ALTERNATING PATHS AND CYCLES OF MINIMUM LENGTH

William Evans
University of British Columbia, Canada

Giuseppe Liotta
Universitá degli Studi di Perugia, Italy

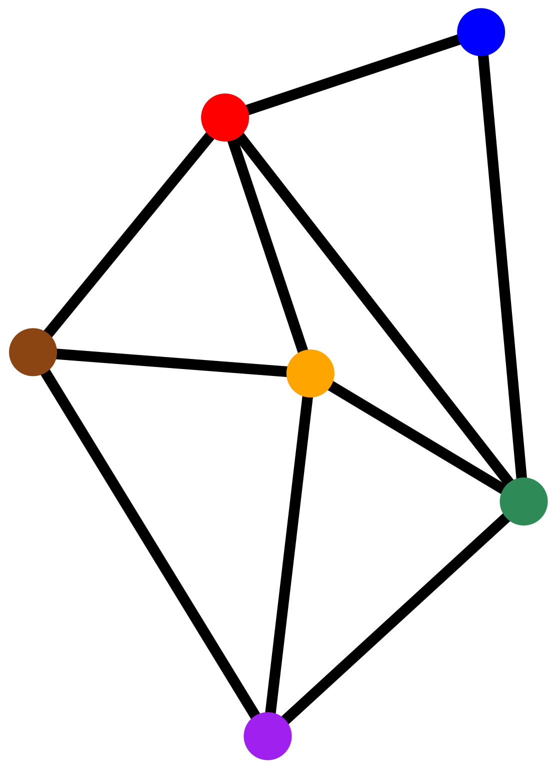
Henk Meijer
U. C. Roosevelt, the Netherlands

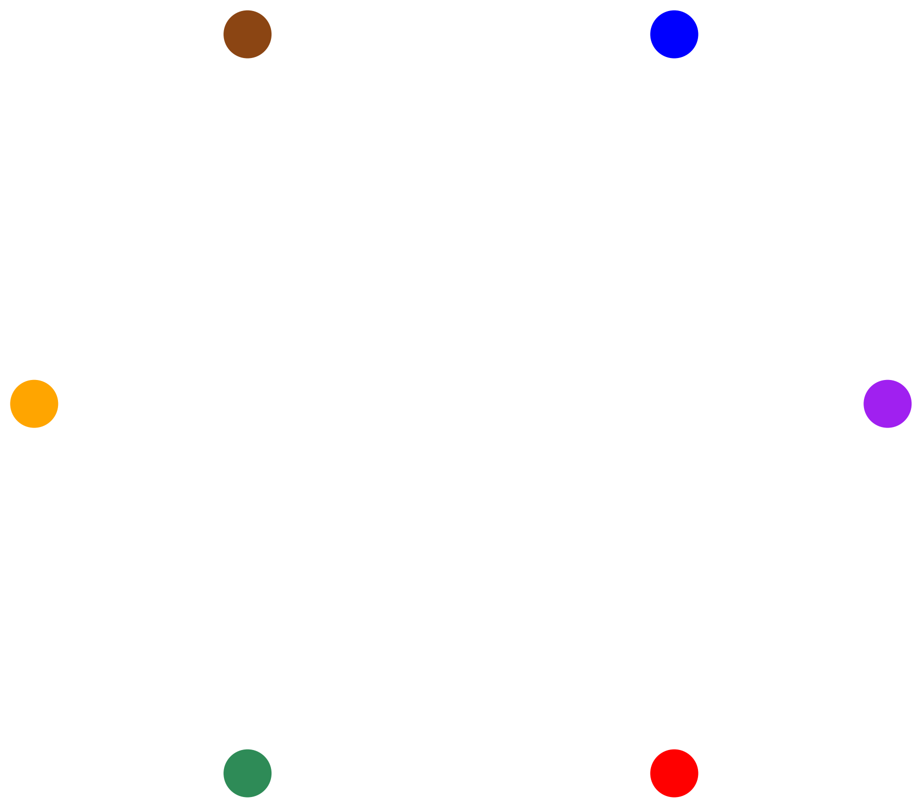
Stephen Wismath
University of Lethbridge, Canada

Motivating problem

Draw planar graph on given point set to minimize total edge length [Chan et al. GD13]

- vertex maps to point of the same color
- each vertex has distinct color





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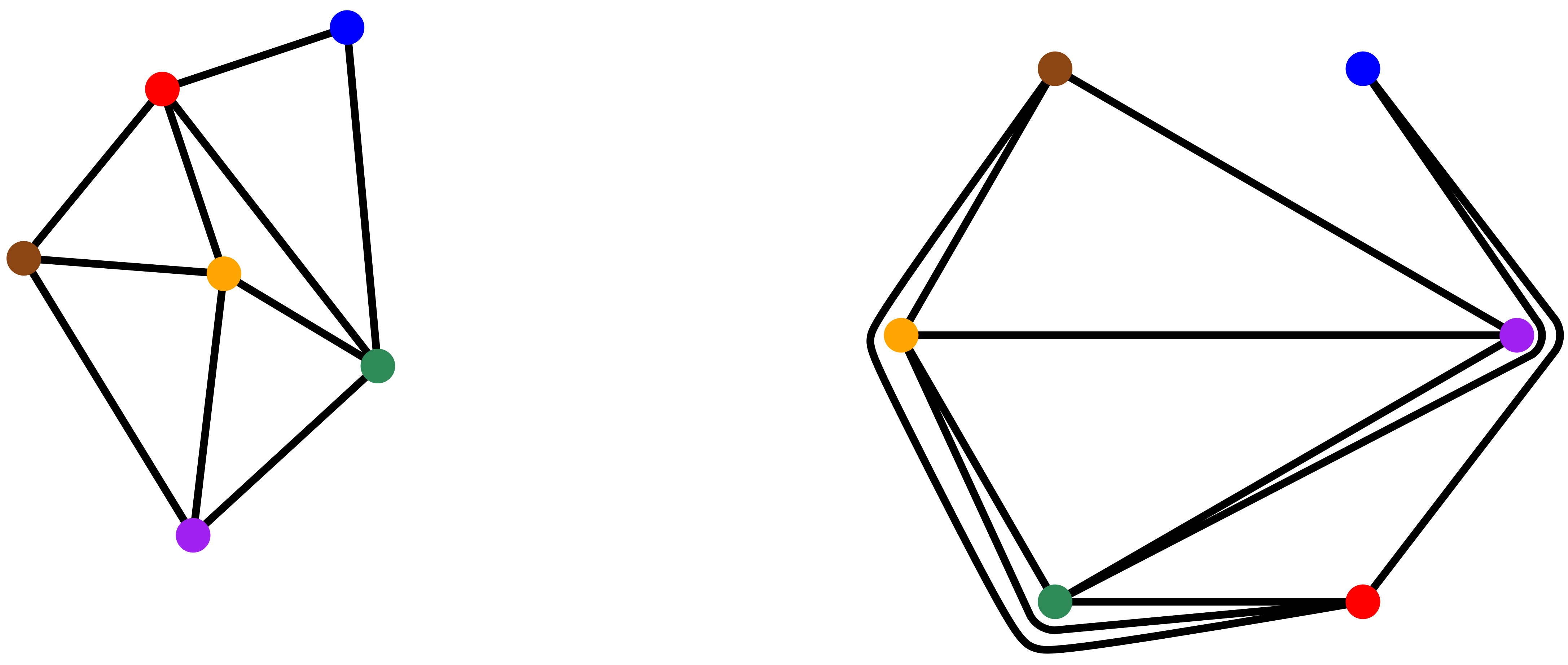
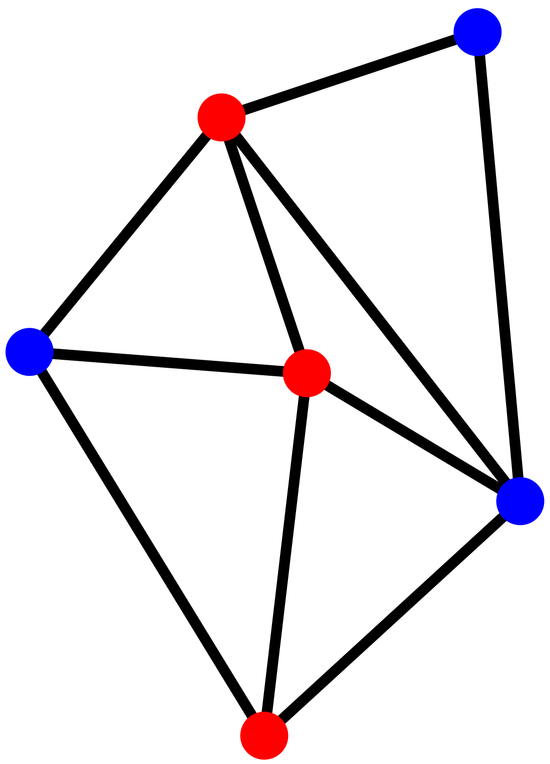


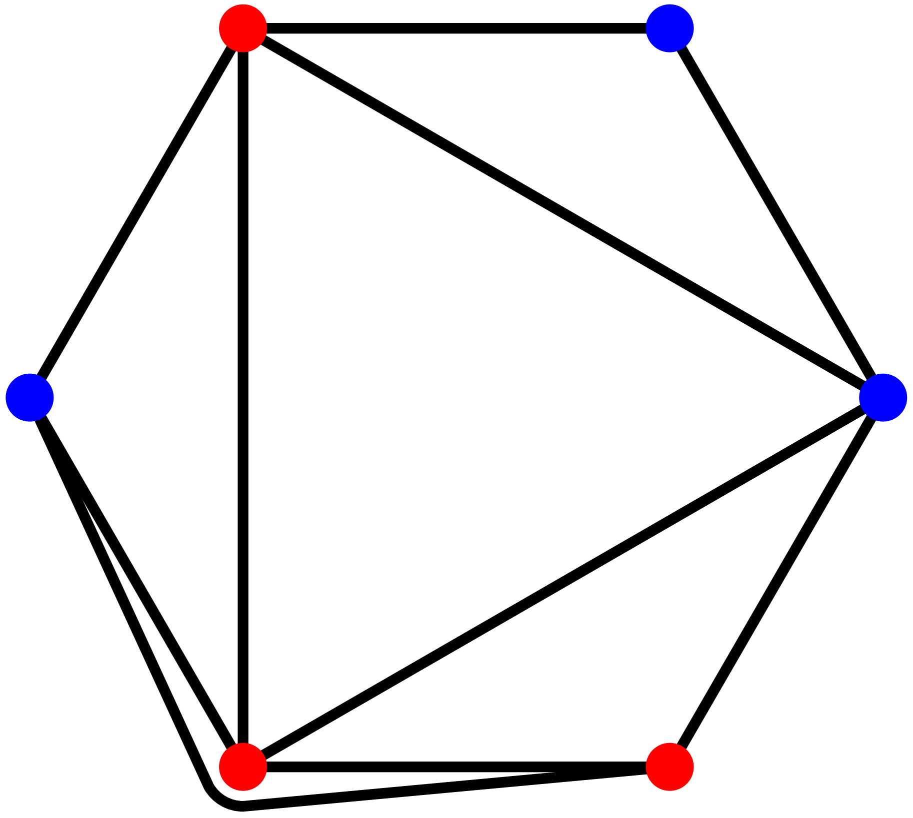
This is NP-hard [Bastert and Fekete 96]

