$ determined by

$$\langle\varsigma_1,\varsigma_1\rangle=-\tau^2,\quad \langle\varsigma_2,\varsigma_2\rangle=\langle\varsigma_3,\varsigma_3\rangle=1,\quad \langle\varsigma_i,\varsigma_j\rangle=0,\quad \text{if}\quad i\neq j. \tag{2.3}$$

We then define a left-invariant Lorentzian metric tensor $g_\tau=\langle\cdot,\cdot\rangle$ in $SU_{1,1}$ by imposing

$$\langle Z\varsigma_i, Z\varsigma_j\rangle=\langle\varsigma_i,\varsigma_j\rangle \tag{2.4}$$

what implies that the left-invariant vector fields

$$E_1=\frac{1}{\tau}Z\varsigma_1,\quad E_2=Z\varsigma_2,\quad E_3=Z\varsigma_3 \tag{2.5}$$

form, at each point, an orthonormal basis with respect to this Lorentzian metric. We denote by $\bar{\nabla}$ and $\bar{R}$ the Levi-Cività connection and curvature tensor associated to this metric.

The matrix Lie bracket in $SU_{1,1}$ is determined by the split-quaternionic algebraic relations

$$[\varsigma_1,\varsigma_2]=2\varsigma_3,\quad [\varsigma_2,\varsigma_3]=-2\varsigma_1,\quad [\varsigma_3,\varsigma_1]=2\varsigma_2. \tag{2.6}$$

For $\tau = 1$, the metric we defined above corresponds to that one obtained by restricting the quadratic form q to $T\bar{M}$. In this case, $\bar{M}$ is then isometrically embedded in $\mathbb{R}^4$ endowed with the flat metric q with signature $(++--)$. This is no longer true for $\tau \neq 1$ as we easily verify by means of Gauss-Codazzi equations.

Fixed the frame $\{E_a\}_{a=1}^3$, one computes the curvature tensor using the structural constants which appear in

$$[E_a, E_b] = \sigma_{ab}^c E_c, \tag{2.7}$$

that is,

$$\sigma_{12}^3 = \frac{2}{\tau}, \quad \sigma_{23}^1 = -2\tau, \quad \sigma_{31}^2 = \frac{2}{\tau}. \tag{2.8}$$

Let $\{\theta^a\}_{a=1}^3$ be the coframe dual to $\{E_a\}_{a=1}^3$. The connection forms associated to these frames are defined by

$$\bar{\nabla} E_a = E_b\, \theta_b^a. \tag{2.9}$$

These forms satisfy the first structural equation

$$d\theta^a + \theta_b^a \wedge \theta^b = 0. \tag{2.10}$$

By Koszul's formula, one has

$$\theta_b^a = \frac{1}{2}\tau_{bc}^a \theta^c, \tag{2.11}$$

where

$$\tau_{bc}^a = \sigma_{cb}^a + \sigma_{ac}^b + \sigma_{ab}^c. \tag{2.12}$$

Then, one gets

$$\theta_2^1 = \theta_1^2 = \tau\theta^3, \quad \theta_3^1 = \theta_1^3 = -\tau\theta^2, \quad \theta_3^2 = -\theta_2^3 = -(\frac{2}{\tau} - \tau)\theta^1. \tag{2.13}$$

The curvature 2-forms

$$\Theta_b^a = \theta^a\big(\bar{R}(\cdot,\cdot)e_b\big) \tag{2.14}$$

satisfy the second structural equations

$$d\theta_b^a + \theta_c^a \wedge \theta_b^c = \Theta_b^a. \tag{2.15}$$

Hence, one has

$$\Theta_b^a = \frac{1}{4}\big(\tau_{be}^a \tau_{cd}^e + \tau_{ec}^a \tau_{bd}^e\big)\theta^c \wedge \theta^d \tag{2.16}$$

and then

$$\Theta_2^1 = \Theta_1^2 = -\tau^2 \theta^1 \wedge \theta^2, \tag{2.17}$$

$$\Theta_3^1 = \Theta_1^3 = -\tau^2 \theta^1 \wedge \theta^3, \tag{2.18}$$

$$\Theta_3^2 = -\Theta_2^3 = (3\tau^2 - 4)\theta^2 \wedge \theta^3. \tag{2.19}$$

3. SURFACES IN AdS SPACES

3.1. **Further geometric facts.** Since we are dealing with a three-dimensional ambient, it holds that the curvature is entirely encoded in Ricci tensor. It follows from (2.17)-(2.19) that the Ricci tensor

$$\bar{R}(V,W) = \theta^a(\bar{R}(V,E_a)W) = \Theta^a_b(V,E_a)\theta^b(W), \quad V,W \in \Gamma(T\bar{M}), \tag{3.1}$$

has components

$$\begin{aligned}&\bar{R}(E_1,E_1) = -2\tau^2, \quad \bar{R}(E_2,E_2) = 4-2\tau^2, \quad \bar{R}(E_3,E_3) = 4-2\tau^2,\\ &\bar{R}(E_1,E_2) = \bar{R}(E_1,E_3) = \bar{R}(E_2,E_3) = 0.\end{aligned}$$

Therefore, the scalar curvature is

$$\bar{R} = -\bar{R}(E_1,E_1) + \bar{R}(E_2,E_2) + \bar{R}(E_3,E_3) = 8-2\tau^2. \tag{3.2}$$

Hence, the metric (2.3)-(2.4) satisfies the following version of $(2+1)$-dimensional Einstein's equation

$$\bar{R}(V,W) - \frac{\bar{R}}{2}\langle V,W\rangle = -\tau^2\langle V,W\rangle + 4(1-\tau^2)T(V,W), \tag{3.3}$$

where the energy-momentum tensor T is the rank two covariant tensor

$$T(V,W) = \theta^1(V)\theta^1(W)$$

and the constant $-\tau^2$ plays the role of a cosmological constant. The tensor T is divergence-free. In fact, in Lemma 2 we prove that

$$\bar{\nabla}_V E_1 = \tau V \times E_1, \tag{3.4}$$

where we are considering the Lorentzian cross product defined by

$$[E_a, E_b] = 2E_a \times E_b. \tag{3.5}$$

Using these expressions, one computes

$$\begin{aligned}\bar{\nabla}_V T(V,W) &= V(T(V,W)) - T(\bar{\nabla}_V V, W) - T(V, \bar{\nabla}_V W)\\ &= \langle V, \bar{\nabla}_V E_1\rangle\langle W, E_1\rangle + \langle \bar{\nabla}_V E_1, W\rangle\langle V, E_1\rangle\\ &= \tau\langle V\times E_1, W\rangle\langle V, E_1\rangle\end{aligned}$$

what implies that

$$\bar{\nabla}_{E_a} T(E_a, W) = 0, \quad a = 1,2,3. \tag{3.6}$$

Thus, equation (3.3) may be derived from a variational principle based on an Einstein-Hilbert action. This genre of calculations may be found for the case $\tau = 1$ in [2], where one of the main issues is to produce black-hole solutions for $(2+1)$-gravity through identifications provided by ambient isometries generated by a time-like Killing vector field.

In our case, it is easily seen that E_1 is a time-like Killing vector field. Thus, $\bar{M}$ has a stationary metric which may be described by means of the Hopf fibration $\pi : SU_{1,1} \to \mathbb{H}^2$. The adjoint representation $\mathrm{Ad} : SU_{1,1} \to O_{++}$ is a double covering map of the connected component of the identity O_{++} of the Lorentz group. An orthonormal matrix $A \in O_{++}$ is defined by the relation $A^T\eta A = \eta$ where

$$\eta = \mathrm{diag}(-1,1,1),$$

This implies that the following relations hold

$$-(A^1_1)^2 + (A^2_1)^2 + (A^3_1)^2 = -1, \quad -(A^1_2)^2 + (A^2_2)^2 + (A^3_2)^2 = 1, \quad -(A^1_3)^2 + (A^2_3)^2 + (A^3_3)^2 = 1 \tag{3.7}$$

and

$$-A_i^1 A_j^1 + A_i^2 A_j^2 + A_i^3 A_j^3 = 0, \quad \text{if} \quad i \neq j. \tag{3.8}$$

The formula (3.4) is interpreted as measuring expansion and twist of the congruence of time-like geodesics in the direction of the vector field E_1 (see [15], p. 217).

