Drawing graphs using a small number of obstacles[1]

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[1]Part of the research was conducted during the workshop Homonolo 2014 supported by the European Science Foundation as a part of the EuroGIGA collaborative research program (Graphs in Geometry and Algorithms).

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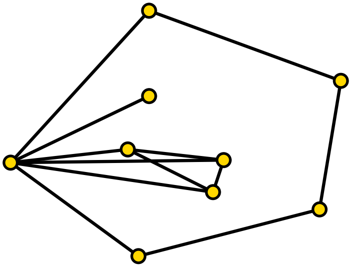

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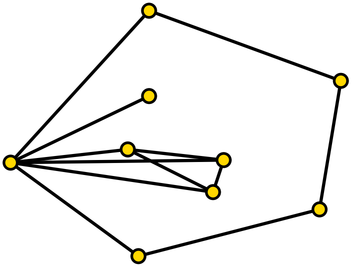

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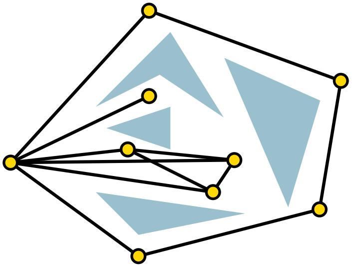

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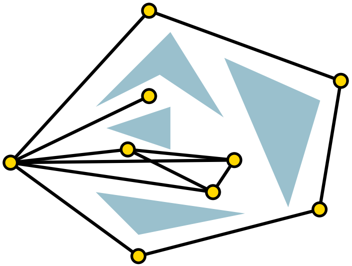

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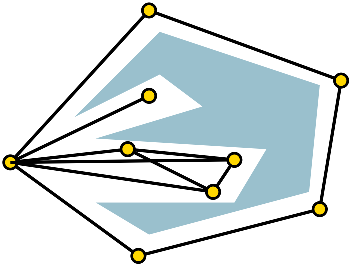

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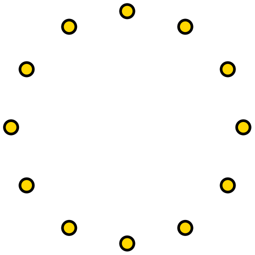

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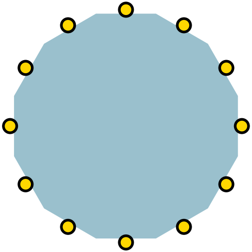

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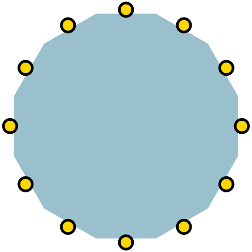

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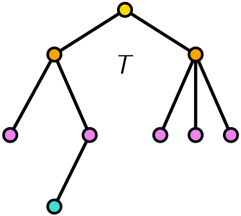

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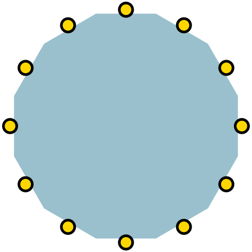

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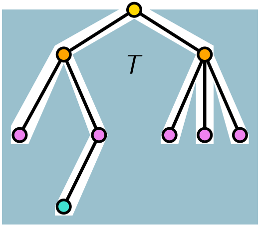

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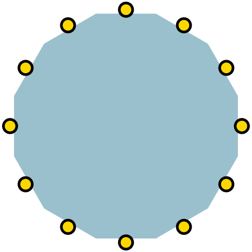

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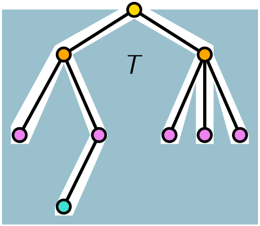

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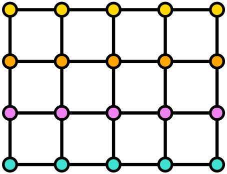

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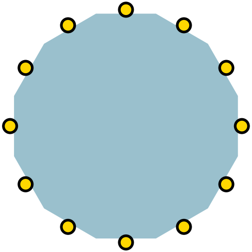