Our problem

Draw planar graph on given point set to minimize total edge length

- vertex maps to point of the same color
- ~~each vertex has distinct color~~



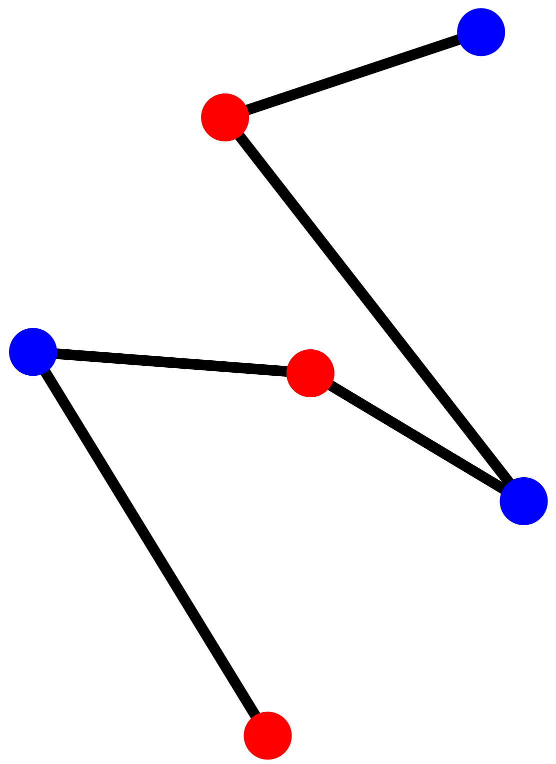


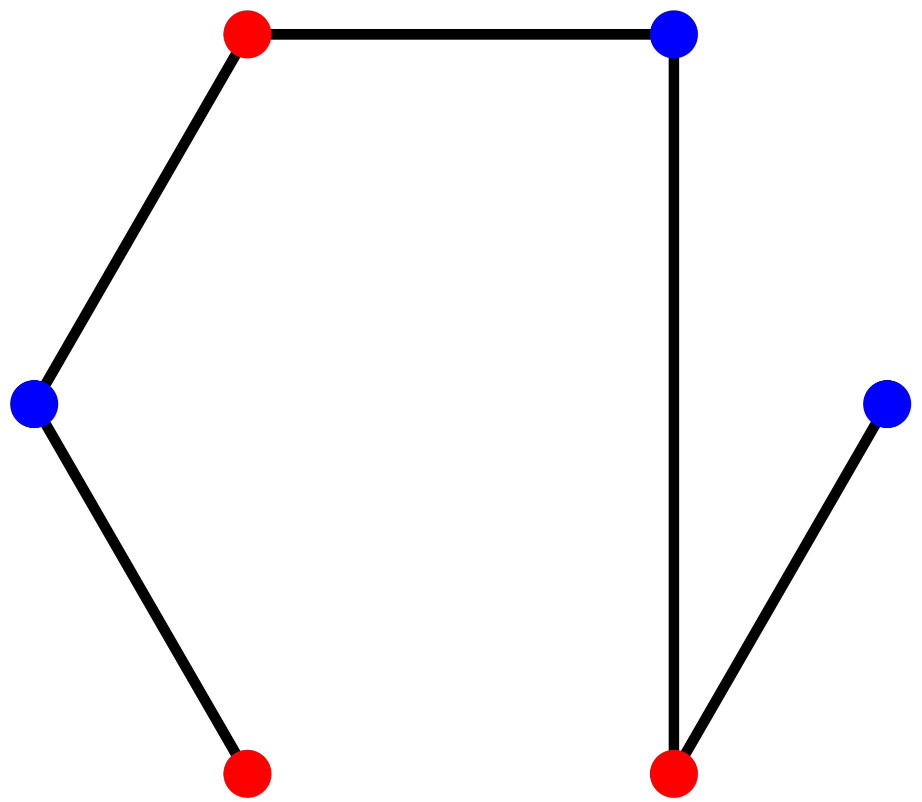
Our problem

alternating path/cycle

Draw planar ~~graph~~ on given point set to minimize total edge length

- vertex maps to point of the same color
- ~~each vertex has distinct color~~



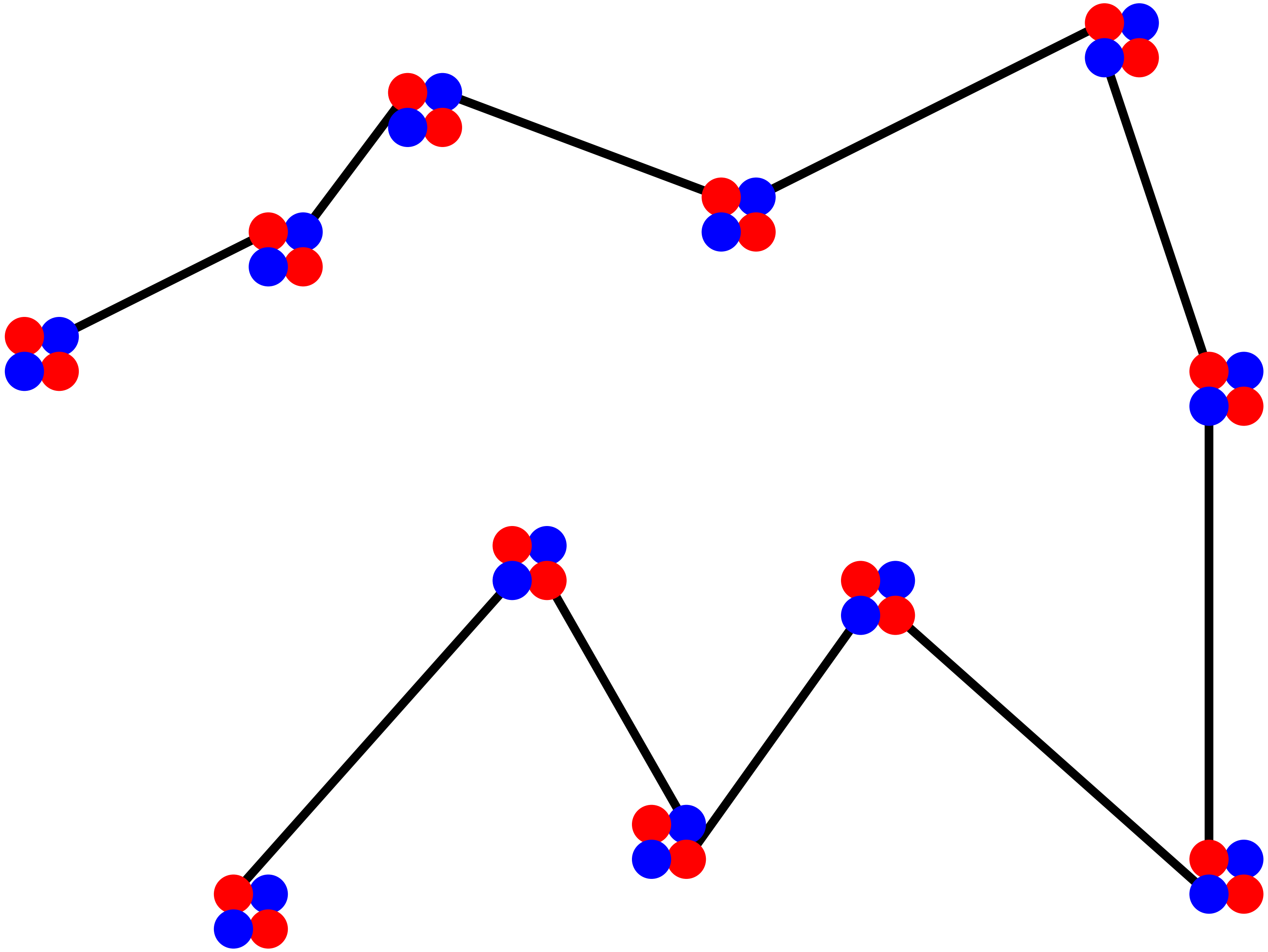


Our problem is NP-hard

Draw planar alternating path/cycle on given point set to minimize total edge length

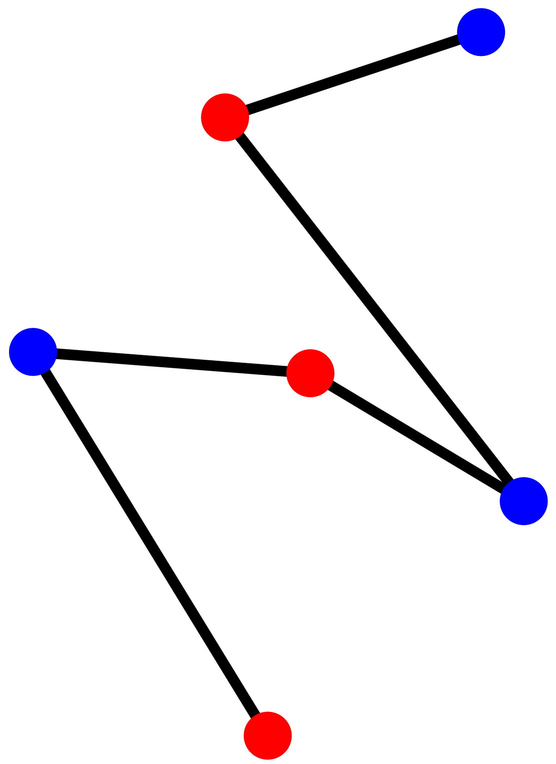
Idea: Reduce from EXACT COVER

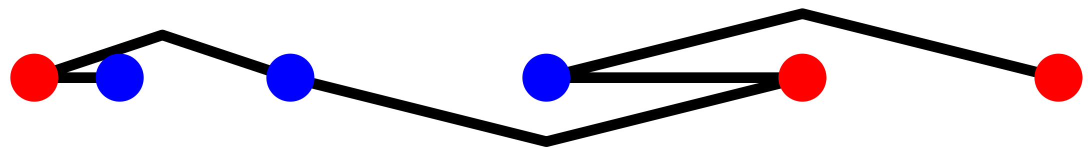
Use (modified) reduction to Euclidean TSP [Papadimitriou 77]