3.2. **Surfaces in AdS.** We now consider a simply connected oriented surface Σ and a space-like isometric immersion $f : \Sigma \to \bar{M}$. The immersed surface may be oriented by an unit time-like vector field N. The metric induced in Σ by f is denoted by I_f. The second fundamental form is by definition the symmetric rank two covariant tensor

$$II_f(V, W) = I_f(\bar{\nabla}_V W, N), \quad V, W \in \Gamma(T\Sigma). \tag{3.9}$$

The Weingarten map K_f is the $(1, 1)$ tensor obtained by raising an index in II_f. Hence, one has

$$K_f = -\bar{\nabla} N. \tag{3.10}$$

The extrinsic Gaussian curvature is

$$K_{\text{ext},f} = \det K_f. \tag{3.11}$$

Gauss and Codazzi equations are respectively written as

$$\langle R(V, W)V, W\rangle - \langle \bar{R}(V, W)V, W\rangle = II_f(V, V)II_f(W, W) - II_f(V, W)^2 \tag{3.12}$$

and

$$\nabla_U II_f(V, W) - \nabla_V II_f(U, W) = \langle \bar{R}(U, V)N, W\rangle, \tag{3.13}$$

for any vector fields $U, V, W \in \Gamma(T\Sigma)$. Here, ∇ and R are respectively the Riemannian connection and the curvature tensor corresponding to the induced metric I_f in Σ.

We fix a local orthonormal adapted frame $\{e_a\}_{a=1}^3$, that is, a local section of the frame bundle f^*O_{++}, so that e_1 is a time-like unit normal vector field along f. Let $\{\omega^a\}_{a=1}^3$ be the coframe dual to the adapted frame and let $\{\omega_b^a\}_{a,b=1}^3$ be the connection forms associated to this coframe. Then, one has

$$d\omega^a + \omega_b^a \wedge \omega^b = 0.$$

Since $\omega^1 = 0$ along f one has $\omega_2^1 \wedge \omega^2 + \omega_3^1 \wedge \omega^3 = 0$ and the Cartan's lemma assures that

$$\omega_i^1 = K_{i2}\omega^2 + K_{i3}\omega^3, \quad i = 2, 3. \tag{3.14}$$

Thus, the first and second fundamental forms may be recovered in terms of these 1-forms as

$$I_f = \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$$

and

$$II_f = K_{22}\omega^2 \otimes \omega^2 + 2K_{23}\omega^2 \otimes \omega^3 + K_{33}\omega^3 \otimes \omega^3.$$

The curvature forms calculated with respect to the adapted frame are defined by

$$d\omega_b^a + \omega_c^a \wedge \omega_b^c = \Omega_b^a. \tag{3.15}$$

The form $d\omega_3^2$ is the intrinsic curvature form in the Riemannian manifold (Σ, I_f). Hence, applying both sides of (3.15), when $a = 2$ and $b = 3$, to the tangent vector fields e_2, e_3, one deduces the following alternative presentation of (3.12)

$$K_{\text{int}} - \bar{K} = K_{\text{ext},f},$$

where

$$K_{\text{int}} = d\omega_3^2(e_2, e_3) \quad \text{and} \quad \bar{K} = \langle \bar{R}(e_2, e_3)e_2, e_3\rangle.$$

Differentiating (3.14), one gets

$$d\omega_i^1 + \omega_j^1 \wedge \omega_i^j = \nabla K_{ij} \wedge \omega^j \tag{3.16}$$

and then rearranging indices

$$d\omega_i^1 + \omega_j^1 \wedge \omega_i^j = \frac{1}{2}\big(\nabla K_{ik}(e_j) - \nabla K_{ij}(e_k)\big)\omega^j \wedge \omega^k$$

what gives an alternative way of writing Codazzi's equation (3.13), namely

$$\Omega_i^1 = \frac{1}{2}\big(\nabla K_{ik}(e_j) - \nabla K_{ij}(e_k)\big)\omega^j \wedge \omega^k. \tag{3.17}$$

Gauss and Codazzi equations are used to deduce the constraint equations and dynamical equations in the Hamiltonian formalism of $(2+1)$-gravity. The initial value formulation of $(2+1)$-gravity involve *lapse* and *shift* terms.

In our setting, these data concern the normal and tangential components of the flow lines of E_1 crossing the initial surface Σ transversely. We may write

$$E_1 = \alpha N + \beta$$

where $\alpha \in C^\infty(\Sigma, \mathbb{R})$ and $\beta \in \Gamma(TM)$ are respectively the lapse and shift data.

In the case of the canonical metric $\tau = 1$, this formalism is well-developed in references as [2], [13] and [14].

4. Some Auxiliary Tensors

There exists a map $A : \Sigma \to O_{++}$ so that, fixing an arbitrary point $x \in \Sigma$, the frames $\{E_a\}_{a=1}^3$ and $\{e_a\}_{a=1}^3$ at $f(x)$ are related by

$$e_a = E_b\, A_a^b, \tag{4.1}$$

where $A(x) = (A_a^b)_{a,b=1}^3$. The dual and connection forms are respectively related by

$$\omega^a = (A^{-1})_b^a \theta^b \tag{4.2}$$

and

$$\omega_b^a = A^{-1}dA + A^{-1}\theta A. \tag{4.3}$$

4.1. **Christoffel tensor.** In this section, we compute the $\mathfrak{o}_{++}$-valued 1-form

$$\lambda := A^{-1}\theta A \;=\; \begin{pmatrix} A_1^1 & -A_1^2 & -A_1^3 \\ -A_2^1 & A_2^2 & A_2^3 \\ -A_3^1 & A_3^2 & A_3^3 \end{pmatrix} \begin{pmatrix} 0 & \theta_2^1 & \theta_3^1 \\ \theta_1^2 & 0 & \theta_3^2 \\ \theta_1^3 & \theta_2^3 & 0 \end{pmatrix} \begin{pmatrix} A_1^1 & A_2^1 & A_3^1 \\ A_1^2 & A_2^2 & A_3^2 \\ A_1^3 & A_2^3 & A_3^3 \end{pmatrix}. \tag{4.4}$$

One denotes by M_j^i the minor of the matrix A obtained by suppression of the i-th row and j-th column. We then verify using Cramer's rule

$$(A^{-1})_j^i = (-1)^{i+j} M_i^j \tag{4.5}$$

and $\det A = 1$ that

$$\begin{aligned}
&\lambda_1^1 = \lambda_2^2 = \lambda_3^3 = 0,\\
&\lambda_2^1 = \lambda_1^2 = (\frac{2}{\tau} - \tau)(A^{-1})_1^3\theta^1 + \tau(A^{-1})_2^3\theta^2 + \tau(A^{-1})_3^3\theta^3,\\
&\lambda_3^1 = \lambda_1^3 = -(\frac{2}{\tau} - \tau)(A^{-1})_1^2\theta^1 - \tau(A^{-1})_2^2\theta^2 - \tau(A^{-1})_3^2\theta^3,\\
&\lambda_3^2 = -\lambda_2^3 = -(\frac{2}{\tau} - \tau)(A^{-1})_1^1\theta^1 - \tau(A^{-1})_2^1\theta^2 - \tau(A^{-1})_3^1\theta^3.
\end{aligned}$$

However, in view of (4.2), one deduces that

$$\begin{aligned}
&\lambda_1^1 = \lambda_2^2 = \lambda_3^3 = 0,\\
&\lambda_2^1 = \lambda_1^2 = 2(\frac{1}{\tau} - \tau)(A^{-1})_1^3\theta^1 + \tau\omega^3,\\
&\lambda_3^1 = \lambda_1^3 = -2(\frac{1}{\tau} - \tau)(A^{-1})_1^2\theta^1 - \tau\omega^2,\\
&\lambda_3^2 = -\lambda_2^3 = -2(\frac{1}{\tau} - \tau)(A^{-1})_1^1\theta^1 - \tau\omega^1.
\end{aligned}$$

However, motivated by the decomposition

$$E_1 = e_1 \underbrace{(A^{-1})_1^1}_{\text{lapse}} + \underbrace{e_2(A^{-1})_1^2 + e_3(A^{-1})_2^3}_{\text{shift}}, \tag{4.6}$$

we denote

$$U^1 = (A^{-1})_1^1, \quad U^2 = (A^{-1})_1^2, \quad U^3 = (A^{-1})_1^3. \tag{4.7}$$

Therefore

$$\begin{aligned}
\theta^1 &= A_1^1\omega^1 + A_2^1\omega^2 + A_3^1\omega^3 = (A^{-1})_1^1\omega^1 - (A^{-1})_1^2\omega^2 - (A^{-1})_1^3\omega^3\\
&= U^1\omega^1 - U^2\omega^2 - U^3\omega^3
\end{aligned}$$

With this notation, we have

Lemma 1. *The form $\lambda = A^{-1}\theta A$ satisfies*

$$\lambda_1^1 = \lambda_2^2 = \lambda_3^3 = 0, \tag{4.8}$$

$$\lambda_2^1 = \lambda_1^2 = 2(\frac{1}{\tau} - \tau)U^3\chi + \tau\omega^3, \tag{4.9}$$

$$\lambda_3^1 = \lambda_1^3 = -2(\frac{1}{\tau} - \tau)U^2\chi - \tau\omega^2, \tag{4.10}$$

$$\lambda_3^2 = -\lambda_2^3 = -2(\frac{1}{\tau} - \tau)U^1\chi - \tau\omega^1, \tag{4.11}$$

where $U^a = (A^{-1})_1^a$ and $\chi = U^1\omega^1 - U^2\omega^2 - U^3\omega^3$.