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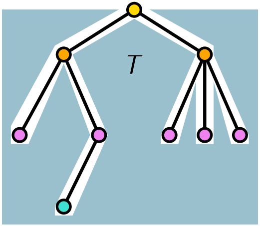

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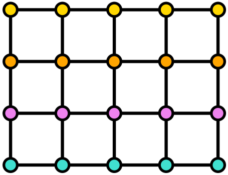

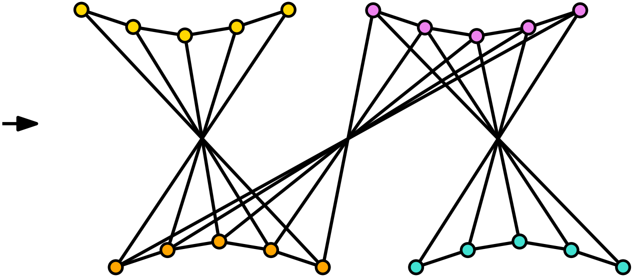

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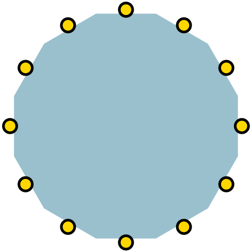

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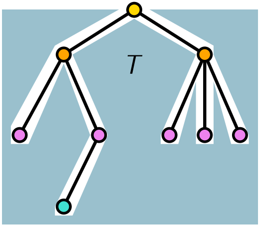

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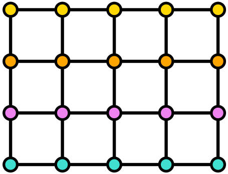
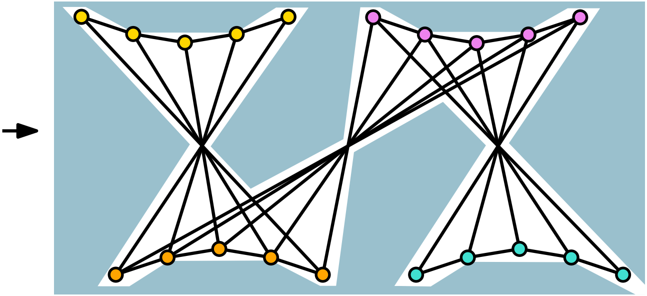


$\text{obs}(P_4 \times P_5) = 1$
(Fabrizio Frati)



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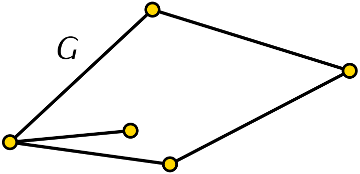

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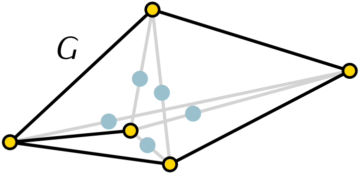

$\mathrm{obs}(G) \leq \binom{n}{2} - |E(G)|$

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The bounds apply even if the obstacles are required to be convex.

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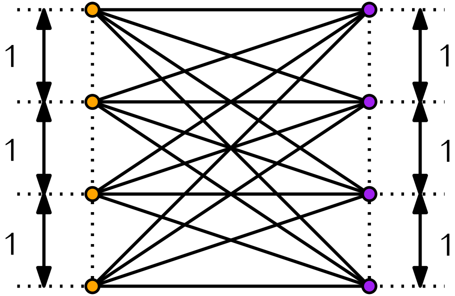

Regular drawing of $K_{4,4}$

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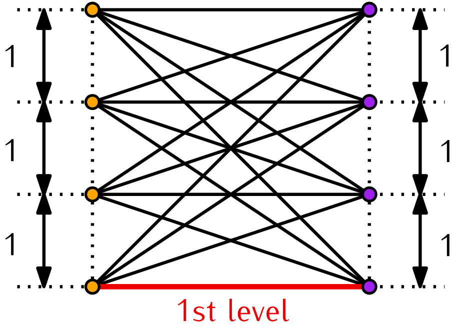