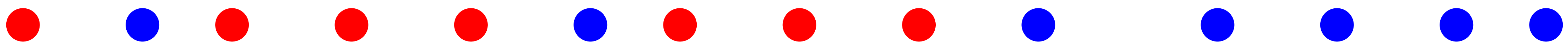
Our problem

Draw planar alternating path/cycle on given colinear point set to minimize total edge length





Cycles: A lower bound



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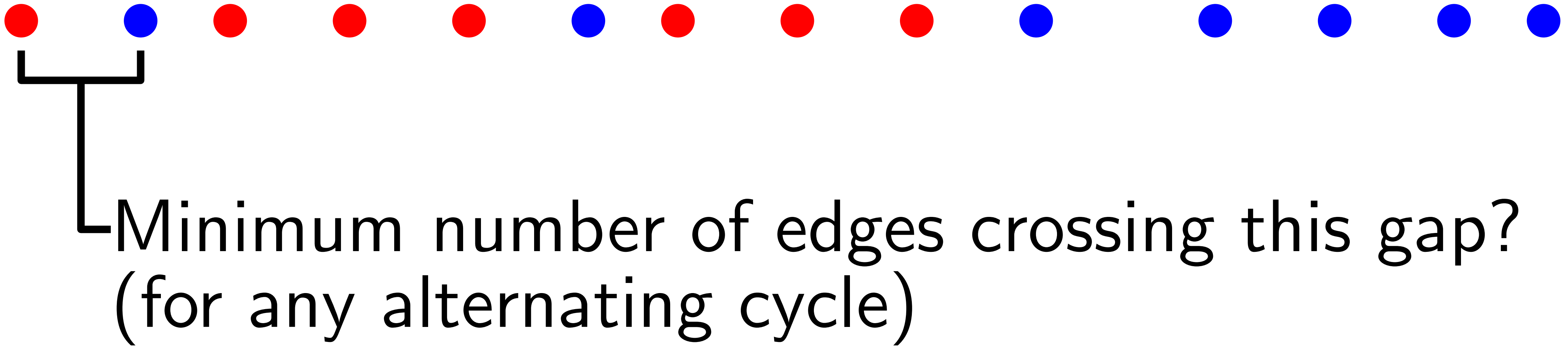

Cycles: A lower bound

Minimum number of edges crossing this gap?
(for any alternating cycle)

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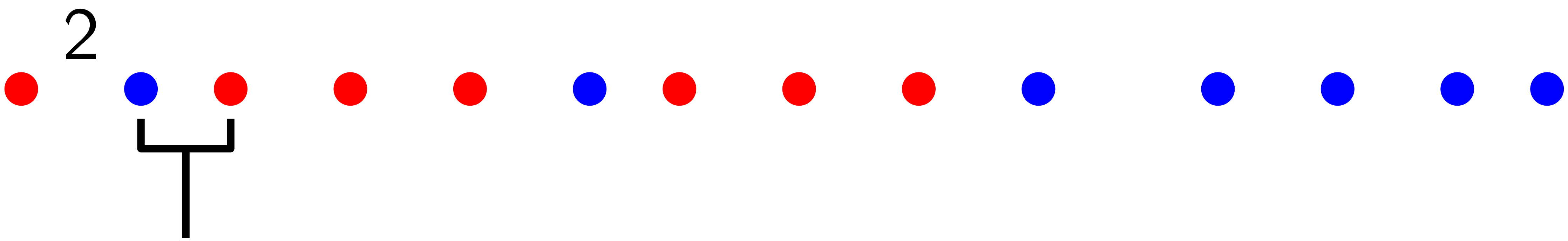

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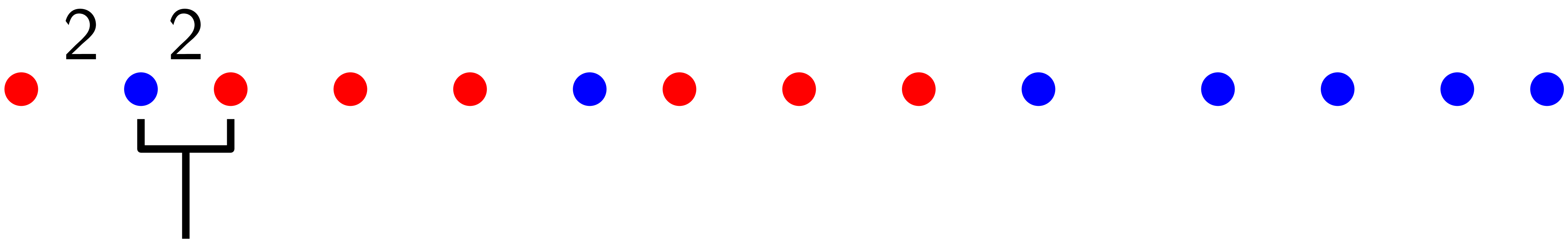

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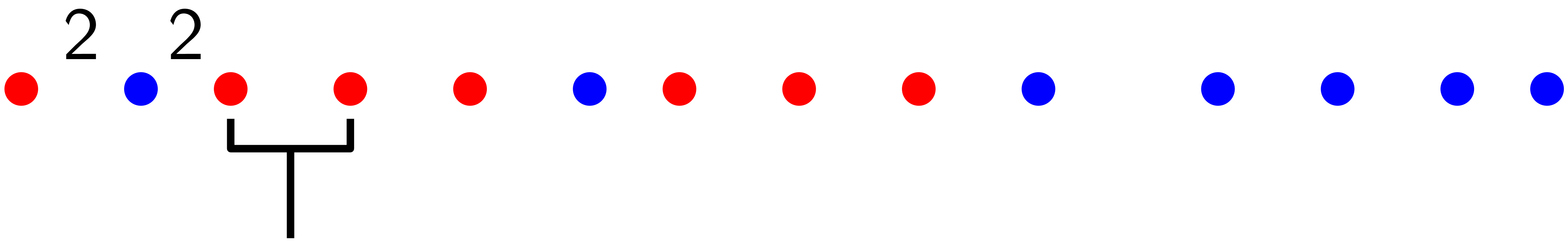

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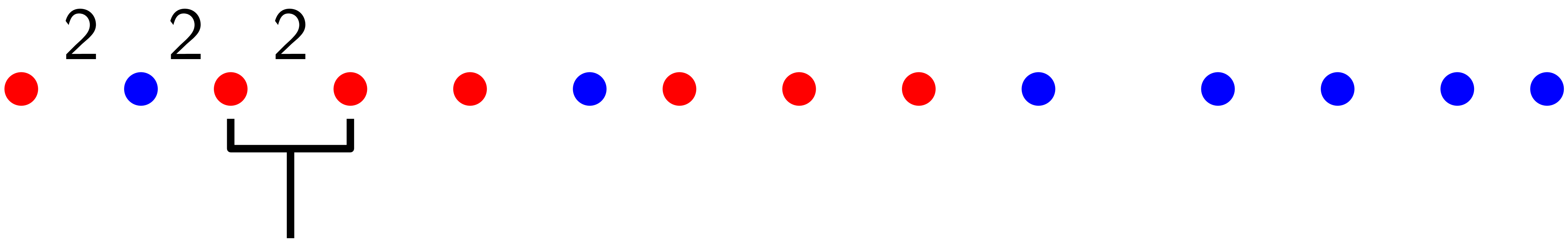

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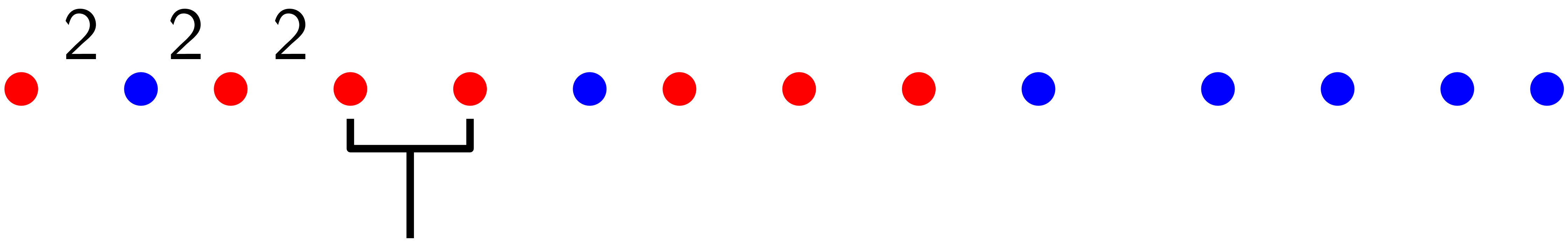

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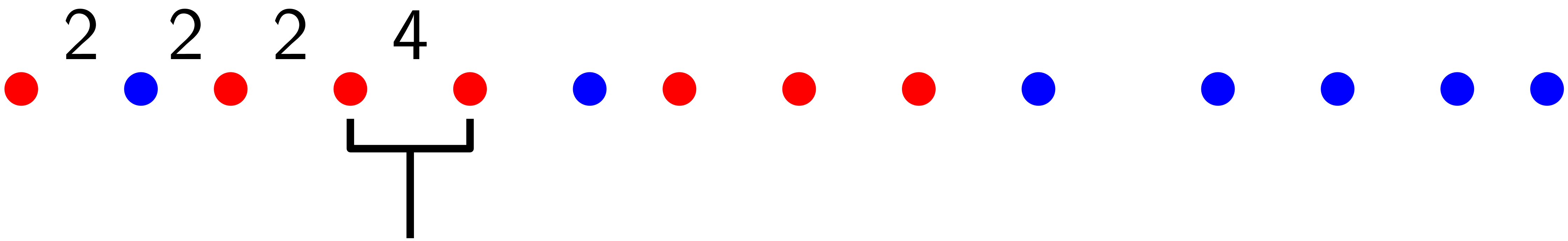