4.2. **Curvature forms.** We now determine Ω, using the formula

$$\Omega = A^{-1}\Theta A. \tag{4.12}$$

Setting $a=-\tau^2\theta^1\wedge\theta^2, b=-\tau^2\theta^1\wedge\theta^3$ and $c=(3\tau^2-4)\theta^2\wedge\theta^3$, one has using again (4.5)

$$\Omega_2^1 = A_3^3 a - A_3^2 b + A_3^1 c, \tag{4.13}$$

$$\Omega_3^1 = -A_2^3 a + A_2^2 b - A_2^1 c, \tag{4.14}$$

$$\Omega_3^2 = A_1^3 a - A_1^2 b + A_1^1 c. \tag{4.15}$$

In view of (4.2) one has denoting $r=\omega^1\wedge\omega^2, s=\omega^1\wedge\omega^3$ and $t=\omega^2\wedge\omega^3$,

$$\begin{aligned}\theta^1\wedge\theta^2 &= (A_1^1A_2^2 - A_2^1A_1^2)\omega^1\wedge\omega^2 + (A_1^1A_3^2 - A_3^1A_1^2)\omega^1\wedge\omega^3 + (A_2^1A_3^2 - A_3^1A_2^2)\omega^2\wedge\omega^3\\ &= M_3^3\omega^1\wedge\omega^2 + M_2^3\omega^1\wedge\omega^3 + M_1^3\omega^2\wedge\omega^3\\ &= A_3^3 r - A_2^3 s - A_1^3 t \end{aligned}\tag{4.16}$$

and similarly

$$\theta^1\wedge\theta^3 = -A_3^2 r + A_2^2 s + A_1^2 t \tag{4.17}$$

$$\theta^2\wedge\theta^3 = -A_3^1 r + A_2^1 s + A_1^1 t. \tag{4.18}$$

Hence, one obtains

$$\begin{aligned}\Omega_2^1 &= A_3^3 a - A_3^2 b + A_3^1 c\\ &= -\tau^2 A_3^3(A_3^3 r - A_2^3 s - A_1^3 t) + \tau^2 A_3^2(-A_3^2 r + A_2^2 s + A_1^2 t)\\ &\quad + A_3^1(3\tau^2-4)(-A_3^1 r + A_2^1 s + A_1^1 t)\\ &= (-\tau^2((A_3^3)^2 + (A_3^2)^2 + 3(A_3^1)^2) + (A_3^1)^2)r + (\tau^2(A_2^3A_3^3 + A_3^2A_2^2 + 3A_3^1A_2^1) - 4A_3^1A_2^1)s\\ &\quad + (\tau^2(A_3^3A_1^3 + A_3^2A_1^2 + 3A_3^1A_1^1) - 4A_3^1A_1^1)t\\ &= (-\tau^2(1+4(A_3^1)^2) + 4(A_3^1)^2)r + (-\tau^2(A_2^iA_3^i + 4A_3^1A_2^1) - 4A_3^1A_2^1)s\\ &\quad + (\tau^2(A_3^iA_1^i + 4A_3^1A_1^1) - 4A_3^1A_1^1)t\end{aligned}$$

and using (3.8), one concludes that

$$\Omega_2^1 = (4(A_3^1)^2(1-\tau^2) - \tau^2)r + (4(A_3^1(A_2^1)(\tau^2-1))s + (4A_3^1A_1^1(\tau^2-1))t.$$

Thus, denoting $\hat\tau = \dfrac{1}{\tau} - \tau$, it follows that

$$\Omega_2^1 = (4(U^3)^2\hat\tau\tau - \tau^2)\omega^1\wedge\omega^2 - 4U^2U^3\hat\tau\tau\omega^1\wedge\omega^3 + 4U^3U^1\hat\tau\tau\omega^2\wedge\omega^3. \tag{4.19}$$

We also deduce, following similar computations,

$$\Omega_3^1 = -4\hat\tau\tau U^2U^3\omega^1\wedge\omega^2 + (4\hat\tau\tau(U^2)^2 - \tau^2)\omega^1\wedge\omega^3 - 4\hat\tau\tau U^1U^2\omega^2\wedge\omega^3 \tag{4.20}$$

and

$$\Omega_3^2 = -4\hat\tau\tau U^1U^3\omega^1\wedge\omega^2 + 4U^1U^2\hat\tau\tau U^1U^2\omega^1\wedge\omega^3 - (4\hat\tau\tau(U^1)^2 + \tau^2)\omega^2\wedge\omega^3. \tag{4.21}$$

4.3. **Relating Christoffel and curvature tensors.** We define for further purposes the tensor

$$Q_b^a = d\lambda_b^a - \lambda_c^a\wedge\lambda_b^c + \lambda_c^a\wedge\omega_b^c + \omega_c^a\wedge\lambda_b^c. \tag{4.22}$$

We then prove

Lemma 2. *The forms Q and Ω defined respectively by (4.22) and (4.19)-(4.21) satisfy $Q=\Omega$.*

Proof. One has

$$E_1 = e_a(A^{-1})^a_1.$$

Thus, denoting $(A^{-1})^a_1 = U^a$,

$$\bar{\nabla} E_1 = \bar{\nabla} e_a U^a = (\bar{\nabla} e_a)U^a + e_a dU^a = e_a(dU^a + \omega^a_b U^b).$$

On the other hand, given a vector field V, it follows from (2.13)

$$\bar{\nabla}_V E_1 = \tau\big(E_2\theta^3(V) - E_3\theta^2(V)\big)$$

However, Lorentzian cross product gives

$$V \times E_1 = \big(E_2\theta^2(V) + E_3\theta^3(V)\big) \times E_1 = -E_3\theta^2(V) + E_2\theta^3(V)$$

which implies that

$$\bar{\nabla}_V E_1 = \tau V \times E_1. \tag{4.23}$$

Hence, one has

$$\begin{aligned}
\bar{\nabla}_V E_1 &= \tau V \times (e_1U^1 + e_2U^2 + e_3U^3) = \tau U^1(e_2\omega^2(V) + e_3\omega^3(V)) \times e_1 \\
&= \tau U^2(e_1\omega^1(V) + e_3\omega^3(V)) \times e_2 + \tau U^3(e_1\omega^1(V) + e_2\omega^2(V)) \times e_3 \\
&= \tau e_1(U^2\omega^3(V) - U^3\omega^2(V)) + \tau e_2(U^1\omega^3(V) - U^3\omega^1(V)) \\
&\quad + \tau e_3(U^2\omega^1(V) - U^1\omega^2(V)).
\end{aligned}$$

Therefore, we have

$$dU^1 + \omega^1_2U^2 + \omega^1_3U^3 = \tau U^2\omega^3 - \tau U^3\omega^2, \tag{4.24}$$

$$dU^2 + \omega^2_1U^1 + \omega^2_3U^3 = \tau U^1\omega^3 - \tau U^3\omega^1, \tag{4.25}$$

$$dU^3 + \omega^3_1U^1 + \omega^3_2U^2 = \tau U^2\omega^1 - \tau U^1\omega^2, \tag{4.26}$$

what is equivalent to

$$dU^1 + (\omega^1_2 - \tau\omega^3)U^2 + (\omega^1_3 + \tau\omega^2)U^3 = 0, \tag{4.27}$$

$$dU^2 + (\omega^2_1 - \tau\omega^3)U^1 + (\omega^2_3 + \tau\omega^1)U^3 = 0, \tag{4.28}$$

$$dU^3 + (\omega^3_1 + \tau\omega^2)U^1 + (\omega^3_2 - \tau\omega^1)U^2 = 0. \tag{4.29}$$