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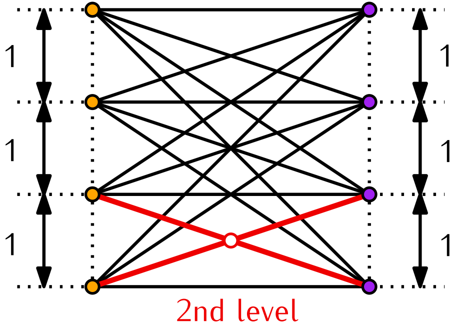

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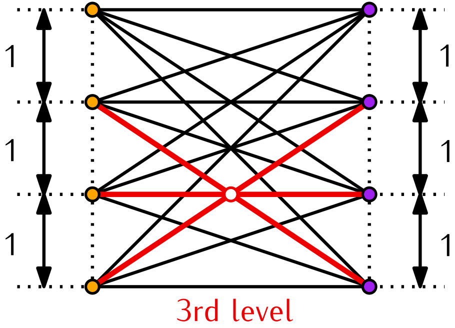

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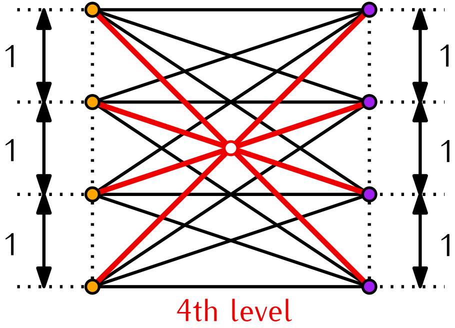

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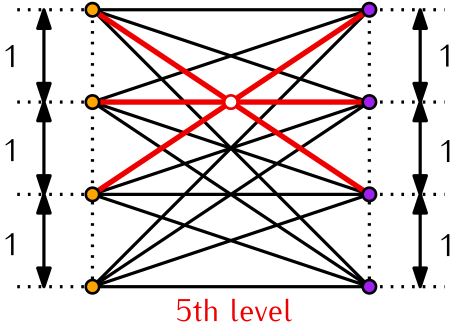

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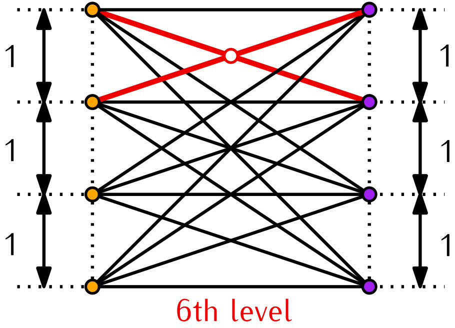

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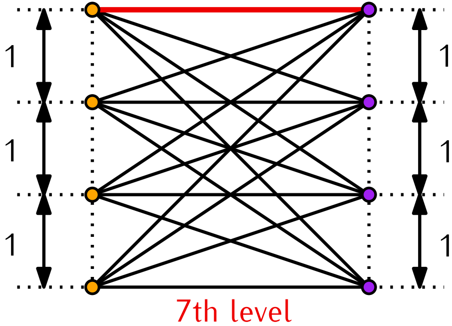

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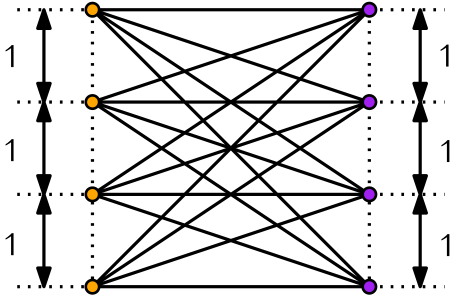

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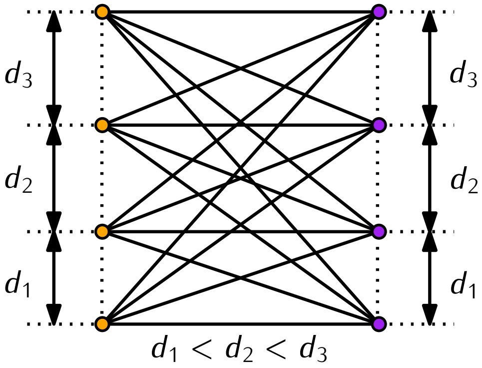