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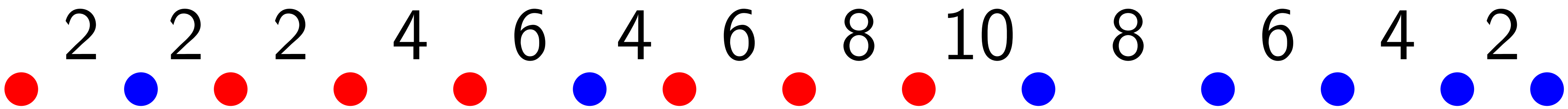
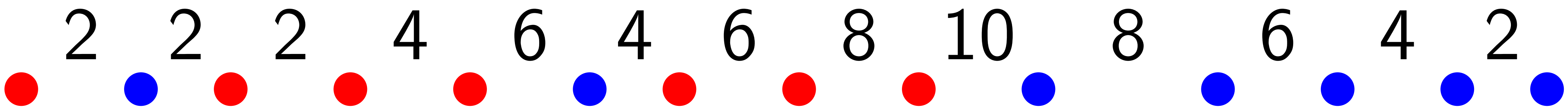


Lemma 1. Minimum number of edges crossing gap$_i$ is

$$c_i = 2\max\{|r_i - b_i|, 1\}$$

r_i = red, b_i = blue points before gap$_i$

Cycles: A lower bound



Lemma 1. Minimum number of edges crossing gap_i is

$$c_i = 2\max\{|r_i - b_i|, 1\}$$

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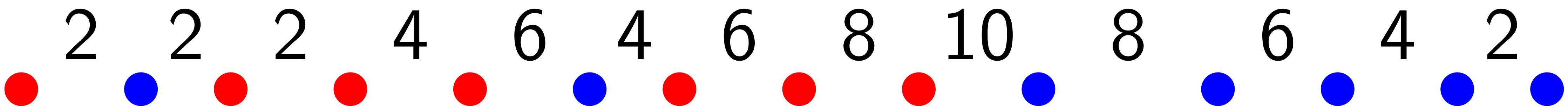
Proof: Each cycle component to the left of gap_i has the same number of red and blue points ± 1.

Cycles: A lower bound



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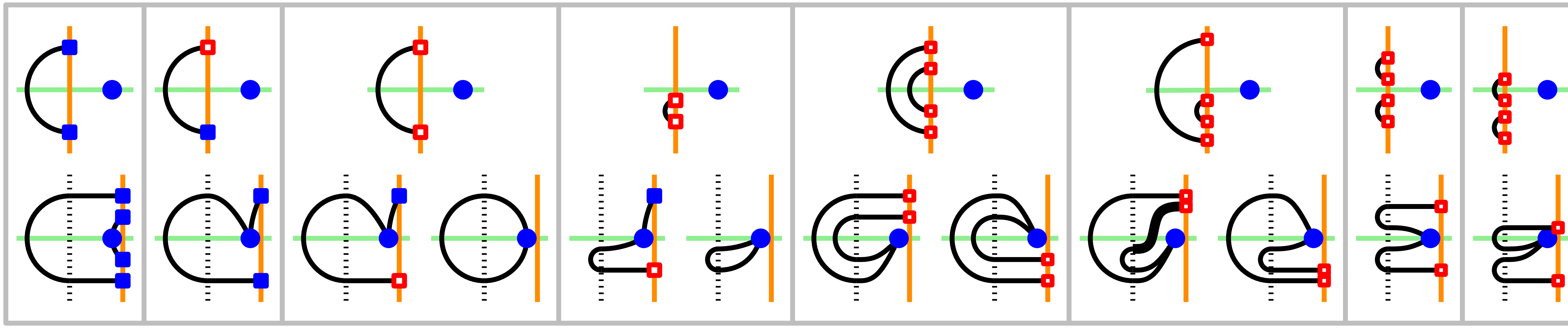
Cycle length $\geq \sum_i c_i |\text{gap}_i|$

Cycles: Matching the lower bound with 2 bends

Drawing Rules



Invariants

At gap$_i$:

1. Number of red components $= \max\{r_i - b_i, 0\}$.
2. Number of blue components $= \max\{b_i - r_i, 0\}$.
3. If $r_i = b_i$, one red/blue component spans spine.
4. Two closest components to spine are not both above or below.

Cycles: Matching the lower bound with 2 bends

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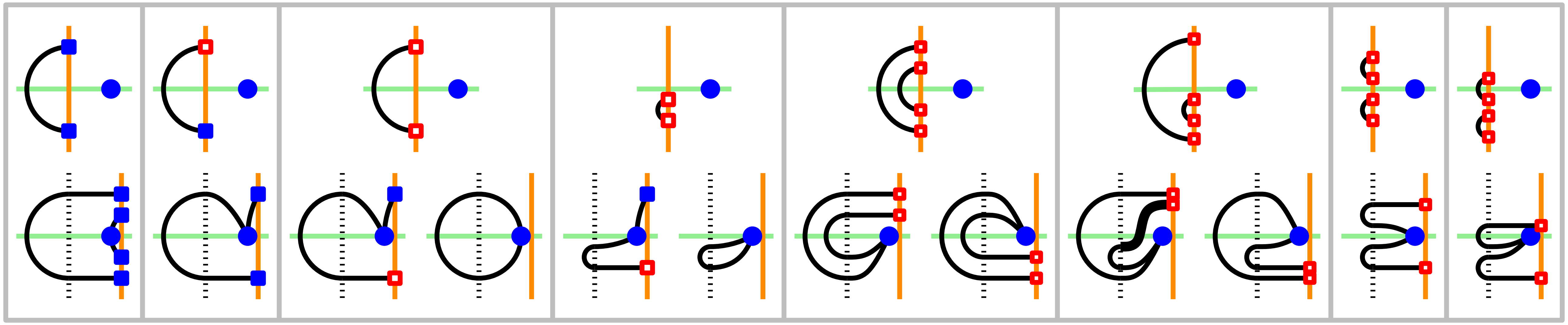

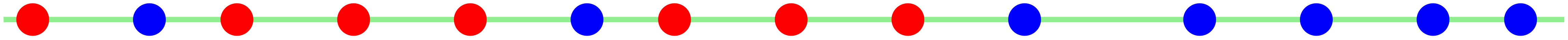

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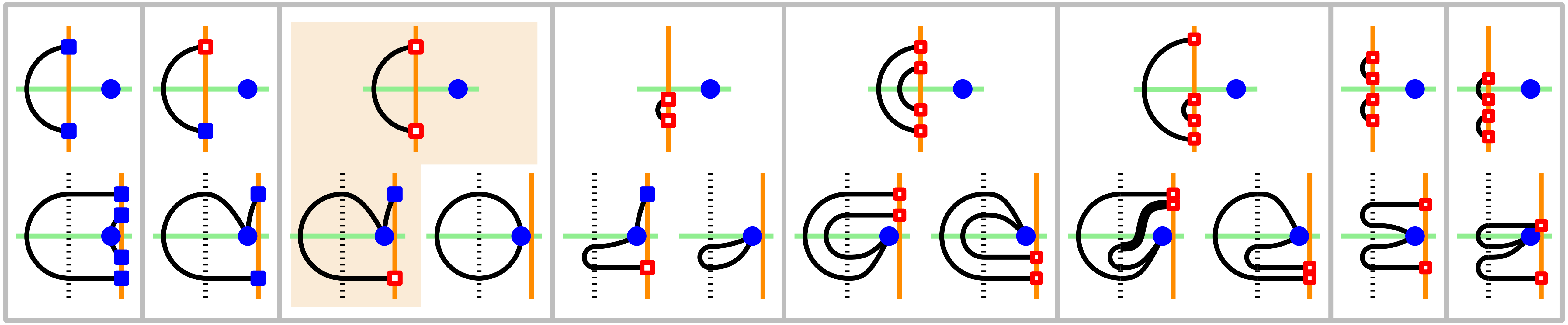

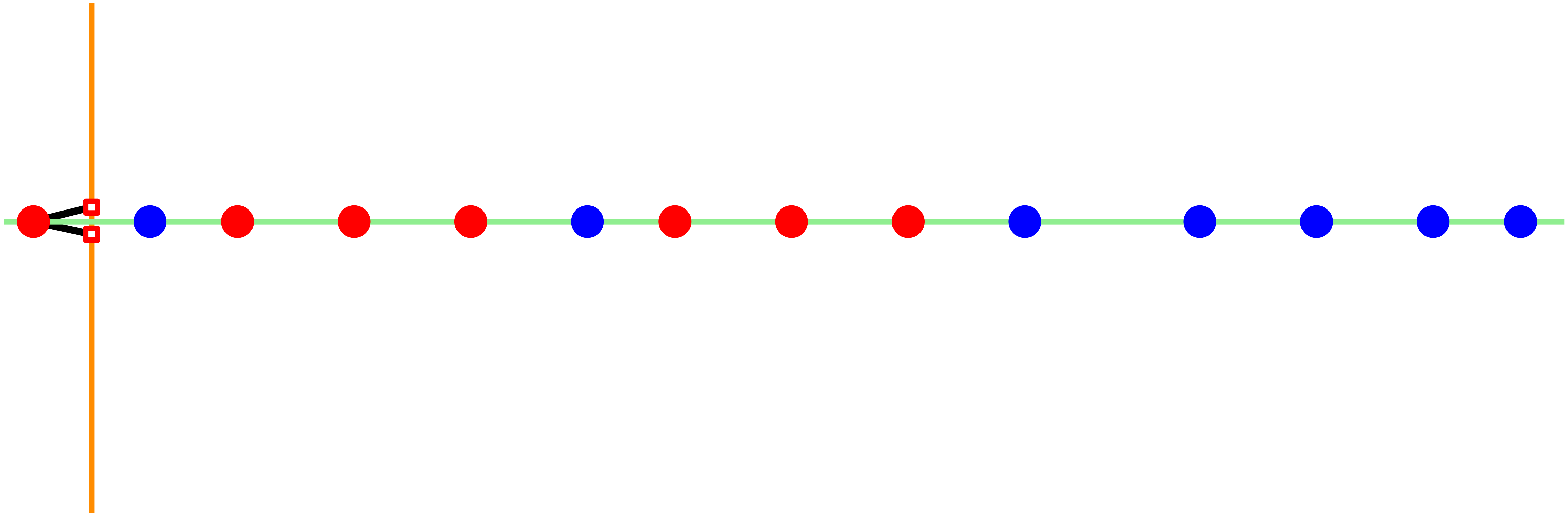

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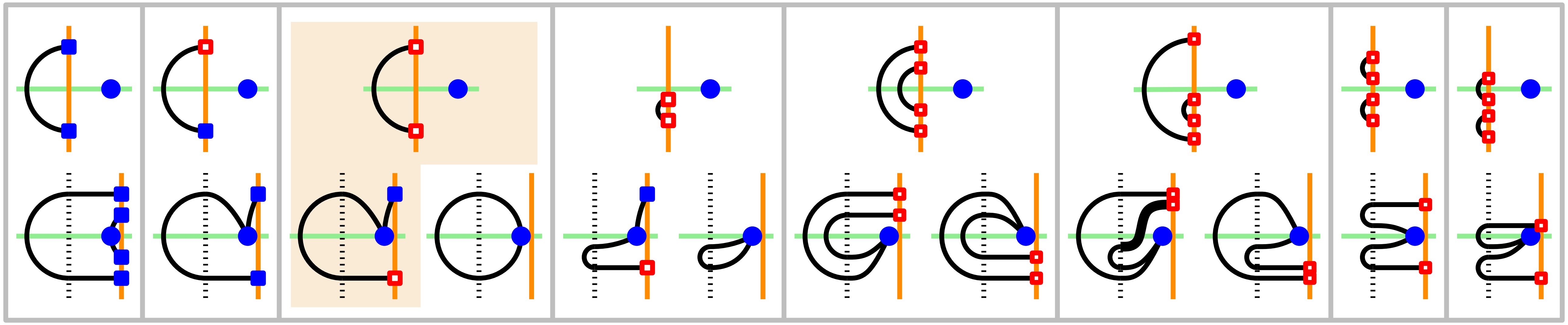

