Crossing the Bridge to

MATHEMATICS EDUCATORS AGREE THAT ratio and proportion are important middle school mathematics topics. In fact, researchers have stated that proportional reasoning involves "watershed concepts" that are at the "cornerstone of higher mathematics" (Lamon 1994; Lesh, Post, and Behr 1988). Yet assisting students in developing robust understanding of the many concepts and procedures that are related to using ratios, rates, and proportions is not straightforward. For example, being able to reason proportionally and being able to represent that reasoning symbolically do not always go hand-in-hand. As with many complex topics, students' understanding grows with time and experience.

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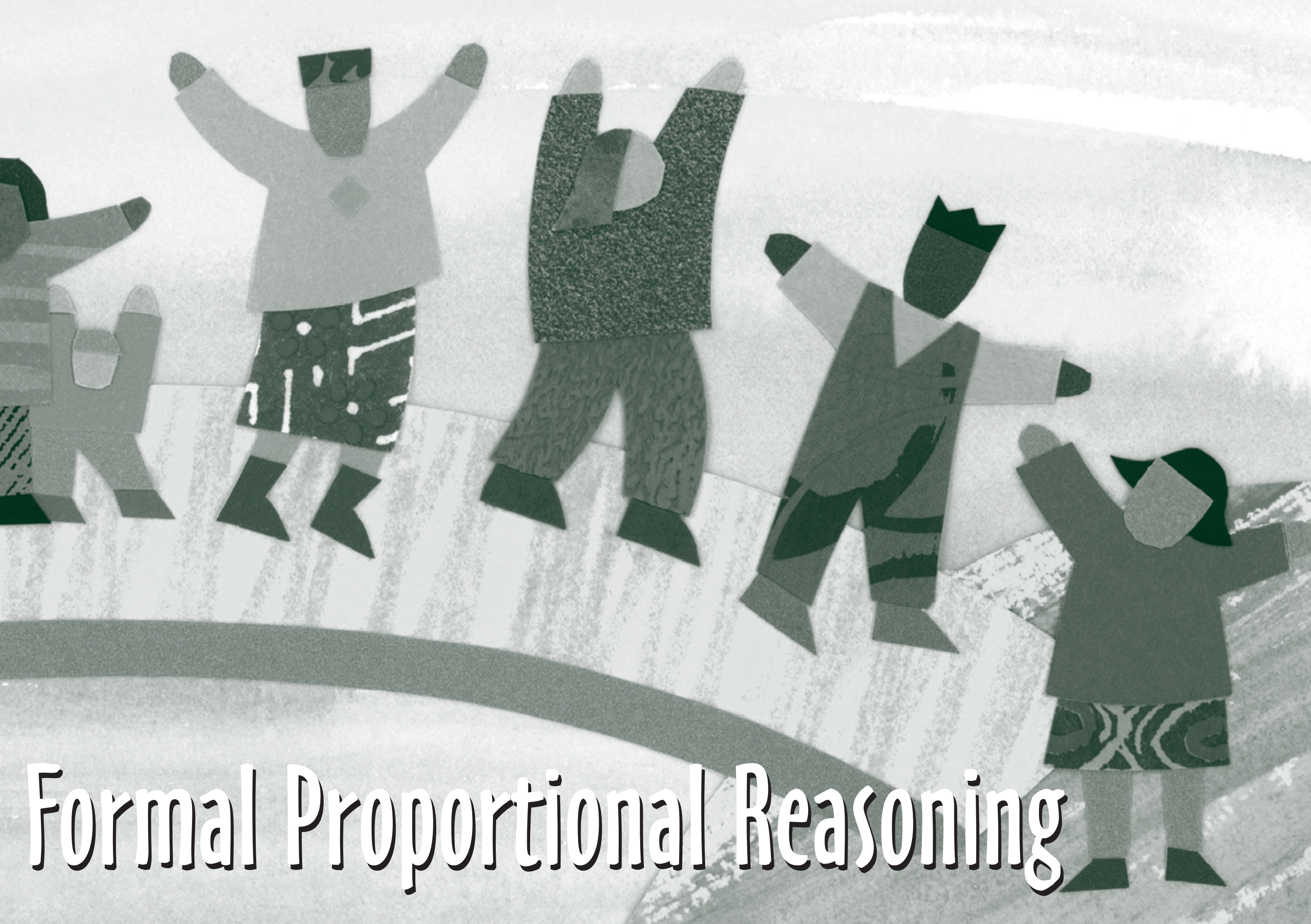
Langrall and Swafford (2000) have classified students' proportional reasoning solution strategies into four levels. Students at level 0 do not use any proportional reasoning strategies. They tend to use additive strategies and randomly selected methods and arrive at the correct solution only by luck. At level 1, students use pictures, models, or manipulative materials to solve proportion problems. At level 2, students might continue to use materials to make sense of a problem, but they then use numeric calculations, particularly multiplication and division, to arrive at a solution. At level 3, students' solution strategies are formalized; students set up and solve proportions using cross-multiplication or equal ratios.

