Mathematics 150B
Semester Review Problems

1. For each of the following, find $f'(x)$:

   (a) $f(x) = \dfrac{e^{4x}}{x^2+1}$

   (b) $f(x) = \ln(1+e^x)$

   (c) $f(x) = x^2 \ln x$

   (d) $f(x) = \arcsin \sqrt{x}$

   (e) $f(x) = (1 + \tan^{-1} x)^{10}$

   (f) $f(x) = (x^2+3)^x$

2. Find the equation of the line tangent to the curve $4x^2 = y^2 + \ln(3x - y)$ at the point $(1, 2)$.

3. For each of the following, locate all intervals of increase or decrease, all intervals of concavity, all local maxima and minima, and all inflection points:

   (a) $f(x) = x^2 e^{-2x}$

   (b) $f(x) = x^2 - 8 \ln x$

4. Find

   (a) $\displaystyle\int \frac{x^2+1}{x^2-3x+2}\,dx$

   (b) $\displaystyle\int \sin^3 x\,dx$

   (c) $\displaystyle\int \frac{x}{x^2+2x+2}\,dx$

   (d) $\displaystyle\int x^3 \ln x\,dx$

   (e) $\displaystyle\int x(2x+1)^{1/2}\,dx$

   (f) $\displaystyle\int \frac{x^2}{(x^2+1)^2}\,dx$

   (g) $\displaystyle\int \frac{x+5}{x^3-2x^2+x}\,dx$

   (h) $\displaystyle\int (1-\sec x)^2\,dx$

   (i) $\displaystyle\int \sec^3 x\, \tan^3 x\,dx$

   (j) $\displaystyle\int_{-\pi/4}^{\pi/4} x^2 \sin x\,dx$

   (k) $\displaystyle\int e^x \cos x\,dx$

   (l) $\displaystyle\int x \cos^2 x\,dx$

5. Find the area under the curve $y = \sqrt{4 - x^2}$ that lies between $x = 1$ and $x = 2$.

6. Find the area of the region in the first quadrant that is between the curves $y = \dfrac{x}{x^2+1}$ and $y = \dfrac{5x}{(x^2+1)^2}$.

7. Using the trapezoidal rule with $n = 4$ and Simpson's rule with $n = 4$, approximate $\displaystyle\int_0^4 2^x\,dx$.

8. Evaluate each of the following limits:

   (a) $\displaystyle\lim_{x\to 0} \frac{1-\cos 2x}{1-\cos 3x}$

   (b) $\displaystyle\lim_{x\to\infty} (x^2+1)^{1/\ln x}$

9. For each of the following improper integrals:
   1) state whether it is convergent, and — if convergent —
   2) find its value

   (a) $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}$

   (b) $\displaystyle\int_0^6 \frac{1}{(9-2x)^2}\,dx$

   (c) $\displaystyle\int_4^\infty \frac{dx}{x^2-1}$

   (d) $\displaystyle\int_3^\infty \frac{dx}{9+x^2}$

   (e) $\displaystyle\int_4^\infty \frac{x}{x^2-1}\,dx$

10. Express the length of each of the following curves as an integral (but do not evaluate).

    (a) $y = x^4$ for $1 \le x \le 2$

    (b) $x = t^2$, $y = \cos \pi t$ for $0 \le t \le 1$

11. A curve is given parametrically by

$$x = 4\cos(3t+\pi),\ \ y = 4\sin(3t+\pi)$$

for $0 \le t \le \pi$. Find $\dfrac{dy}{dx}$ in terms of t. For which values of t does $\dfrac{dy}{dx}$ not exist? Sketch the curve and identify the points on the curve for which $\dfrac{dy}{dx}$ does not exist.

12. Consider the curve given parametrically by $x = 1 + t + t^3$, $y = 2t^4 - t^5$, for $0 \le t \le 2$.

   (a) Make a rough sketch of the curve.

   (b) Find the equation of the tangent line to the curve at the point $(3, 1)$.

   (c) Find the area of the region between the curve and the x-axis.