One also computes

$$\begin{aligned}
d\theta^1 &= -\theta^1_2 \wedge \theta^2 - \theta^1_3 \wedge \theta^3 = 2\tau\theta^2 \wedge \theta^3 \\
&= 2\tau(A^2_2A^3_3 - A^2_3A^3_2)\omega^2 \wedge \omega^3 + 2\tau(A^2_1A^3_3 - A^2_3A^3_1)\omega^1 \wedge \omega^3 + 2\tau(A^2_1A^3_2 - A^2_2A^3_1)\omega^1 \wedge \omega^2 \\
&= 2\tau(A^{-1})^1_1\omega^2 \wedge \omega^3 - 2\tau(A^{-1})^2_1\omega^1 \wedge \omega^3 + 2\tau(A^{-1})^3_1\omega^1 \wedge \omega^2 \\
&= 2\tau U^1\omega^2 \wedge \omega^3 - 2\tau U^2\omega^1 \wedge \omega^3 + 2\tau U^3\omega^1 \wedge \omega^2.
\end{aligned}$$

One has, denoting $\hat{\tau} = \frac{1}{\tau} - \tau$,

$$\begin{aligned}
(\lambda \wedge \omega)^1_2 &= \lambda^1_3 \wedge \omega^3_2 = -2\hat{\tau}U^2\theta \wedge \omega^3_2 + \tau\omega^3_2 \wedge \omega^2, \\
(\omega \wedge \lambda)^1_2 &= \omega^1_3 \wedge \lambda^3_2 = -2\hat{\tau}U^1\theta \wedge \omega^1_3 + \tau\omega^3_1 \wedge \omega^1, \\
(\lambda \wedge \omega)^1_3 &= \lambda^1_2 \wedge \omega^2_3 = 2\hat{\tau}U^3\theta \wedge \omega^2_3 - \tau\omega^2_3 \wedge \omega^3, \\
(\omega \wedge \lambda)^1_3 &= \omega^1_2 \wedge \lambda^2_3 = 2\hat{\tau}U^1\theta \wedge \omega^1_2 - \tau\omega^2_1 \wedge \omega^1, \\
(\lambda \wedge \omega)^2_3 &= \lambda^2_1 \wedge \omega^1_3 = 2\hat{\tau}U^3\theta \wedge \omega^1_3 - \tau\omega^1_3 \wedge \omega^3, \\
(\omega \wedge \lambda)^2_3 &= \omega^2_1 \wedge \lambda^1_3 = 2\hat{\tau}U^2\theta \wedge \omega^2_1 - \tau\omega^1_2 \wedge \omega^2.
\end{aligned}$$

Moreover

$$(\lambda\wedge\lambda)^1_2=\lambda^1_3\wedge\lambda^3_2=-4\hat\tau^2U^1U^2\theta\wedge\theta-2\tau\hat\tau U^2\theta\wedge\omega^1+2\tau\hat\tau U^1\theta\wedge\omega^2+\tau^2\omega^1\wedge\omega^2,$$
$$(\lambda\wedge\lambda)^1_3=\lambda^1_2\wedge\lambda^2_3=-4\hat\tau^2U^1U^3\theta\wedge\theta-2\tau\hat\tau U^3\theta\wedge\omega^1+2\tau\hat\tau U^1\theta\wedge\omega^3+\tau^2\omega^1\wedge\omega^3,$$
$$(\lambda\wedge\lambda)^2_3=\lambda^2_1\wedge\lambda^1_3=-4\hat\tau^2U^2U^3\theta\wedge\theta-2\tau\hat\tau U^3\theta\wedge\omega^2+2\tau\hat\tau U^2\theta\wedge\omega^3+\tau^2\omega^2\wedge\omega^3.$$

Finally, using (4.27)-(4.29), one gets

$$d\lambda^1_2=2\hat\tau U^3d\theta+2\hat\tau\theta\wedge\big((\omega^3_1+\tau\omega^2)U^1+(\omega^3_2-\tau\omega^1)U^2\big)-\tau(\omega^3_1\wedge\omega^1+\omega^3_2\wedge\omega^2),$$
$$d\lambda^1_3=-2\hat\tau U^2d\theta-2\hat\tau\theta\wedge\big((\omega^2_1-\tau\omega^3)U^1+(\omega^2_3+\tau\omega^1)U^3\big)+\tau(\omega^2_1\wedge\omega^1+\omega^2_3\wedge\omega^3),$$
$$d\lambda^2_3=-2\hat\tau U^1d\theta-2\hat\tau\theta\wedge\big((\omega^1_2-\tau\omega^3)U^2+(\omega^1_3+\tau\omega^2)U^3\big)\tau(\omega^1_2\wedge\omega^2+\omega^1_3\wedge\omega^3).$$

Therefore

$$d\lambda^1_2-(\lambda\wedge\lambda)^1_2=2\hat\tau U^3d\theta+4\hat\tau^2U^1U^2\theta\wedge\theta+2\hat\tau U^1\theta\wedge\omega^3_1+2\hat\tau U^2\theta\wedge\omega^3_2-\tau(\omega^3_1\wedge\omega^1+\omega^3_2\wedge\omega^2)$$
$$-\tau^2\omega^1\wedge\omega^2$$

and

$$Q^1_2=d\lambda^1_2-(\lambda\wedge\lambda)^1_2+(\lambda\wedge\omega)^1_2+(\omega\wedge\lambda)^1_2=2\hat\tau U^3d\theta-\tau^2\omega^1\wedge\omega^2.$$

Thus,

$$Q^1_2 \quad=\quad 4\tau\hat\tau U^1U^3\omega^2\wedge\omega^3-4\tau\hat\tau U^2U^3\omega^1\wedge\omega^3+(4\tau\hat\tau(U^3)^2-\tau^2)\omega^1\wedge\omega^2. \tag{4.30}$$

One also has

$$Q^1_3=-4\tau\hat\tau U^1U^2\omega^2\wedge\omega^3+(4\tau\hat\tau(U^2)^2-\tau^2)\omega^1\wedge\omega^3-4\tau\hat\tau U^2U^3\omega^1\wedge\omega^2 \tag{4.31}$$

and

$$Q^2_3=(-4\tau\hat\tau(U^1)^2-\tau^2)\omega^2\wedge\omega^3+4\tau\hat\tau U^1U^2\omega^1\wedge\omega^3-4\tau\hat\tau U^1U^3\omega^1\wedge\omega^2. \tag{4.32}$$

This finishes the proof of the lemma. □

Using expressions (4.24)-(4.26), one deduces the following technical result.

Lemma 3. *The map $U\in C^\infty(\Sigma,\mathbb{R}^3)$ given by*

$$U(x)=(U^1(x),-U^2(x),-U^3(x)) \tag{4.33}$$

satisfies the equation

$$dU-U\omega=U\lambda, \tag{4.34}$$

what is equivalent to

$$dU^1+\omega^1_2U^2+\omega^1_3U^3=\lambda^1_2U^2+\lambda^1_3U^3,$$
$$dU^2+\omega^2_1U^1+\omega^2_3U^3=\lambda^2_1U^1+\lambda^2_3U^3,$$
$$dU^3+\omega^3_1U^1+\omega^3_2U^2=\lambda^3_1U^1+\lambda^3_2U^2.$$

Proof. For proving this lemma, it suffices to verify that

$$\lambda^1_2U^2+\lambda^1_3U^3=\tau\omega^3U^2-\tau\omega^2U^3,$$
$$\lambda^2_1U^1+\lambda^2_3U^3=\tau\omega^3U^1-\tau\omega^1U^3,$$
$$\lambda^3_1U^1+\lambda^3_2U^2=-\tau\omega^2U^1+\tau\omega^1U^2.$$

what may be easily checked by the reader. □

5. Existence of an Isometric Immersion

From now on, we determine sufficient conditions for the existence of an isometric space-like immersion of an oriented simply-connected surface Σ in AdS space $\bar{M}$ endowed with the metric tensor defined in Section 2.