In our work with middle school students, we have wrestled with how to further students' understanding of ratio and proportion. Vergnaud (1994) points



Formal Proportional Reasoning

SUZANNE H. CHAPIN AND NANCY CANAVAN ANDERSON

out that for students to conceptualize concepts properly, they have to apply them to new domains of experience. Thus, creating classroom situations that extend students' knowledge and help them recognize the same mathematical "structure" in different contexts can support transfer and generalization. When teachers assist students in classifying and formalizing these structures and explicitly linking them to existing knowledge, learning is both strengthened and deepened. One topic that is often new to students in the middle grades and involves proportional reasoning is similarity. We have used this topic as a venue for discussing how level 2 solution strategies link to the formal solution strategies of level 3. When examining the symbolic representation of problems, we have also facilitated discussions about the multiplicative relationships within and between ratios to assist students in expanding their solution methods.

In this article, we share some of our experiences working with a group of seventh-grade students who were part of Project Challenge, a federally funded collaboration between a university and a public school system. We first provide some general background information about solving proportions and about the concept of similarity. We then discuss how students made the transition from informal conceptual methods of solving proportional reasoning problems to a more formalized method of solving proportions, using a similarity problem from the *Stretching and Shrinking* unit in the Connected Mathematics series. (Anderson is the lead teacher of the seventh graders, and Chapin is the project director.)

Relationships within and between Ratios

ALL RATIOS HAVE MULTIPLICATIVE RELATIONSHIPS between the measures. For example, in the ratio 2:10, if we multiply the first value (2) by 5, we obtain the second value (10); if we multiply the second value (10) by 1/5, we obtain the first value (2). In some ratios, these multiplicative relationships are easy to identify using mental mathematics, but in other ratios, the relationships that exist between the values are not as obvious and must be determined through multiplication or division. For example, in the ratio 8:36, if we divide 8 by 36, we find that 8 is 2/9 the size of 36. If we divide 36 by 8, we find that 36 is 9/2, or 4.5 times, the size of 8.

When two ratios are equal, the multiplicative relationships "within" each of the individual ratios are the same, or invariant. As the example below shows, multiplying the first number in both ratios by 5 results in the second number in each ratio.

$$\overset{\times 5}{2:10} = \overset{\times 5}{3:15} \quad \text{or} \quad 5 \times \left(\frac{2}{10} = \frac{3}{15} \right) \times 5$$

When students understand that the multiplicative relationships are the same within each ratio in a proportion, they can use this knowledge to find a missing value, a solution method often referred to as the *scalar method*. For example, to determine the value of x below, we identify the multiplicative relationship within the first ratio, which is multiplying by 5, then multiply the given value in the second ratio, 6.8, by the same factor ($6.8 \times 5 = 34$). In this ratio, x equals 34.