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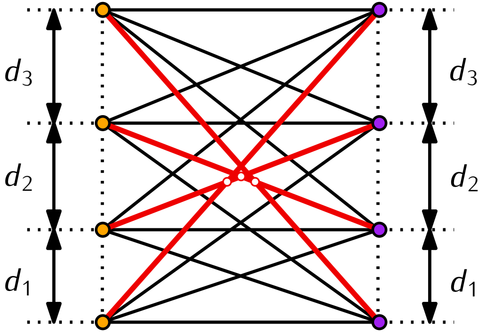

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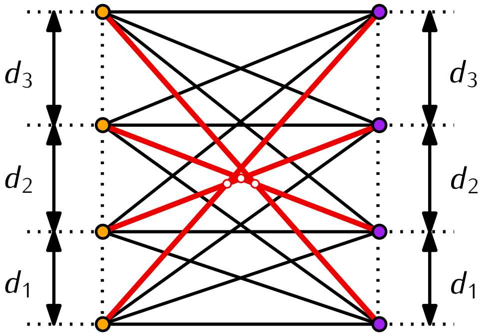

Levels form caps.

Key observation

(a) If $d_1 < \cdots < d_{n-1}$, then all levels of a drawing D of $K_{n,n}$ form caps.

(b) If D is ε-dilated, that is, we also have $d_{n-1} < (1+\varepsilon)d_1$, and $\varepsilon > 0$ is small, then all edges of every cap are incident to the same face of D.

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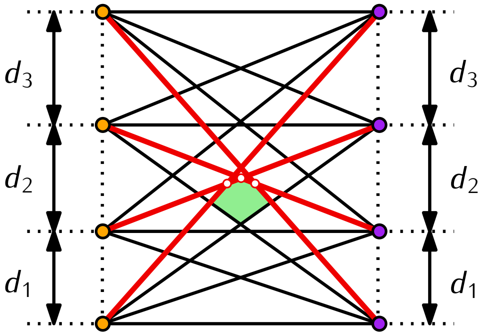

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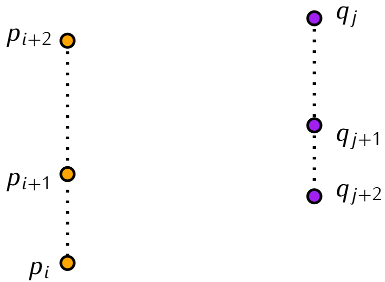

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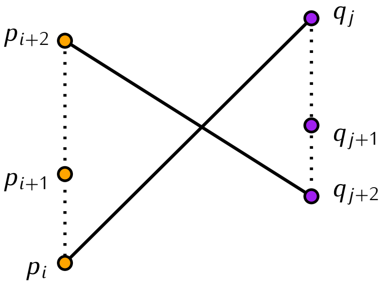

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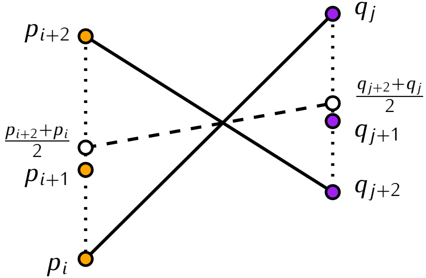

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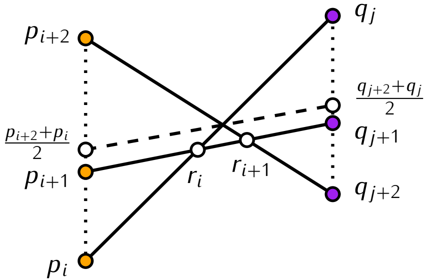