We fix a Riemannian structure in Σ given by a metric tensor I. We also prescribe a rank two symmetric tensor $II \in \Gamma(T^*\Sigma \otimes T^*\Sigma)$. Let K be a $(1,1)$ tensor field K defined by

$$II(V,W) = I(K(V),W), \quad V,W \in \Gamma(T\Sigma). \tag{5.1}$$

We consider a tangent vector field $\beta \in \Gamma(T\Sigma)$ and a real function $\alpha \in C^\infty(\Sigma,\mathbb{R})$ so that

$$-\alpha^2 + I(\beta,\beta) = -1. \tag{5.2}$$

We refer to α and β respectively as *lapse* and *shift*.

Let ∇ be the Riemannian covariant derivative associated to I and denote by R the corresponding curvature tensor. We denote by K_{int} the intrinsic curvature in the Riemannian surface (Σ, I). We also denote $K_{\text{ext}} = \det K$. The complex structure in (Σ, I) is denoted in what follows by J.

Suppose that the equations

$$K_{\text{int}} - K_{\text{ext}} = 4\tau\hat{\tau}\alpha^2 + \tau^2 \tag{5.3}$$

and

$$(\nabla_V K)W - (\nabla_W K)V = 4\alpha\tau\hat{\tau} I(JV,W)J\beta \tag{5.4}$$

hold, for any $V,W \in \Gamma(T\Sigma)$. Suppose also that I, II, α and β satisfy the following set of additional conditions

$$\alpha KV + \nabla_V \beta = \tau JV \tag{5.5}$$

and

$$d\alpha(V) + II(V,\beta) = \tau\langle JV,\beta\rangle \tag{5.6}$$

We refer to these conditions (5.3)-(5.6) as *integrability conditions*. Then, we are able to state the following

Theorem 5.1. *Let Σ be an oriented simply connected surface and consider a metric I in Σ, a symmetric tensor field $II \in \Gamma(T^*\Sigma \otimes T^*\Sigma)$, a vector field $\beta \in \Gamma(T\Sigma)$ and a real function $\alpha \in C^\infty(\Sigma,\mathbb{R})$ satisfying the integrability conditions above. Then, there exists an isometric immersion $f : \Sigma \to \bar{M}$ so that*

$$I(V,W) = \langle f_*V, f_*W\rangle, \tag{5.7}$$

$$II(V,W) = \langle \bar{\nabla}_{f_*V} f_*W, N\rangle, \tag{5.8}$$

where $V,W \in \Gamma(T\Sigma)$ and $x \in \Sigma \mapsto N|_{f(x)} \in T_{f(x)}\bar{M}$ is an unit normal vector field along f. Moreover, restricting the vector field E_1, one has

$$E_1|_{f(x)} = \alpha(x)N|_{f(x)} + \beta(x). \tag{5.9}$$

Before proving this theorem, we translate all relevant information about the data I, II, α and β for the language of moving frames. In terms of differential forms, the integrability conditions determine an exterior differential system.

Theorem 5.2. *Let $\omega^1 = 0, \omega^2, \omega^3$ and $\omega = (\omega^a_b)^3_{a,b=1}$ be 1-forms defined in Σ satisfying*

$$I = \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3 \tag{5.10}$$

and

$$d\omega^a + \omega^a_b \wedge \omega^b = 0. \tag{5.11}$$

Let $U \in C^\infty(\Sigma, \mathbb{R}^3)$ with components $U(x) = (U^1(x), -U^2(x), -U^3(x))$. Let $\Omega \in \Lambda^2(\Sigma, O_{++})$ be given by

$$\hat{\Omega}^1_2 = 4\tau\hat{\tau}U^1U^3\omega^2 \wedge \omega^3 - 4\tau\hat{\tau}U^2U^3\omega^1 \wedge \omega^3 + (4\tau\hat{\tau}(U^3)^2 - \tau^2)\omega^1 \wedge \omega^2, \tag{5.12}$$

$$\hat{\Omega}^1_3 = -4\tau\hat{\tau}U^1U^2\omega^2 \wedge \omega^3 + (4\tau\hat{\tau}(U^2)^2 - \tau^2)\omega^1 \wedge \omega^3 - 4\tau\hat{\tau}U^2U^3\omega^1 \wedge \omega^2, \tag{5.13}$$

$$\hat{\Omega}^2_3 = (-4\tau\hat{\tau}(U^1)^2 - \tau^2)\omega^2 \wedge \omega^3 + 4\tau\hat{\tau}U^1U^2\omega^1 \wedge \omega^3 - 4\tau\hat{\tau}U^1U^3\omega^1 \wedge \omega^2. \tag{5.14}$$

Suppose that

$$d\omega^a_b + \omega^a_c \wedge \omega^c_b = \hat{\Omega}^a_b \tag{5.15}$$

and that

$$dU^1 + \omega^1_2 U^2 + \omega^1_3 U^3 = \tau U^2\omega^3 - \tau U^3\omega^2, \tag{5.16}$$

$$dU^2 + \omega^2_1 U^1 + \omega^2_3 U^3 = \tau U^1\omega^3 - \tau U^3\omega^1, \tag{5.17}$$

$$dU^3 + \omega^3_1 U^1 + \omega^3_2 U^2 = \tau U^2\omega^1 - \tau U^1\omega^2. \tag{5.18}$$

Then, there exists an isometric immersion $f : \Sigma \to \bar{M}$ with induced metric I so that the extrinsic curvature is given by

$$\omega^1_i = K_{i2}\omega^2 + K_{i3}\omega^3, \quad i = 2, 3. \tag{5.19}$$

Moreover, one has along f,

$$E_1|_{f(x)} = \alpha N + \beta, \tag{5.20}$$

where N is an unit normal map along f and

$$\alpha = U^1, \quad \omega^2(\beta) = U^2, \quad \omega^3(\beta) = U^3.$$

The hypothesis in the statements of these two theorems may be related according the following calculations.

Let $e_1 = N$ be an unit normal vector field along f and let e_2, e_3 be a local orthonormal tangent frame in Σ so that $e_2 \times e_3 = e_1$. Since $\omega^1 = 0$, given vector fields $V, W \in \Gamma(T\Sigma)$, one has

$$\hat{\Omega}^2_3 = -(4\tau\hat{\tau}(U^1)^2 + \tau^2)\omega^2 \wedge \omega^3(V, W)$$

and then

$$\langle R(V, W)V, W\rangle = \omega^2 \wedge \omega^3(V, W)\Omega^3_2(V, W) = \big(4\tau\hat{\tau}(U^1)^2 + \tau^2\big)\big(\omega^2 \wedge \omega^3(V, W)\big)^2$$
$$= \big(4\tau\hat{\tau}(U^1)^2 + \tau^2\big)\langle N, V \times W\rangle^2.$$

Hence, it follows that

$$K_{\text{int}} = \langle R(e_2, e_3)e_2, e_3\rangle = 4\tau\hat{\tau}(U^1)^2 + \tau^2. \tag{5.21}$$

Thus, the equation

$$d\omega^2_3 + \omega^2_1 \wedge \omega^1_3 = \hat{\Omega}^2_3$$

becomes

$$K_{\text{int}} - K_{\text{ext}} = 4\tau\hat{\tau}\alpha^2 + \tau^2.$$

Now, one computes

$$\bar{R}(V,W)N = \omega^2(\bar{R}(V,W)N)e_2 + \omega^3(\bar{R}(V,W)N)e_3 = \Omega_1^2(V,W)e_2 + \Omega_1^3(V,W)e_3$$
$$= 4U^1\tau\hat{\tau}\omega^2 \wedge \omega^3(V,W)(U^3e_2 - U^2e_3) = 4\alpha\tau\hat{\tau}\langle N, V\times W\rangle J\beta = 4\alpha\tau\hat{\tau}\langle JV, W\rangle J\beta.$$

Using the decomposition

$$E_1 = \alpha N + \beta$$

and the formula (4.23), one also computes

$$\tau V \times (\alpha N + \beta) = \bar{\nabla}_V(\alpha N + \beta) = d\alpha(V)N + \alpha\bar{\nabla}_V N + \nabla_V\beta + II(V,\beta)N. \tag{5.22}$$

However, the left-hand side may be rewritten as

$$\tau V \times (\alpha N + \beta) = \tau\alpha V \times N + \tau V \times \beta = \tau JV + \tau V \times \beta. \tag{5.23}$$

Comparing the last two expressions, one deduces that the following equations hold true

$$\alpha KV + \nabla_V\beta = \tau JV,$$
$$d\alpha(V) + II(V,\beta) = \tau\langle JV, \beta\rangle,$$

where we used again the fact that

$$\langle V\times\beta, N\rangle = \langle JV, \beta\rangle.$$

Throughout the next sections, we will prove Theorem 5.2. This requires as first step assuring the existence an adapted frame, a notion that will be clarified below.