$$\overset{\times 5}{2:10} = \overset{\times 5}{6.8:x} \quad \text{or} \quad 5 \times \left(\frac{2}{10} = \frac{6.8}{x} \right) \times 5 \Rightarrow x = 34$$

Multiplicative relationships "between" the two ratios in a proportion also exist. In the first example above, if we multiply both numbers in the ratio 2:10 by 1.5, we obtain the values in the second ratio, 3:15. We can use this method, often referred to as the *functional method*, to find x in the proportion below.

$$2:10 \underset{\times 3.4}{\overset{\times 3.4}{=}} 6.8:x \quad \text{or} \quad \frac{2}{10} \underset{\times 3.4}{\overset{\times 3.4}{=}} \frac{6.8}{x}$$

The number 6.8 is 3.4 times as large as 2. Thus, we multiply 10 by the same factor, 3.4, to determine that the value of x is 34.

Students can use relationships within ratios and between two ratios to solve missing-value proportion problems. Frequently, however, students do not analyze the relationships among symbols, do not recognize patterns, and thus, do not use the relationships within and between ratios. Instead, they tend to use the cross-product method. The cross-product method is most appropriate when multiplicative relationships within or between ratios are not obvious, but it should be one of a number of methods that students use, not the only one. We believe that instruction should help students to identify the most straightforward relationship that requires the least amount of computation, then use that relationship to solve the problem by finding the missing value. This approach allows students to use their conceptual understanding of proportional relationships to develop procedural fluency with proportions.

Why Similarity?

THE CONCEPT OF SIMILARITY IS AN IMPORTANT one that is encountered in many situations, such as enlargements, reductions, scale factors, projections, and indirect measurement. Understanding similarity helps students not only to link geometry with number but also to develop an understanding of the geometry in their environment. Furthermore, because the lengths of the corresponding sides of similar figures are proportional, the many relationships within and between ratios can be investigated and discussed.

For example, in **figure 1,** triangle ABC is similar to triangle DEF because the measures of their corresponding angles are equal and the lengths of their corresponding sides increase by the same factor, called the *scale factor;* in other words, the lengths are proportional. The scale factor from $\triangle ABC$ to $\triangle DEF$ is 3; multiplying the length of each side of $\triangle ABC$ by 3 results in the lengths of the corresponding sides of $\triangle DEF$. Likewise, the scale factor from $\triangle DEF$ to $\triangle ABC$ is 1/3 because multiplying the length of each side of $\triangle DEF$ by 1/3 gives the lengths of the corresponding sides of $\triangle ABC$.

One reason we introduce the topic of similar figures to shift students into using level 3 solution methods is that many equivalent ratios exist that relate similar triangles. For example, one ratio that can be constructed compares the length of a side of one triangle to the length of the corresponding side of the similar triangle, that is, $\overline{AB}:\overline{DE}$. This ratio is equivalent to the comparison of the two hypotenuses of the triangles, or $\overline{AC}:\overline{DF}$. One way to determine the length of $\overline{DF}$ is to set up a proportion, note that the "within" multiplicative relationship in the first ratio is the same as the scale factor, which is 3, and realize that the relationships in the second ratio are invariant and, thus, will also be linked by the scale factor. Hence, we solve for x by multiplying 2.5 by 3 to get 7.5.

$$\frac{AB}{DE} = \frac{AC}{DF} \quad \text{or} \quad \times 3 \left(\frac{1.5}{4.5} = \frac{2.5}{x} \right) \times 3 \quad x = 7.5$$