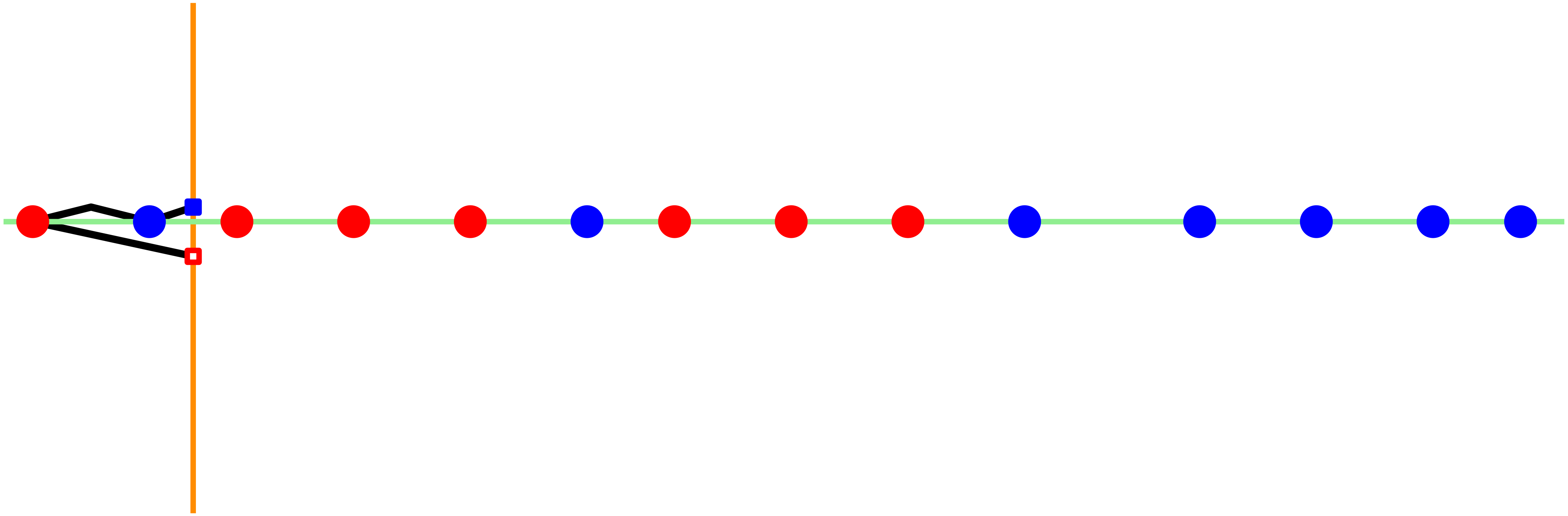

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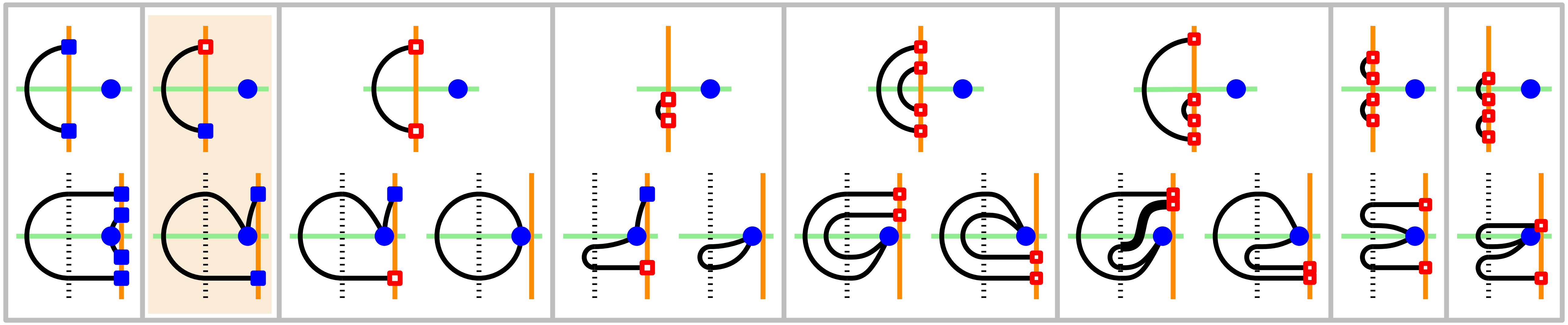

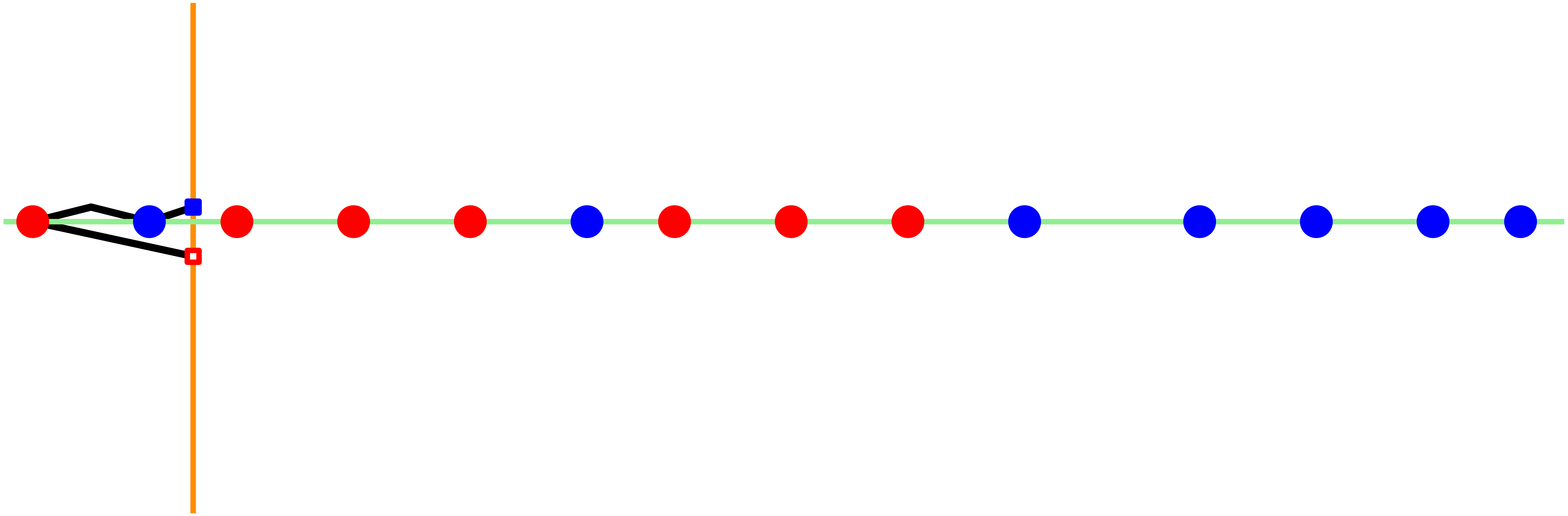

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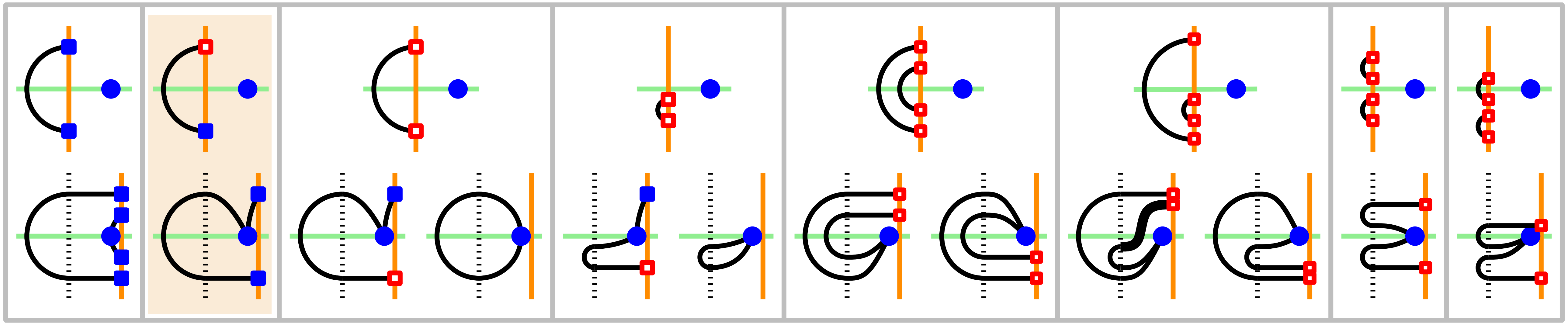