5.1. **Existence of an adapted frame.** Given a map $U \in C^\infty(\Sigma, \mathbb{R}^3)$ with

$$U(x) = (U^1(x), -U^2(x), -U^3(x)),$$

a map $A \in C^\infty(\Sigma, O_{++})$ is said to be *admissible* when it is of the form

$$A(x) = \begin{pmatrix} U(x) \\ * \end{pmatrix}. \tag{5.24}$$

If we denote by $\mu : M(3,\mathbb{R}) \to \mathbb{R}^3$ the projection on the first line, the condition (5.24) means that $\mu(A(x)) = U(x)$. The set of admissible maps consists of the three-dimensional submanifold of $\Sigma \times O_{++}$

$$\mathcal{U} = \left\{(x,A) : A = \begin{pmatrix} U(x) \\ * \end{pmatrix}\right\} \tag{5.25}$$

whose tangent space at a point (x, A) is

$$T_{(x,A)}\mathcal{U} = \left\{(v,\mathbf{B}) : \mathbf{B} = \begin{pmatrix} dU(x)\cdot v \\ * \end{pmatrix}\right\}. \tag{5.26}$$

One defines the following set of 1-forms

$$\hat{\lambda}_1^1 = \hat{\lambda}_2^2 = \hat{\lambda}_3^3 = 0, \tag{5.27}$$

$$\hat{\lambda}_2^1 = \hat{\lambda}_1^2 = 2(\frac{1}{\tau} - \tau)U^3\chi + \tau\omega^3, \tag{5.28}$$

$$\hat{\lambda}_3^1 = \hat{\lambda}_1^3 = -2(\frac{1}{\tau} - \tau)U^2\chi - \tau\omega^2, \tag{5.29}$$

$$\lambda_3^2 = -\hat{\lambda}_2^3 = -2(\frac{1}{\tau} - \tau)U^1\chi - \tau\omega^1, \tag{5.30}$$

where $\chi = U^1\omega^1 - U^2\omega^2 - U^3\omega^3$.

A suitable version of Lemma 2 holds true.

Proposition 1. *One has*

$$d\hat{\lambda}^a_b - \hat{\lambda}^a_c \wedge \hat{\lambda}^c_b + \hat{\lambda}^a_c \wedge \omega^c_b + \omega^a_c \wedge \hat{\lambda}^c_b = \hat{\Omega}^a_b, \quad a, b = 1, 2, 3. \tag{5.31}$$

Proof. In order to verify (5.31) it suffices to follow the proof of the Lemma 2. □

We then prove the following result.

Proposition 2. *Under the hypothesis of Theorem 5.2, there exists an admissible map $A \in C^\infty(\Sigma, O_{++})$ so that*

$$A^{-1}dA = \omega - \hat{\lambda} \tag{5.32}$$

with initial condition $A(x_0) = Id$, for a given $x_0 \in \Sigma$.

Proof. It suffices to show that the hypothesis in the statement imply the hypothesis in Proposition 5 in Appendix. This allows us to assure existence of an admissible map so that

$$A^{-1}dA = \hat{\omega}, \tag{5.33}$$

where we set

$$\hat{\omega} = \omega - \hat{\lambda}. \tag{5.34}$$

Denoting

$$\Upsilon = A^{-1}dA - \hat{\omega}, \tag{5.35}$$

one computes

$$\begin{aligned} d\Upsilon &= -A^{-1}dA \wedge A^{-1}dA - d\hat{\omega} \\ &= -\big(\Upsilon + \hat{\omega}\big) \wedge \big(\Upsilon + \hat{\omega}\big) - d\hat{\omega} \\ &= -\Upsilon \wedge \Upsilon - \Upsilon \wedge \hat{\omega} - \hat{\omega} \wedge \Upsilon - d\hat{\omega} - \hat{\omega} \wedge \hat{\omega}. \end{aligned}$$

So, using $\hat{\omega} = \omega - \hat{\lambda}$, one obtains the following equation modulo Υ

$$\begin{aligned} -d\Upsilon &= d\hat{\omega} + \hat{\omega} \wedge \hat{\omega} \\ &= d\omega + \omega \wedge \omega - d\hat{\lambda} + \hat{\lambda} \wedge \hat{\lambda} - \omega \wedge \hat{\lambda} - \hat{\lambda} \wedge \omega. \end{aligned}$$

So from (5.15) and (5.31), one concludes that the equation modulo Υ

$$-d\Upsilon = d\hat{\omega} + \hat{\omega} \wedge \hat{\omega} = 0$$

holds true.

In order to accomplish all conditions of the Proposition 5 in Appendix, it remains to prove that the data defined above satisfy

$$dU - U\omega + U\hat{\lambda} = 0. \tag{5.36}$$

This is verified following the same reasoning as in Lemma 3.

The existence of an admissible map solving (5.33) follows then directly from Proposition 5 in Appendix. This finishes the proof. □

Given an admissible map $A : \Sigma \to O_{++}$ solving (5.32), one defines in Σ another set of dual 1-forms by

$$\hat{\theta}^b = A^b_a \omega^a \tag{5.37}$$

and

$$\hat\theta^a_b = \frac{1}{2}\tau^a_{bc}\hat\theta^c, \tag{5.38}$$

where τ^a_{bc} are defined in (2.12). We then restate Lemma 1, obtaining

Proposition 3. *The admissible frame obtained above as solution of the equation (5.32) satisfies*

$$\hat\lambda = A^{-1}\hat\theta A, \tag{5.39}$$

where $\hat\theta = (\hat\theta^a_b)^3_{a,b=1}$ is defined in (5.38).

Proof. It suffices to mimic the proof of Lemma 1 in Section 4. □

We finally define the following 2-forms

$$\hat\Theta^a_b = \frac{1}{4}\big(\tau^a_{be}\tau^e_{cd} + \tau^a_{ec}\tau^e_{bd}\big)\hat\theta^c \wedge \hat\theta^d. \tag{5.40}$$

Then we are able to prove the following result.

Proposition 4. *The admissible frame defined above as solution of the equation (5.32) satisfies*

$$-\hat\Omega = A^{-1}\hat\Theta A, \tag{5.41}$$

where $\hat\Theta = (\hat\Theta^a_b)^3_{a,b=1}$ is defined in (5.40).

Proof. One has

$$A^{-1}dA = \omega - \hat\lambda \tag{5.42}$$

and

$$d\hat\lambda - \hat\lambda\wedge\hat\lambda + \omega\wedge\hat\lambda + \hat\lambda\wedge\omega = \hat\Omega \tag{5.43}$$

and as we just proved

$$\hat\lambda = A^{-1}\hat\theta A. \tag{5.44}$$

Then

$$\begin{aligned} d\hat\lambda &= dA^{-1}\wedge\hat\theta A + A^{-1}d\hat\theta A - A^{-1}\hat\theta\wedge dA \\ &= -A^{-1}dA\wedge A^{-1}\hat\theta A + A^{-1}d\hat\theta A - A^{-1}\hat\theta A\wedge A^{-1}dA \\ &= -(\omega-\hat\lambda)\wedge\hat\lambda + A^{-1}d\hat\theta A - \hat\lambda\wedge(\omega-\hat\lambda) \\ &= 2\hat\lambda\wedge\hat\lambda - \omega\wedge\hat\lambda - \hat\lambda\wedge\omega + A^{-1}d\hat\theta A. \end{aligned}$$

Therefore

$$\begin{aligned} \hat\Omega &= d\hat\lambda - \hat\lambda\wedge\hat\lambda + \omega\wedge\hat\lambda + \hat\lambda\wedge\omega = \hat\lambda\wedge\hat\lambda + A^{-1}d\hat\theta A \\ &= A^{-1}\hat\theta A\wedge A^{-1}\hat\theta A + A^{-1}d\hat\theta A \\ &= A^{-1}\big(d\hat\theta + \hat\theta\wedge\hat\theta\big)A. \end{aligned}$$