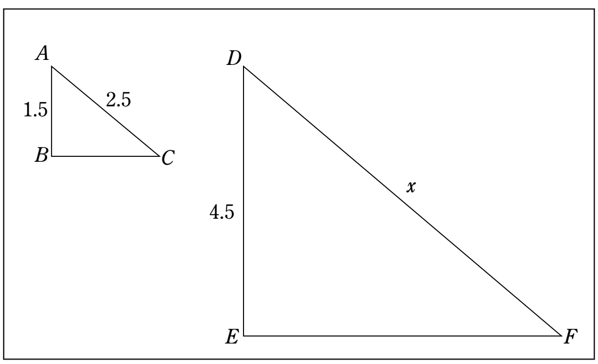


Fig. 1 Similar triangles *ABC* and *DEF*

Another proportion that can be constructed to find the length of $\overline{DF}$ is shown below. In this situation, the "between" multiplicative relationship from one ratio to the other ratio is the scale factor, 3, and the procedure for finding the missing value (DF), is to multiply 2.5 by the scale factor ($2.5 \times 3 = 7.5$).

$$\frac{AB}{AC} = \frac{DE}{DF} \quad \text{or} \quad \frac{1.5}{2.5} \overset{\times 3}{=} \frac{4.5}{x} \quad x = 7.5$$

Classroom Implementation

HOW DO TEACHERS HELP STUDENTS MAKE THE transition from using the informal conceptual solution methods of levels 1 and 2 to the formal proportions of level 3? The next sections describe an activity and discussion from a Project Challenge class that helped students cross this bridge. Before and during the five-week unit on similarity, Project Challenge students had many opportunities to reason about proportional situations; to find unit rates; to construct equivalent ratios; and to link ratios, percents, and fractions. They were using level 2 solution strategies, even though some instruction had focused on setting up and solving proportions. Discussions of solution strategies emphasized students' understanding of the multiplicative relationships inherent in the problems. By the end of the similarity unit, students seemed ready to link informal and formal strategies. One particular problem emerged as being extremely useful in helping students link what they knew about proportions across contexts and was discussed for two hour-long mathematics periods.

Exploring a Problem

THE FOLLOWING PROBLEM FROM *STRETCHING and Shrinking* was used to explore proportionality:

> Mr. Anwar's class is using the shadow method to estimate the height of their school building. They have made the following measurements and sketch [see **fig. 2**]:
>
> Length of the meterstick = 1 m
> Length of the meterstick shadow = 0.2 m
> Length of the building shadow = 7 m
>
> Use what you know about similar triangles to find the building's height from the given measurements. Explain your work. (Lappan et al. 1997, p. 60)

The lesson began with a short discussion about whether the triangles were similar. Once the students had determined that the angles in the two triangles were congruent and, thus, that the triangles were similar, they set to work using what they knew about similar shapes to find the missing height. Working in pairs, most students decided to use the scale factor to find the missing height. Many students used these number sentences to solve the problem:

$0.2 \times ? = 7 \qquad 7 \div 0.2 = 35 \qquad 1 \times 35 = 35$

The majority of students were using level 2 solution strategies; they recognized the relationships in the problem as multiplicative and used division and multiplication to find the scale factor and missing height. They were not at level 3, because they had not set up a proportion and solved for the unknown. A few pairs of students, however, did use formal proportions to find the height of the building, which enabled us to introduce more students to formal proportional solution methods through discussion. Our goal was for students to link their informal level 2 approaches that used proportional reasoning to the more formal symbolism and process of setting two equivalent ratios equal to each other.

Discussion of Strategies and Solutions

DURING THE FIRST PART OF THE CLASS DISCUSSION, many students explained how they used multiplication to solve the problem. One student said, "I figured out that 0.2×35 is 7, so I multiplied 1×35 to get the height of the building, which is 35." Another classmate elaborated—

> I knew that I had to find the scale factor to find the height. I figured out that 0.2×35 was 7 since $7 \div 0.2$ was 35. This told me that the scale factor for the two triangles was 35. So then I could use the scale factor times 1 to get the height of the building since it corresponds with the meterstick.