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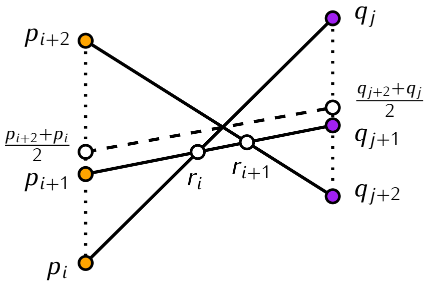

Part (b) follows from the fact that ε-dilated drawings converge to the regular drawing as $\varepsilon \to 0$

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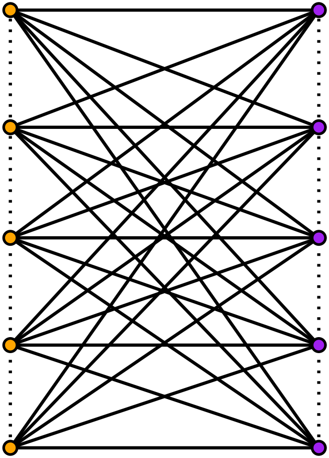

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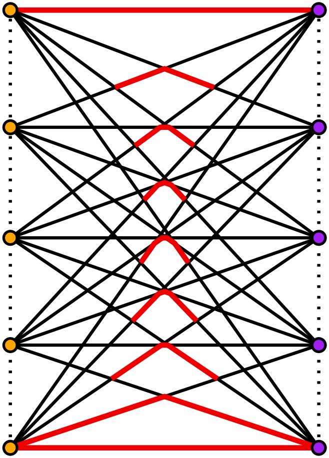

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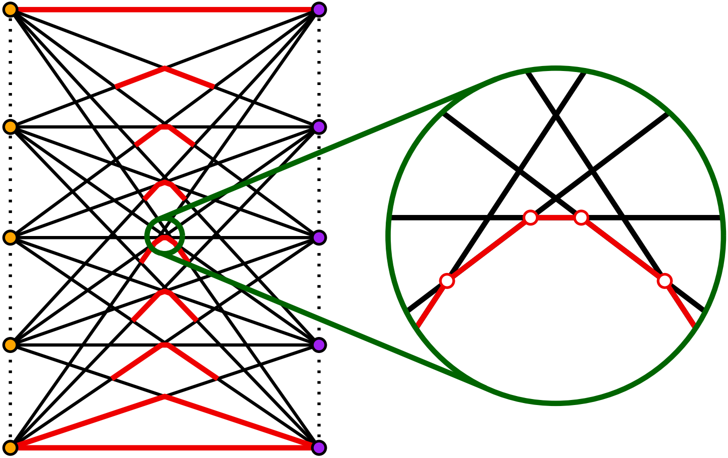

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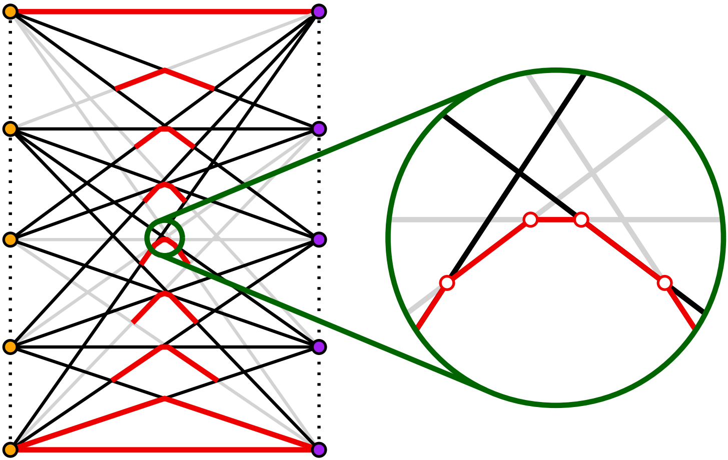