However, one computes directly from the definitions and using (5.11) and (5.32) in the form $dA = A\omega - A\lambda$:

$$\begin{aligned} d\hat{\theta}^a_b &= \frac{1}{2}\tau^a_{bc}(dA^c_d \wedge \omega^d + A^c_d d\omega^d) = \frac{1}{2}\tau^a_{bc}(dA^c_d \wedge \omega^d - A^c_d \omega^d_e \wedge \omega^e) \\ &= \frac{1}{2}\tau^a_{bc}(dA^c_d - A^c_e \omega^e_d) \wedge \omega^d \\ &= -\frac{1}{2}\tau^a_{bc}(A\lambda)^c_d \wedge \omega^d. \end{aligned}$$

However $A\lambda = AA^{-1}\theta A = \hat{\theta}A$. Hence, one gets

$$\begin{aligned} d\hat{\theta}^a_b &= -\frac{1}{2}\tau^a_{bc}(\theta A)^c_d \wedge \omega^d = -\frac{1}{2}\tau^a_{bc}\theta^c_e \wedge A^e_d \omega^d = -\frac{1}{2}\tau^a_{bc}\theta^c_d \wedge \theta^d \\ &= -\frac{1}{4}\tau^a_{bc}\tau^c_{de}\theta^e \wedge \theta^d \end{aligned}$$

and

$$\hat{\theta}^a_c \wedge \hat{\theta}^c_b = \frac{1}{4}\tau^a_{cd}\tau^c_{be}\theta^d \wedge \theta^e.$$

Therefore one concludes that

$$d\hat{\theta} + \hat{\theta} \wedge \hat{\theta} = \hat{\Theta} \tag{5.45}$$

and we deduce as claimed that

$$\hat{\Omega} = d\hat{\lambda} - \hat{\lambda} \wedge \hat{\lambda} + \omega \wedge \hat{\lambda} + \hat{\lambda} \wedge \omega = A^{-1}\hat{\Theta}A.$$

This finishes the proof of the proposition. □

5.2. **Proof of the Theorem 5.2.** In view of the hypothesis, Proposition 2 implies that there exists an admissible map $A : \Sigma \to O_{++}$ which solves (5.32) and satisfies the equations (5.39) and (5.41) for $\{\hat{\theta}^a\}_{a=1}^3$ and $\{\hat{\Theta}^a_b\}_{a,b=1}^3$ defined in (5.38) and (5.40), respectively.

Let $\omega_{\mathfrak{m}}$ be the Maurer-Cartan form in $\bar{M} = SU_{1,1}$. One denotes

$$\bar{e}_a = \omega_{\mathfrak{m}}(E_a), \tag{5.46}$$

where E_a, $a = 1, 2, 3$, are the left-invariant vector fields defined in (2.5). We then define the following 1-form on $\Sigma \times \bar{M}$ with values on $\mathfrak{m} := \mathfrak{su}_{1,1}$

$$\Pi = \pi^*_{\bar{M}}\,\omega_{\mathfrak{m}} - \bar{e}_k(A^k_a \circ \pi_\Sigma)\pi^*_\Sigma\omega^a,$$

where $\pi_{\bar{M}} : \Sigma \times \bar{M} \to \bar{M}$ and $\pi_\Sigma : \Sigma \times \bar{M} \to \Sigma$ are the canonical projections. More succinctly, one may write

$$\Pi = \pi^*_{\bar{M}}\,\omega_{\mathfrak{m}} - \bar{e}_k\pi^*_\Sigma\hat{\theta}^k,$$

where we used (5.37). We then consider the distribution $\mathcal{D} = \ker \Pi$ on $\Sigma \times \bar{M}$. Thus, using (5.11) and (5.32), we calculate (omitting projections)

$$\begin{aligned} d\Pi &= d\omega_{\mathfrak{m}} - \bar{e}_b\, dA^b_a \wedge \omega^a - \bar{e}_b\, A^b_a\, d\omega^a \\ &= -\frac{1}{2}[\omega_{\mathfrak{m}}, \omega_{\mathfrak{m}}] - \bar{e}_b\, dA^b_a \wedge \omega^a + \bar{e}_b A^b_a\, \omega^a_c \wedge \omega^c \\ &= -\frac{1}{2}[\omega_{\mathfrak{m}}, \omega_{\mathfrak{m}}] - \bar{e}_b\, (A\hat{\omega})^b_a \wedge \omega^a + \bar{e}_b A^b_a\, \omega^a_c \wedge \omega^c \\ &= -\frac{1}{2}[\Pi + \bar{e}_b\hat{\theta}^b, \Pi + \bar{e}_c\hat{\theta}^c] - \bar{e}_b\, (A\hat{\omega})^b_a \wedge \omega^a + \bar{e}_b A^b_a\, \omega^a_c \wedge \omega^c \\ &= -\frac{1}{2}[\Pi, \Pi] - \frac{1}{2}[\Pi, \bar{e}_b\theta^b] - \frac{1}{2}[\bar{e}_c\theta^c, \Pi] - \frac{1}{2}[\bar{e}_b\theta^b, \bar{e}_c\theta^c] \\ &\quad -\bar{e}_b\, (A\omega)^b_a \wedge \omega^a + \bar{e}_b\, (A\hat{\lambda})^b_a \wedge \omega^a + \bar{e}_b A^b_a\, \omega^a_c \wedge \omega^c. \end{aligned}$$

Thus considering equality modulo Π it follows that

$$\begin{aligned} d\Pi &= -\frac{1}{2}[\bar{e}_b\hat{\theta}^b, \bar{e}_c\hat{\theta}^c] - \bar{e}_b(A\omega)^b_a \wedge \omega^a + \bar{e}_b(A\hat{\lambda})^b_a \wedge \omega^a + \bar{e}_b A^b_a\, \omega^a_c \wedge \omega^c \\ &= -\frac{1}{2}\hat{\theta}^b \wedge \hat{\theta}^c\, [\bar{e}_b, \bar{e}_c] - \bar{e}_b\, A^b_c\omega^c_a \wedge \omega^a + \bar{e}_b\, A^b_c\hat{\lambda}^c_a \wedge \omega^a + \bar{e}_b\, A^b_c\omega^c_a \wedge \omega^a \\ &= -\frac{1}{2}\hat{\theta}^b \wedge \hat{\theta}^c\, [\bar{e}_b, \bar{e}_c] + \bar{e}_b A^b_c\hat{\lambda}^c_a \wedge \omega^a. \end{aligned}$$

However using (5.39) and (5.37) one obtains

$$\begin{aligned} d\Pi &= -\frac{1}{2}\bar{e}_c\sigma^c_{ab}\, \hat{\theta}^a \wedge \hat{\theta}^b + \bar{e}_d\, A^d_c\hat{\lambda}^c_b(A^{-1})^b_e A^e_a \wedge \omega^a \\ &= -\frac{1}{2}\bar{e}_c\sigma^c_{ab}\, \hat{\theta}^a \wedge \hat{\theta}^b + \bar{e}_b\, (A\hat{\lambda}A^{-1})^b_a \wedge \hat{\theta}^a \\ &= -\frac{1}{2}\bar{e}_c\sigma^c_{ab}\, \hat{\theta}^a \wedge \hat{\theta}^b + \bar{e}_b\hat{\theta}^b_a \wedge \hat{\theta}^a = \bar{e}_a\big(\hat{\theta}^a_b - \frac{1}{2}\,\sigma^a_{cb}\hat{\theta}^c\big) \wedge \hat{\theta}^b. \end{aligned}$$

Therefore $\mathcal{D}$ is involutive since by (5.38) one has

$$\hat{\theta}^a_b = \frac{1}{2}\,\sigma^a_{cb}\hat{\theta}^c + \mu^k_{bc}\hat{\theta}^c, \tag{5.47}$$

where $\mu^a_{bc} = \sigma^b_{ac} + \sigma^c_{ab}$ satisfies $\mu^a_{bc} = \mu^a_{cb}$. Hence, one has

$$\big(\hat{\theta}^a_b - \frac{1}{2}\,\sigma^a_{cb}\hat{\theta}^c\big) \wedge \theta^b = \mu^a_{bc}\hat{\theta}^c \wedge \hat{\theta}^b = 0.$$

We may verify that an integral leaf through the identity ς_0 in $\bar{M}$ is a graph over Σ. The function that graphics this leaf is an isometric immersion $f : \Sigma \to \bar{M}$ with initial condition, say, $f(x_0) = \varsigma_0$, for a given point $x_0 \in \Sigma$.