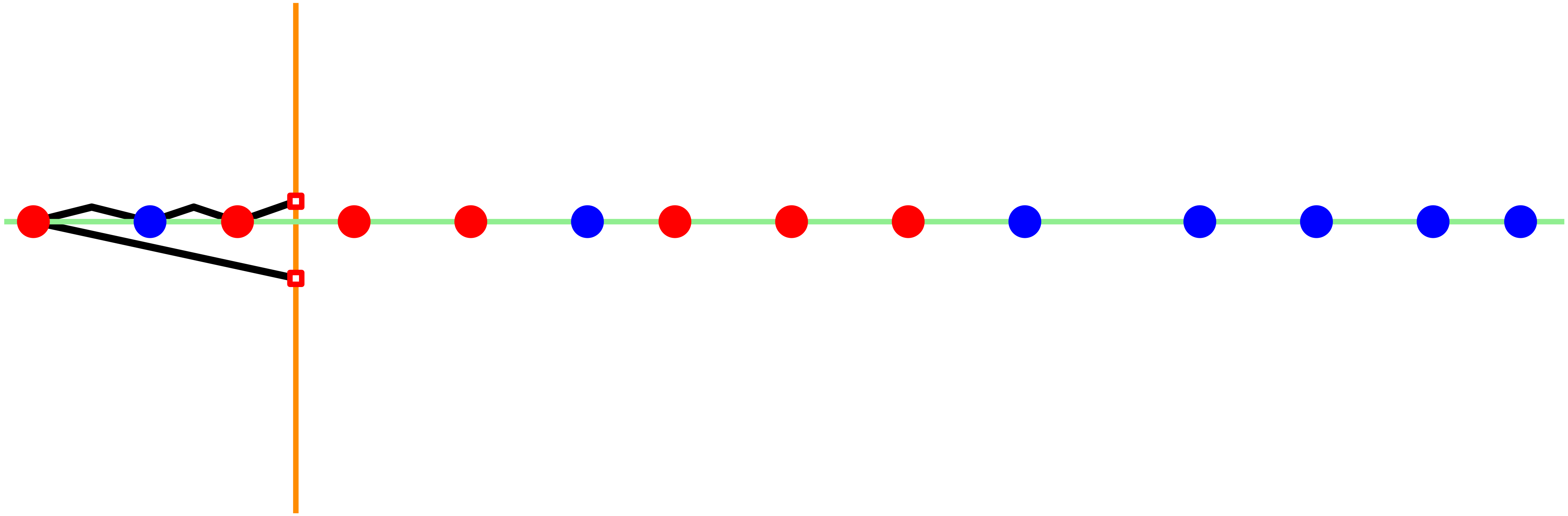

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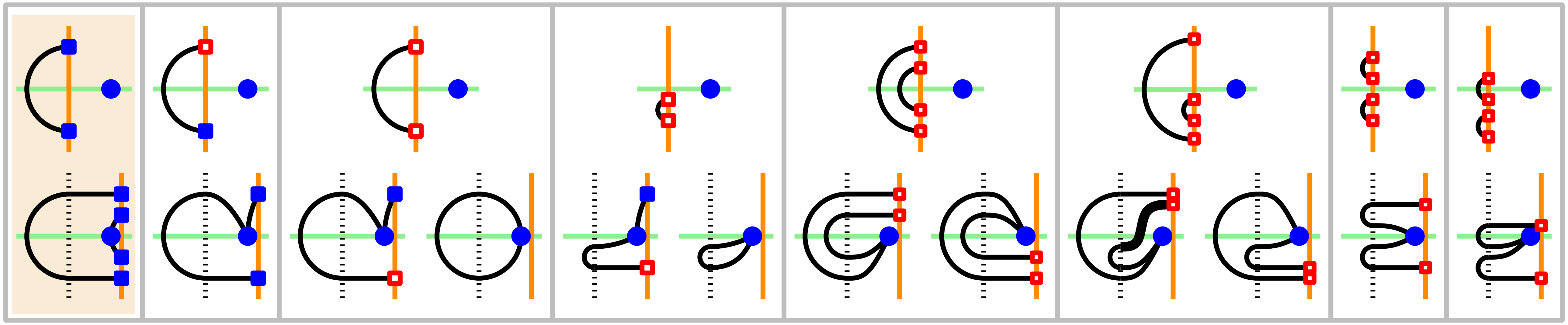

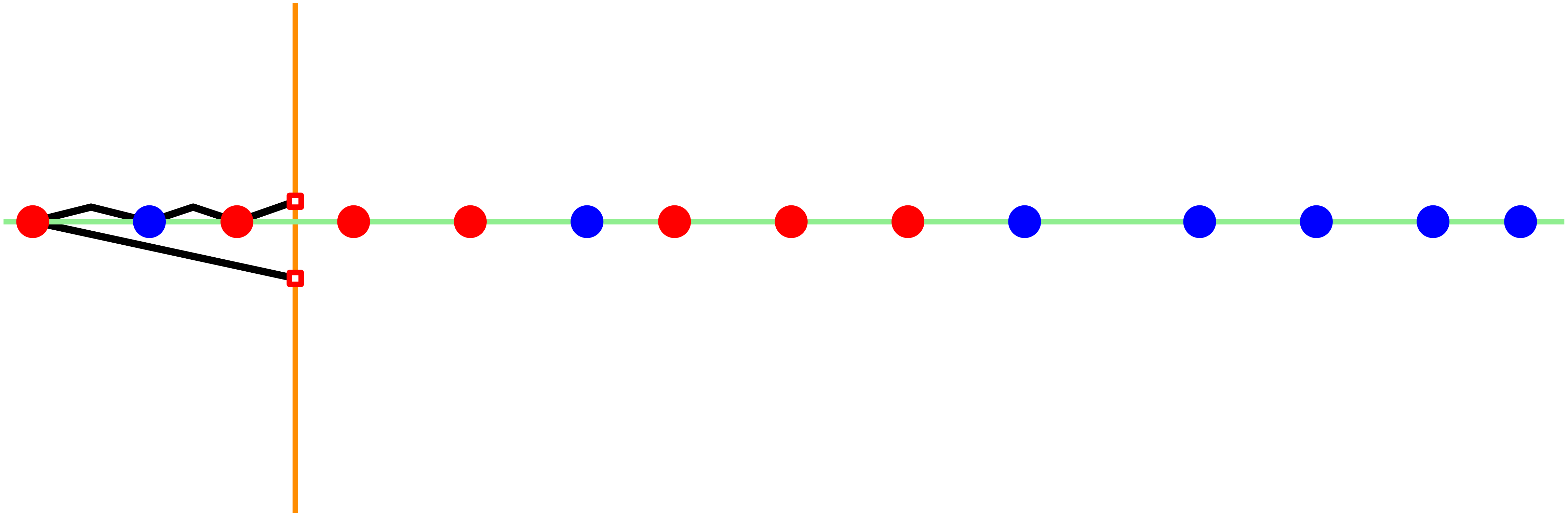

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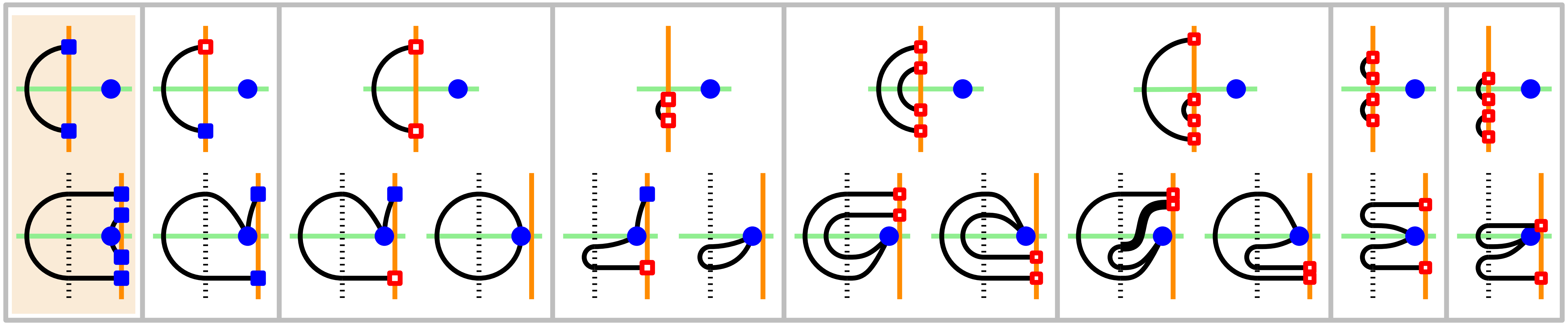

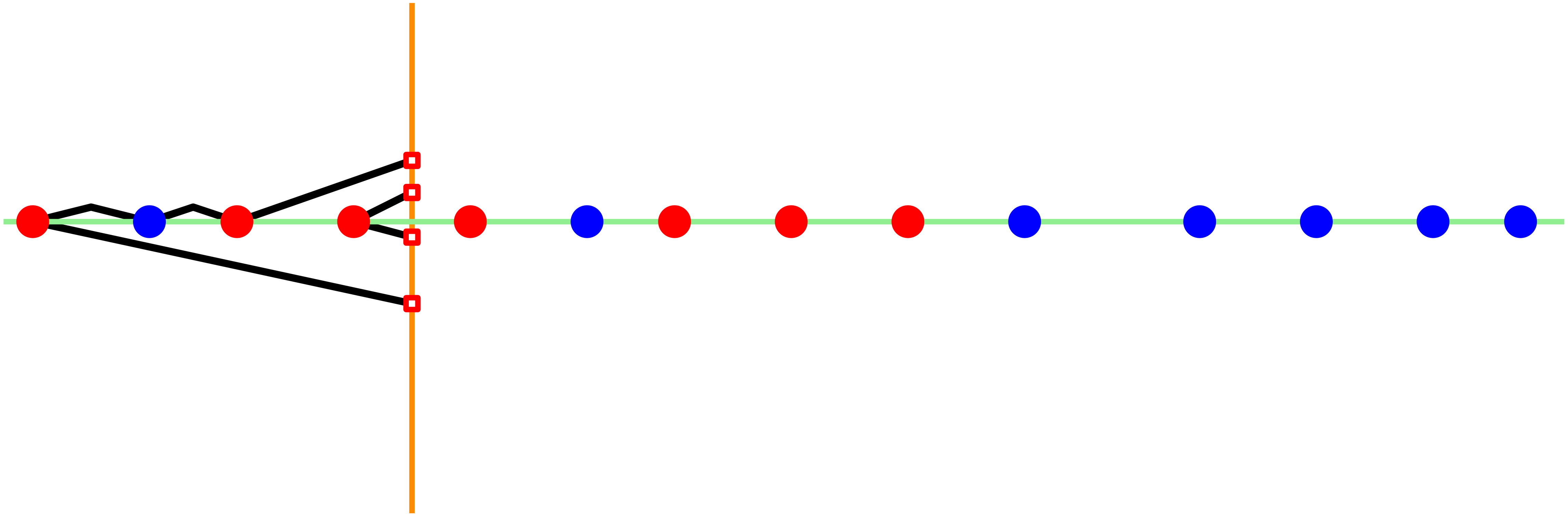

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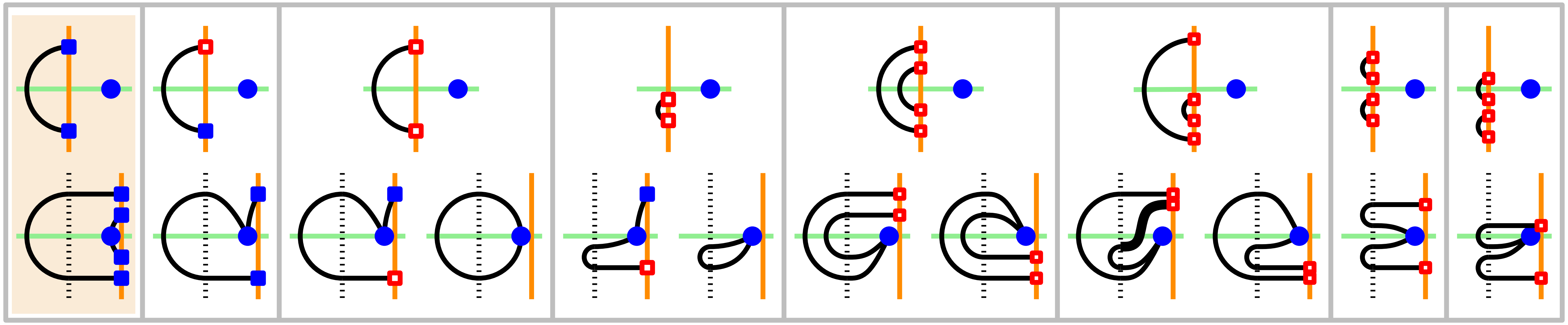