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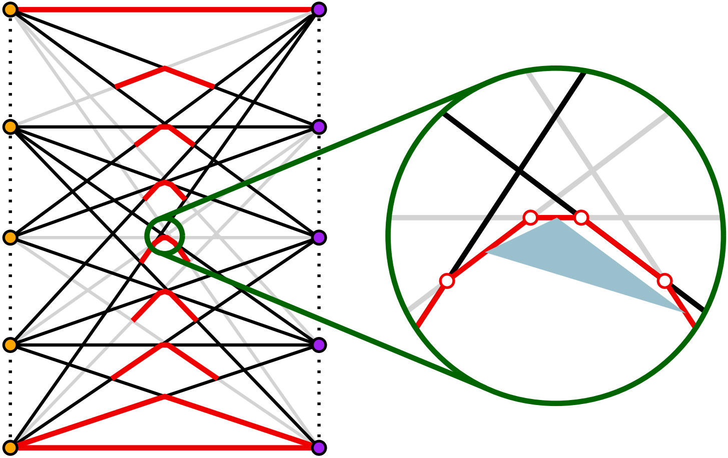

Application I: Obstacle numbers of bipartite graphs

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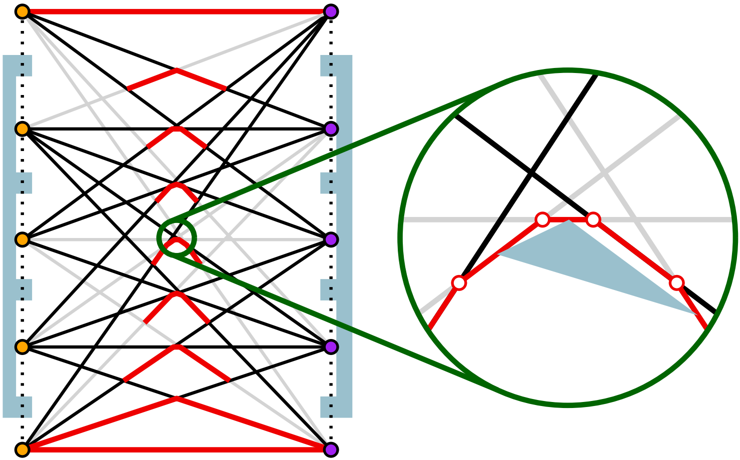

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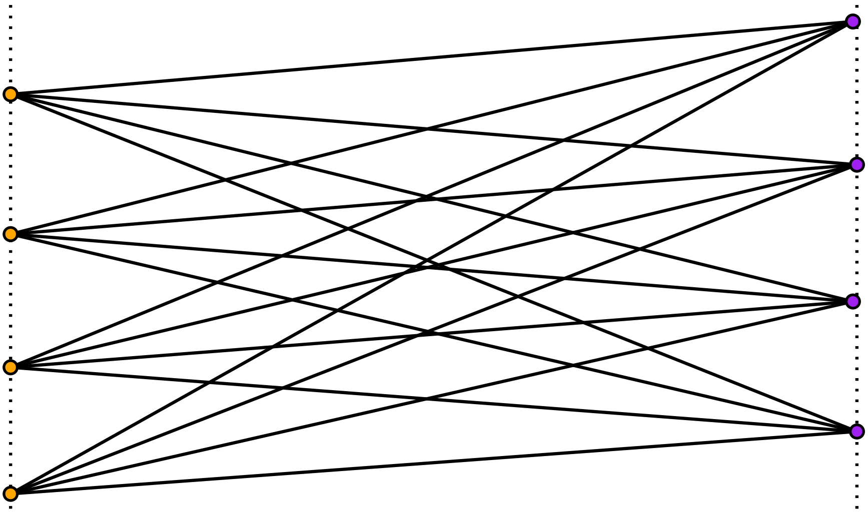