Indeed, a tangent vector $(v, w) \in \mathcal{D}_{(x,Z)}$ satisfies

$$\omega_{\mathfrak{m}}(w) - \varsigma_k A^k_a(x)\omega^a(v) = 0, \tag{5.48}$$

what furnishes after left translating both sides by Z

$$w - E_k|_Z A^k_a(x)\omega^a(v) = 0. \tag{5.49}$$

However, since the leaf is the graph of f, one has

$$f_*(x) \cdot v = w|_{f(x)}, \tag{5.50}$$

where $Z = f(x)$. Thus, we conclude that

$$f_*(x) = E_k|_{f(x)} A^k_a(x)\omega^a. \tag{5.51}$$

Since $A(x) = (A^k_a)^3_{a,k=1} \in O_{++}$, for all $x \in \Sigma$, one deduces that f is an isometric immersion and that

$$e_a|_{f(x)} = E_k|_{f(x)} A^k_a(x), \quad x \in \Sigma,\ a = 1,2,3, \tag{5.52}$$

defines an adapted frame along f with corresponding dual frame given by $\{\omega^a\}^3_{a=1}$. Thus, it follows from (5.11) that $\omega^a_b,\ a = b = 1,2,3$ are the connection 1-forms relative to the adapted frame. Thus the equation (5.32) and (5.37) for A imply that $\{\hat{\theta}^a_b\}^3_{a,b=1}$ are necessarily the connection forms for $\bar{M}$ on the frame $\{E_a\}^3_{a=1}$ along f. From this fact and using (5.40), it follows that $\{\hat{\Theta}^a_b\}^3_{a,b=1}$ are the curvature forms of $\bar{M}$ in this frame. It turns out that the equation (5.41) implies that $\{\hat{\Omega}^a_b\}^3_{a,b=1}$ are the curvature 2-forms of $\bar{M}$ in terms of the adapted frame. Thus, the second fundamental form of f is prescribed by ω^1_i, $i = 2,3$.

The fact that $f(x_0) = \varsigma_0$ is not a serious restriction, since an immersion with initial condition $Z \in \bar{M}$ may be obtained by left-translating by Z the integral leaf through σ_0. Finally, the uniqueness up to rigid motions is deduced from this same reasoning.

This finishes the proof of the Theorem 5.2.

6. Appendix

Let $\bar{\omega} \in \Lambda^1(O_{++}, \mathfrak{o}_{++})$ be the Maurer-Cartan form in O_{++}. Now, we prove

Proposition 5. *Given a 1-form $\hat{\omega} \in \Lambda^1(\Sigma, \mathfrak{o}_{++})$, suppose that*

$$d\hat{\omega} + \hat{\omega} \wedge \hat{\omega} = 0. \tag{6.1}$$

Suppose also that U satisfies the matrix equation

$$dU - U\hat{\omega} = 0. \tag{6.2}$$

Then, there exists an admissible map $A \in C^\infty(\Sigma, O_{++})$ so that

$$\hat{\omega} = A^*\bar{\omega} \tag{6.3}$$

In other terms, there exists an admissible primitive A for the Darboux's derivative $\hat{\omega}$.

Proof. For proving this, we define a 1-form Υ in $\Sigma \times O_{++}$ with values on $\mathfrak{o}_{++}$ by

$$\Upsilon = \pi_1^*\hat{\omega} - \pi_2^*\bar{\omega}, \tag{6.4}$$

where $\pi_1 : \Sigma \times O_{++} \to \Sigma$ and $\pi_2 : \Sigma \times O_{++} \to O_{++}$ are the natural projections. We then define the distribution $\mathcal{D} = \ker \Upsilon$ on $\mathcal{U}$. More precisely

$$(v, \mathbf{B}) \in \mathcal{D}_{(x,A)} \quad \text{if and only if} \quad \hat{\omega}_x(v) = \bar{\omega}_A(\mathbf{B}) \tag{6.5}$$

Recall that if $\mathbf{B} \in T_A O_{++}$ is given by $\mathbf{B} = A\mathcal{B}$ for a certain $\mathcal{B} \in \mathfrak{o}_{++}$, then $\bar{\omega}(\mathbf{B}) = \mathcal{B}$.

In order to prove that (6.5) defines a distribution we must verify that $\ker \Upsilon$ has constant rank. We begin by proving that the differential of π_1 restricted to $\mathcal{D}_{(x,A)}$

$$\pi_{1_*}(x) : \mathcal{D}_{(x,A)} \to T_x\Sigma$$

is a monomorphism. In fact, if $\pi_{1_*}(v, \mathbf{B}) = 0$ for some $(v, \mathbf{B}) \in \mathcal{D}_{(x,A)}$ then $v = 0$. Since $0 = \hat{\omega}(v) = \bar{\omega}(\mathbf{B})$, it follows that $\mathbf{B} = 0$. So,

$$\dim \ker \Upsilon_{(x,A)} \leq 2.$$

Now, given $(v,\mathbf{B})\in T_{(x,A)}\mathcal{U}$ we have

$$\mu\big(A\Upsilon_{(x,A)}(v,\mathbf{B})\big)=\mu\big(A\hat\omega_x(v)-A\bar\omega_A(\mathbf{B})\big)=\mu(A)\hat\omega(v)-\mu\big(\mathbf{B}\big)$$
$$=U\hat\omega_x(v)-dU_x(v)=0$$

where in the last equality we used the hypothesis (6.2). We had verified that

$$\mathrm{Im}\Upsilon_{(x,A)}\subset\{\mathcal{B}\in\mathfrak{o}_{++}:\mu(A\mathcal{B})=0\}$$

Thus, if $\mathcal{B}\in\mathrm{Im}\Upsilon_{(x,A)}$ then $\mathcal{B}=\bar\omega_A(\mathbf{B})$ for some $\mathbf{B}$ tangent to A such that $\mu(\mathbf{B})=0$. This means that

$$\mathrm{Im}\Upsilon_{(x,A)}\subset\bar\omega_A\big(\ker\mu_A\big)$$

where $\ker\mu_A=\{\mathbf{B}\in T_AO_{++}:\mu(\mathbf{B})=0\}$ is as we saw above, an one-dimensional space. Since $\bar\omega_A$ is an isomorphism, it follows that $\bar\omega_A\big(\ker\mu_A\big)$ has also dimension one. Thus,

$$\dim\ker\Upsilon_{(x,A)}\geq 2.$$

Thus, $\mathcal{D}_{(x,A)}$ is two-dimensional, for all $(x,A)\in\Sigma\times O_{++}$.

Now we verify the integrability of $\mathcal{D}$. By hypothesis (6.1), it follows that

$$\begin{aligned}d\Upsilon &= d\hat\omega-d\bar\omega=\hat\omega\wedge\hat\omega-\bar\omega\wedge\bar\omega=(\bar\omega+\Upsilon)\wedge(\bar\omega+\Upsilon)-\bar\omega\wedge\bar\omega\\ &= \bar\omega\wedge\Upsilon+\Upsilon\wedge\bar\omega.\end{aligned}$$

Thus if one calculates $d\Upsilon$ at some vector $(v,\mathbf{B})\in\mathcal{D}_{(x,A)}$ one obtains $\Upsilon(v,\mathbf{B})=0$ and then $d\Upsilon(v,\mathbf{B})=0$ too. So the ideal $\ker\Upsilon$ is differential and then the distribution $\mathcal{D}$ is integrable.

Let then be (x,A) an integral manifold of this distribution passing through (x_0,Id). We must verify that we may write it as a graph $x\mapsto A(x)$ over Σ. In fact we have that π_* is an isomorphism between the tangent space of the leaf (the planes of $\mathcal{D}$) and the tangent space to M. So, π is a local difeomorphism. Since Σ is simply connected this is a global difeomorphism. Let $x\mapsto A(x)$ be the inverse map. Besides this we easily verify that $A^*\bar\omega=\hat\omega$ as desired. Indeed, one has given any $(v,\mathbf{B})\in\mathcal{D}_{(x,A)}$ that

$$0=\Upsilon_{(x,A)}(v,\mathbf{B})=\hat\omega_x(v)-A^{-1}\mathbf{B}$$

On the other hand, since the leaf is the graph of the map $x\mapsto A(x)$ we should have

$$\mathbf{B}=dA(x)\cdot v.$$

Thus, we conclude that

$$\hat\omega_x(v)=A^{-1}dA(x)\cdot v,$$

for any $x\in\Sigma$ and $v\in T_x\Sigma$. Thus $\hat\omega=A^{-1}dA=A^*\bar\omega$. □

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