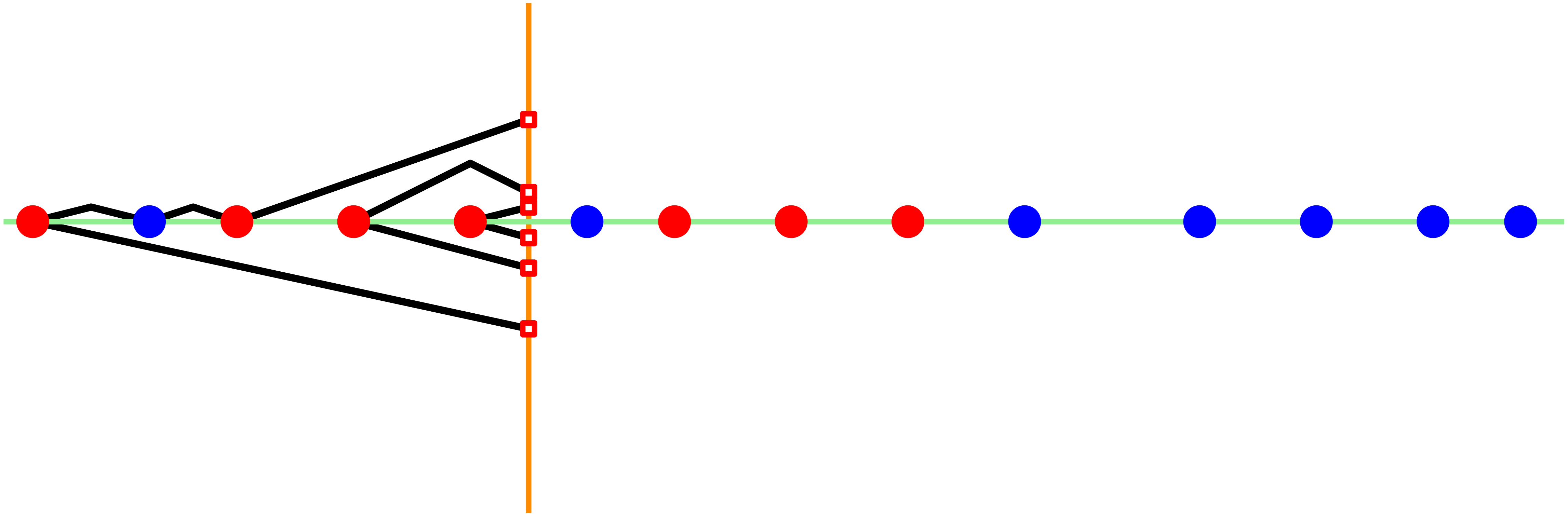

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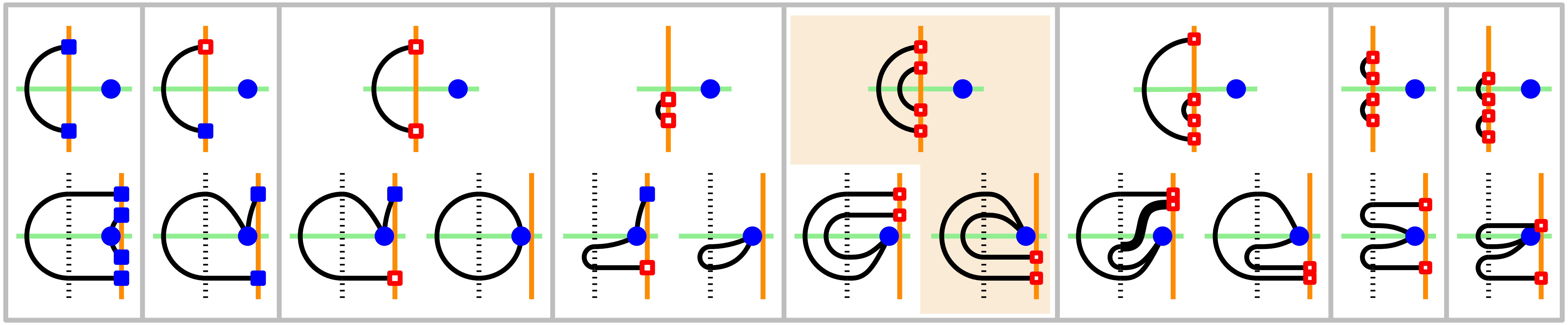

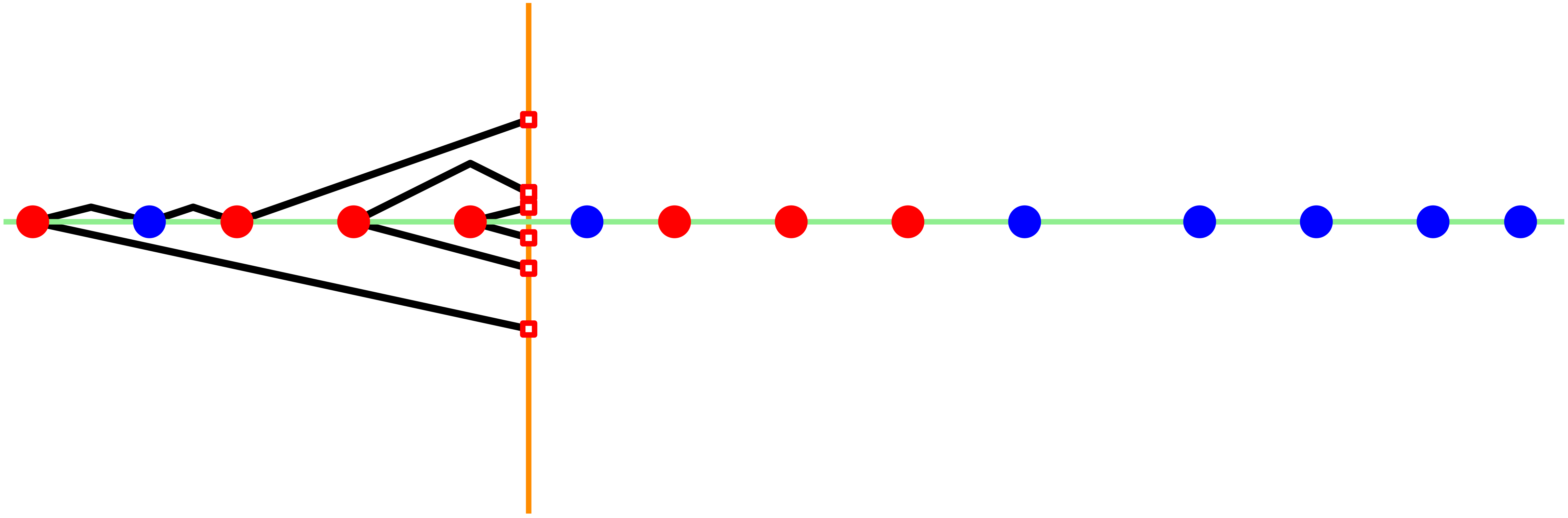

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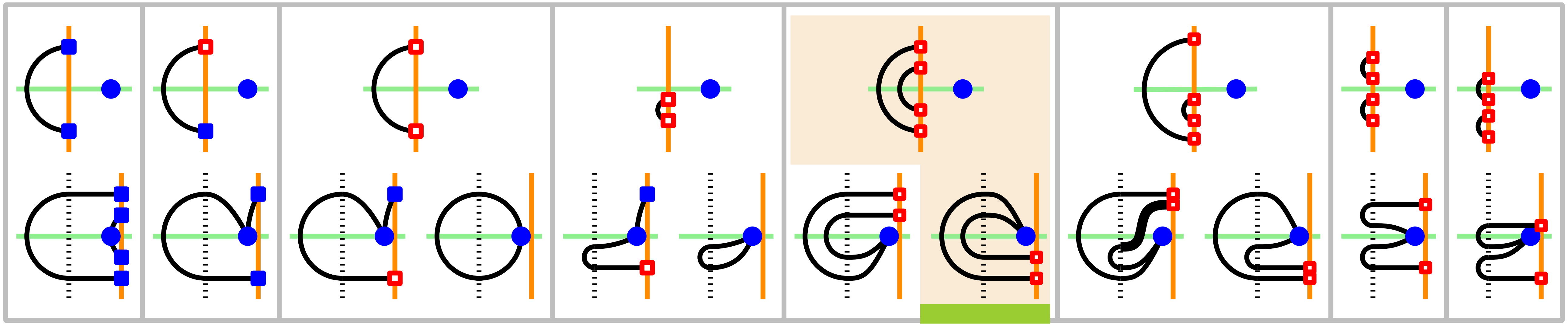