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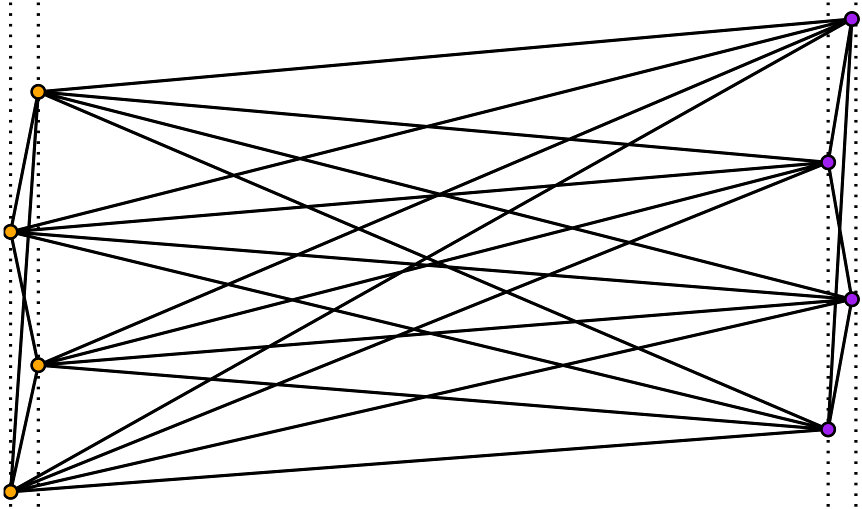

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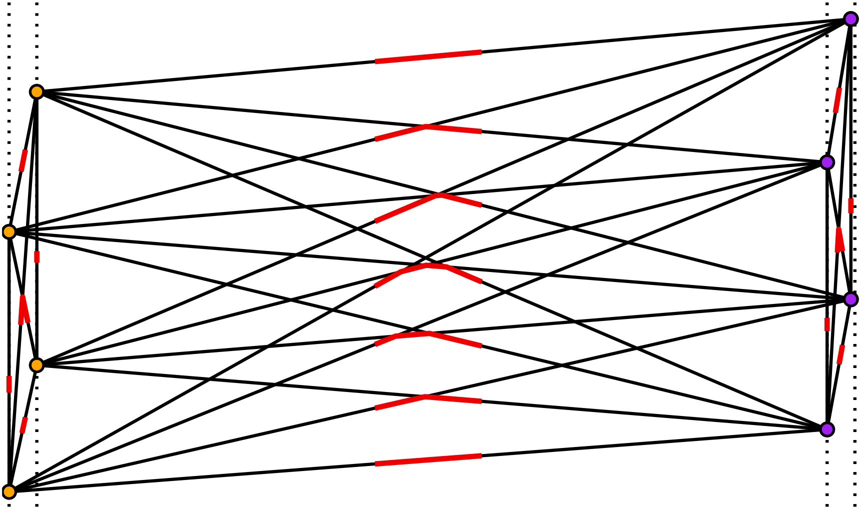

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Theorem

For all $n, h \in \mathbb{N}$ with $h < n$, we have

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Application III: Complexity of faces in arrangements of segments

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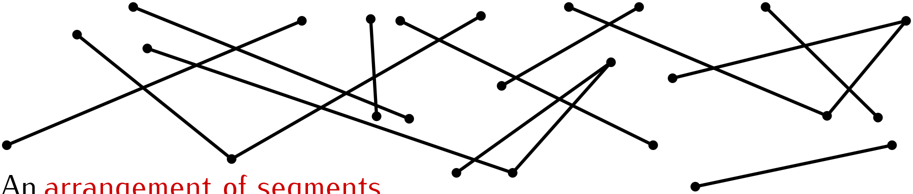

An arrangement of segments

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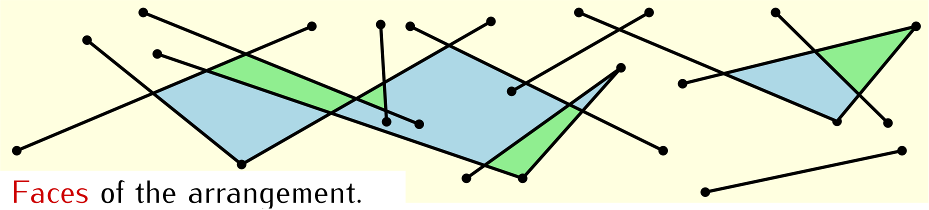