13. Find the area enclosed by the polar curve $r = \sqrt{\sin\theta}$. Sketch the curve.

14. (a) Sketch the curves $r = 1 - \cos\theta$ and $r = \cos\theta$, labeling all intersection points.

   (b) Find the area of the region that is interior to both curves.

15. For each of the following sequences $(n = 1, 2, 3, \ldots)$,
   1) state whether it is ultimately monotonic,
   2) state whether it is bounded, and — if bounded —
   3) give an upper bound and a lower bound
   (No proof is necessary):

   (a) $a_n = \dfrac{n}{n+1}$

   (b) $a_n = n^{\sin\frac{n\pi}{2}}$

   (c) $a_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \dfrac{1}{n}$

   (d) $a_n = \dfrac{\ln n}{n}$

16. For each of the following sequences:
   1) state whether it converges, and — if it converges —
   2) give the limit

   (a) $a_n = \dfrac{(n^2 - n)(2n - 1)}{(3n + 1)(n^2 + 3)}$

   (b) $a_n = \dfrac{\ln(n^2 + 2n + 1)}{n + 1}$

   (c) $a_n = \dfrac{n!}{5^n}$

   (d) $a_n = \left(\dfrac{n}{n+1}\right)^{n+1}$

17. Each of the following infinite series converges. Find its sum.

   (a) $\displaystyle\sum_{k=2}^{\infty} \left(\frac{3}{4}\right)^k$

   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k(k+2)}$

   (c) $\displaystyle\sum_{k=0}^{\infty} \frac{(-2)^k}{k!}$

18. Express the repeating decimal $0.52424\ldots$ as the quotient of two integers.

19. State whether each of the following series is convergent. Give a reason in each case.

(a) $\sum_n \frac{n}{n+10}$

(b) $\sum_k \frac{1}{k^2+3k+2}$

(c) $\sum_k \frac{2^k}{k^4}$

(d) $\sum_k \frac{k!}{k^k}$

(e) $\frac{1}{e} + \frac{4}{e^2} + \frac{9}{e^3} + \frac{16}{e^4} + \cdots$

(f) $\sum_n \frac{(\ln n)^5}{n}$

20. For each of the following series, state whether it diverges, converges conditionally, or converges absolutely.

(a) $\sum_k \frac{(-1)^k k}{2^k}$

(b) $\sin 1 + \sin\frac{1}{4} + \sin\frac{1}{9} + \cdots$

(c) $\sum_k \frac{(-1)^k k}{2k-1}$

(d) $\sum_k \frac{(-1)^k}{3k-2}$

(e) $\sum_n (-1)^n \frac{1}{1+n^{3/2}}$

21. Find the Taylor polynomial of order 3 based at $a = 0$ for each of the following functions:

(a) $\sqrt[3]{1+x}$

(b) $\sin(x^2)$

22. (a) Derive the Maclaurin series for $\ln(1+x)$ by term-by-term antidifferentiation of the series for $\frac{1}{1+x}$.

(b) What is the radius of convergence for the series for $\ln(1+x)$?

23. For each of the following power series, find the convergence set. Be sure to check endpoints.

(a) $\sum_k \frac{(-1)^k x^k}{3k}$

(b) $\sum_k 2^k (x-3)^{2k}$

(c) $\frac{x}{3} + \frac{x^2}{\sqrt{2}\cdot 3^2} + \frac{x^3}{\sqrt{3}\cdot 3^3} + \frac{x^4}{\sqrt{4}\cdot 3^4} + \cdots$

(d) $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \frac{x^4}{17} + \frac{x^5}{26} + \cdots$

(e) $x + \frac{2!}{2^3}x^2 + \frac{3!}{3^3}x^3 + \cdots$

24. Find a power series for $\frac{1}{(1+x)^2}$ by differentiation of a suitable multiple of the power series for $\frac{1}{1+x}$.

25. (a) Derive the Maclaurin series for $f(x) = e^x$, and show for which values of x the series converges to $f(x)$.

(b) Give the Maclaurin series for $f(x) = e^{x^2}$.

26. Use a Maclaurin polynomial to approximate $e^{-1/3}$ with an error less than 0.001. Show how you determined the accuracy.