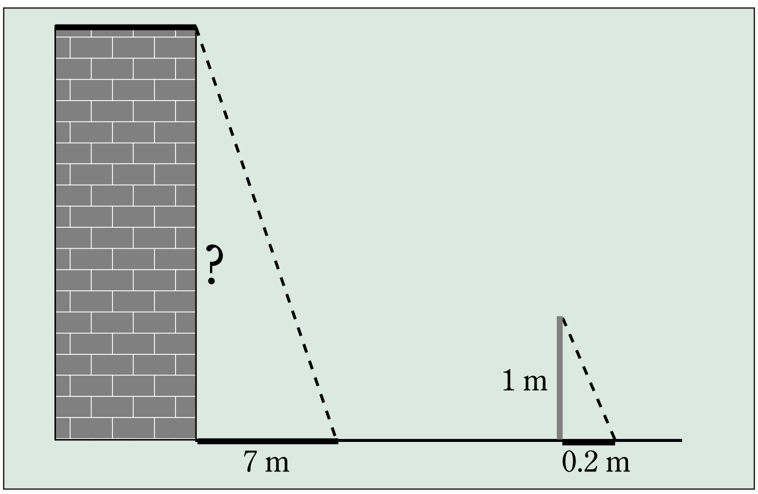


Fig. 2 Similarity problem (adapted from Lappan et al. 1997, p. 60)

A third student drew the picture shown in **figure 3** on the board, linking the legs of the triangles with arrows and the values with expressions.

These responses, as well as others made in pairs and whole-class discussions, indicated that the students had a strong understanding of the components of proportional reasoning. All students knew that the change between corresponding side lengths was multiplicative, not additive in nature. No student tried to find an additive relationship between the side lengths. Instead, all students, through a variety of strategies, tried to identify the factor that described the multiplicative relationship between 0.2 and 7. Further, they knew that the multiplicative change between corresponding sides was invariant. Once students found the multiplicative relationship between the shadows, they immediately reasoned that if the change between shadows was "times 35," then the change between actual heights must also be "times 35" because the scale factor is constant in similar figures.

Having demonstrated such a strong understanding of proportional reasoning while using informal strategies, the students seemed ready to explore more formal level 3 strategies. The teacher called on one pair of students who had solved the problem using a proportion to explain their solution:

> We used a proportion to solve the problem. We wrote 1 over 0.2 equals *N* over 7. You can solve this problem by finding what times 0.2 equals 7. To get this [number], you divide 7 by 0.2, and that's 35. So then, up top, 1×35 equals 35; *N* is 35.

As this student spoke, the teacher recorded her steps on the board (see **fig. 4**).

Once this relationship between ratios was shown with arrows, many other students contributed to the discussion by articulating their understanding of their classmates' methods. Students could explain how to set up the proportion by making sure that corresponding sides were compared accurately. For example, many students stated that one of the most important aspects of setting up this proportion correctly was that "the height has to go with the height and the length has to go with the length." When asked to compare the solution strategy that used multiplication and division number sentences (level 2) with the proportion method (level 3), students made the following observations:

- "The methods look different since the first one uses number sentences, but the other uses ratios. They are similar because, in each, you are multiplying and dividing the same numbers."
- "Both methods use the same lengths. Both use multiplication and division. It's just that in one, you use a proportion."