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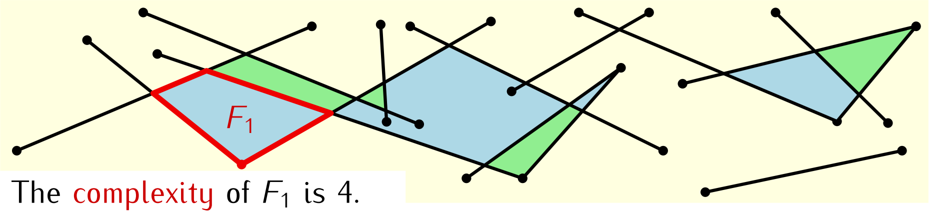

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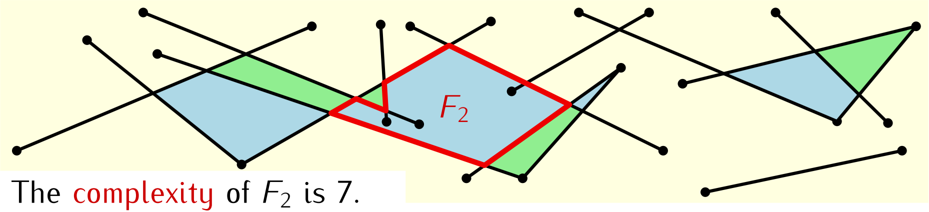

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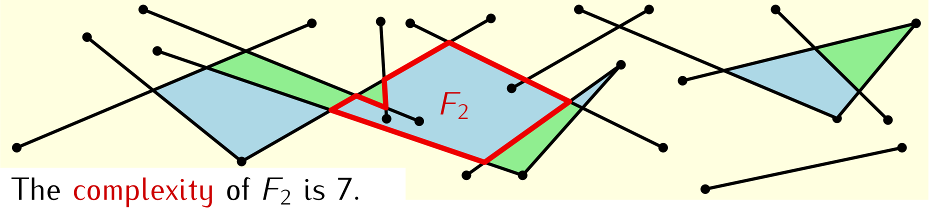

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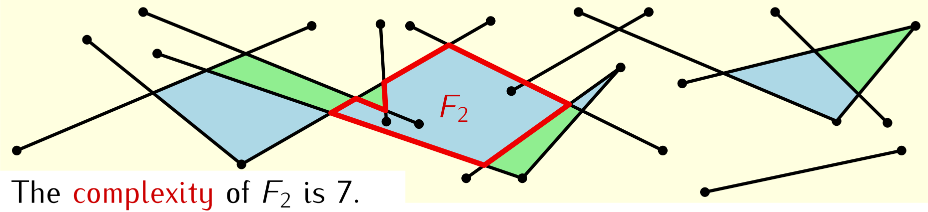

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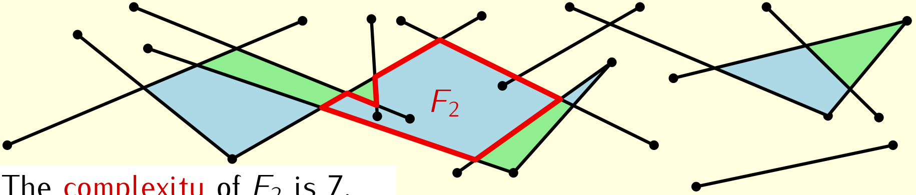

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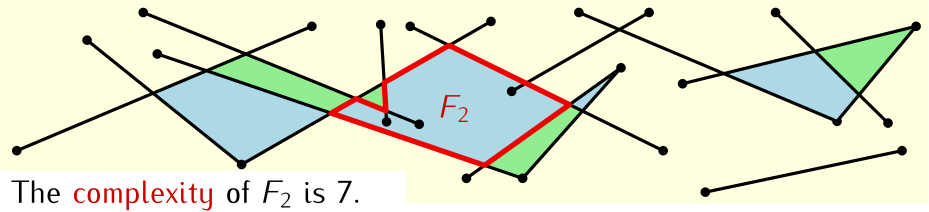

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Tight for $M \geq n \log^{3/2} n$.

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