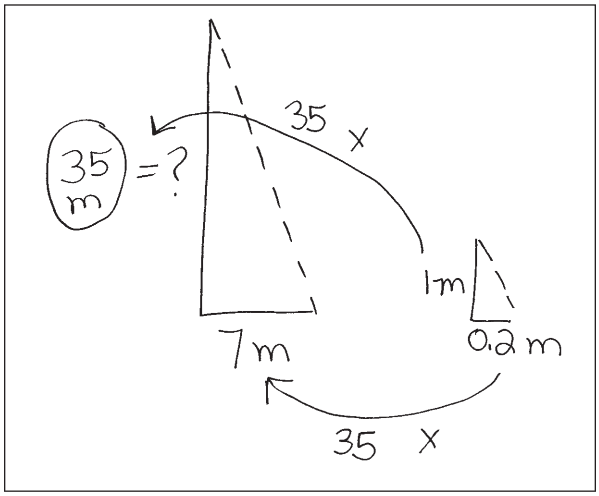


Fig. 3 A student's informal solution method

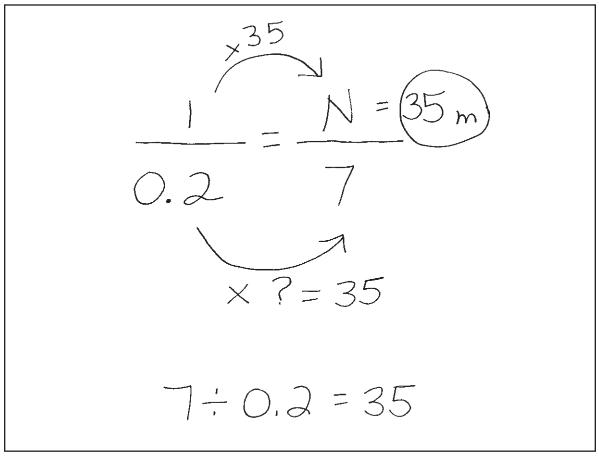


Fig. 4 A student's formal solution method

- "They are similar because, in the end, you end up dividing 7 by 0.2 to get the scale factor."
- "The proportion is more organized. The height is over the length, and you have an arrow going over to the other height and length. In the other method, there's a picture and number sentences all over the place."

The discussion took another turn when one student noted that the relationship between ratios was the scale factor; that is, if you multiply the values in one ratio by 35, the result is the second ratio. Another insightful student asked, "What if you made a different proportion of shadow over shadow equals height over height? Do you still multiply one ratio by the scale factor to get the other ratio?" The class decided to investigate the proportion that followed from this question, as shown below:

$$\frac{0.2}{7} = \frac{1}{N}$$

One student said, "You don't multiply by the scale factor. Since $0.2 \times 5 = 1$, you multiply by 5. So

7×5 equals 35. The answer is the same no matter which way you set up the ratios." Next, the teacher asked, "Where is the scale factor?" Another student responded, "In the ratio of shadow over shadow, you ask yourself what times 0.2 equals 7, and that is 35! So you times 1 by 35 to find *N*. The scale factor is going up and down—between the numbers in the ratio, not across."

The teacher then asked students to compare the two proportions. Students concluded that when they constructed ratios so that corresponding parts of each of the triangles were in different ratios (see **fig. 4**), multiplying by the scale factor connected the ratios. Yet when they constructed ratios that compared corresponding parts of each triangle in the same ratio (see the example on page 424, bottom of page), multiplying the first number in each ratio by the scale factor produced the second number in the same ratio.

Closing Comments

STUDENTS' COMMENTS SUPPORTED the decision to push them beyond level 2 strategies into level 3 strategies. The students could readily identify many relationships between the informal strategy of using multiplication and division and the formal use of proportion. For example, they saw that both methods used the same numbers and the operations of multiplication and division to find the scale factor, relationships between side lengths, and ultimately, the missing height. They understood that depending on how they constructed the ratios, the multiplicative relationship they discovered by dividing (namely, the scale factor) appeared within ratios or between ratios. Students saw how a proportion can be used to organize the multiplicative relationships they identified in the problem and to solve problems. As a result of linking the informal and formal strategies, instead of presenting them as unrelated solutions or focusing only on one, the students could see the benefits of using formal proportions when solving similarity and other types of measurement problems.

Many research studies indicate that students are more likely to use and remember mathematical strategies if they understand them. Students should be encouraged to set up and solve formal proportions once they can recognize a situation as having invariant multiplicative relationships within and between measures. Specifically linking informal and formal procedures also appears to help students move toward symbolic representation of proportionality.

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