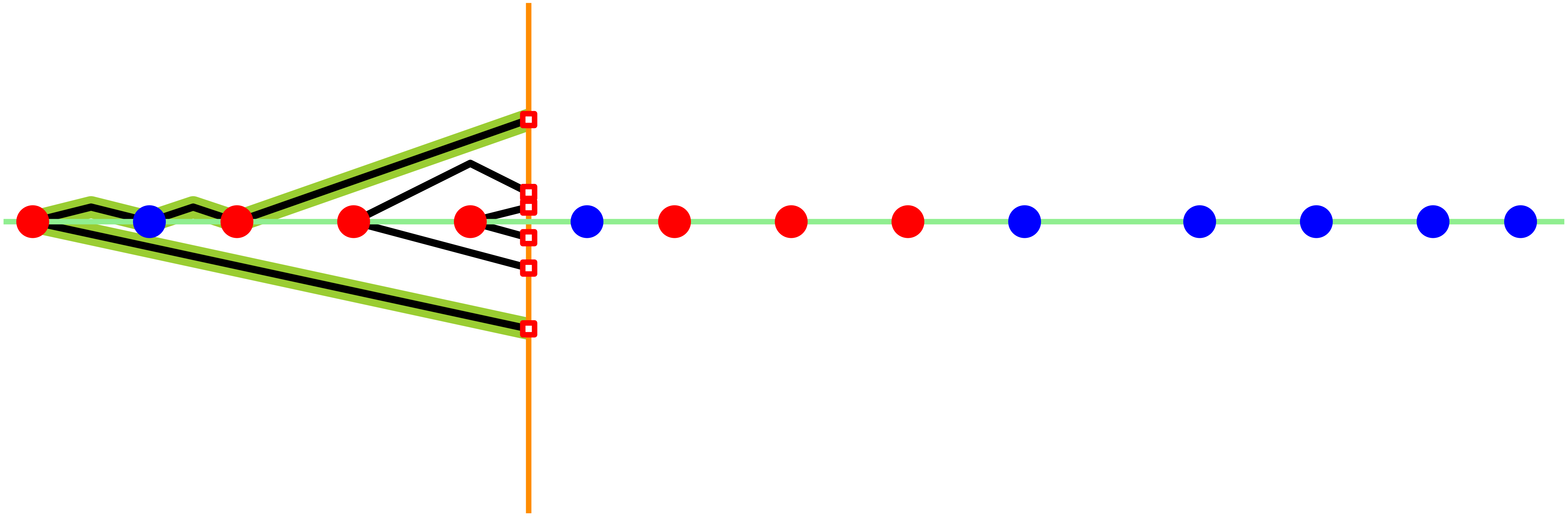

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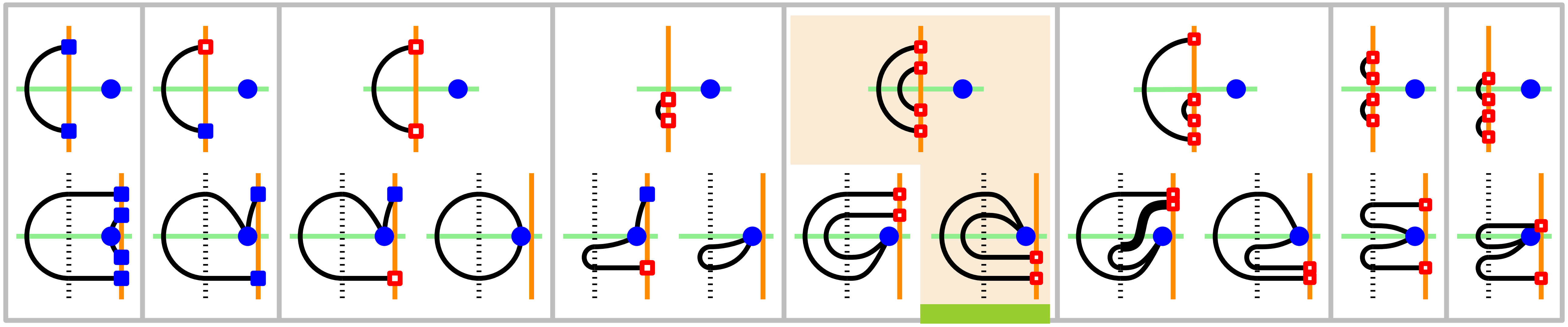

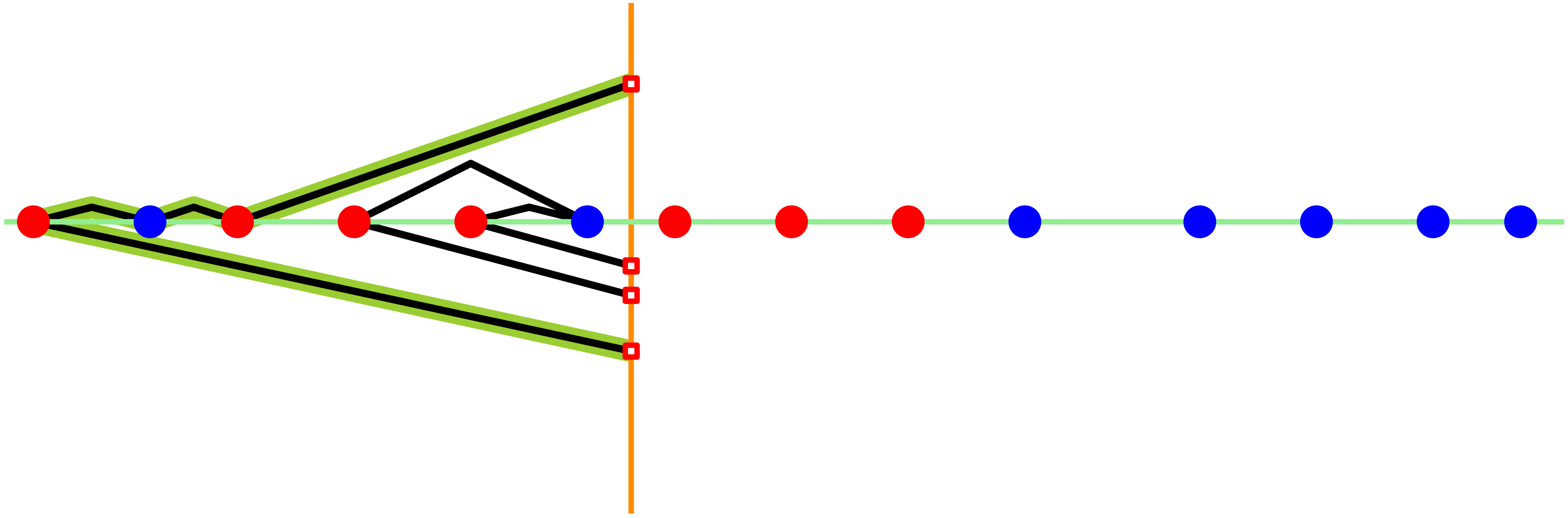

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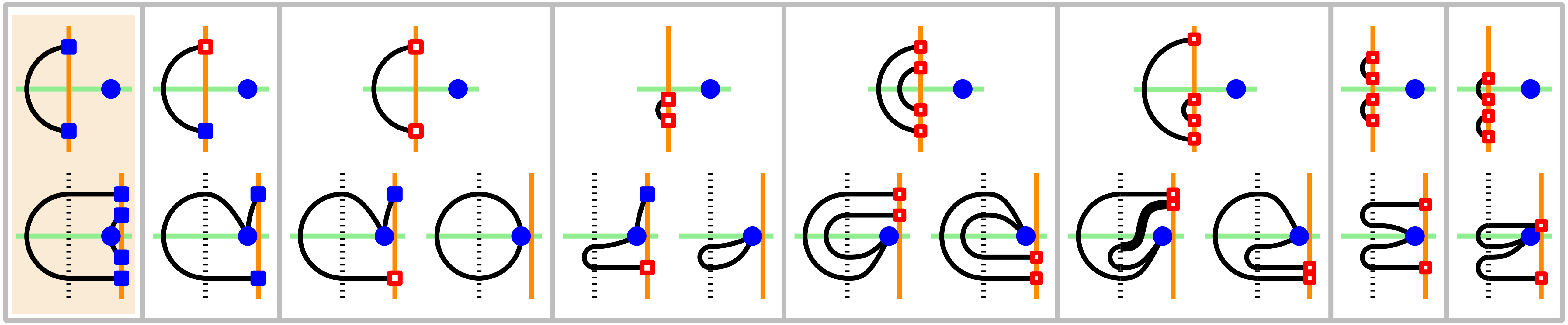

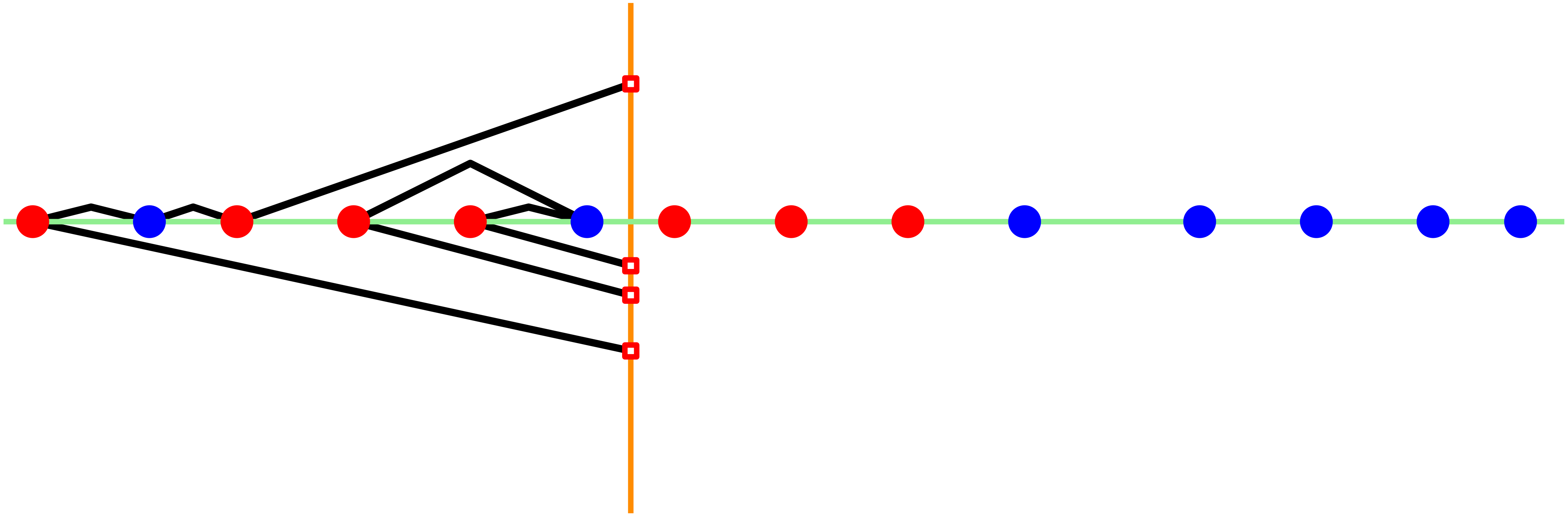

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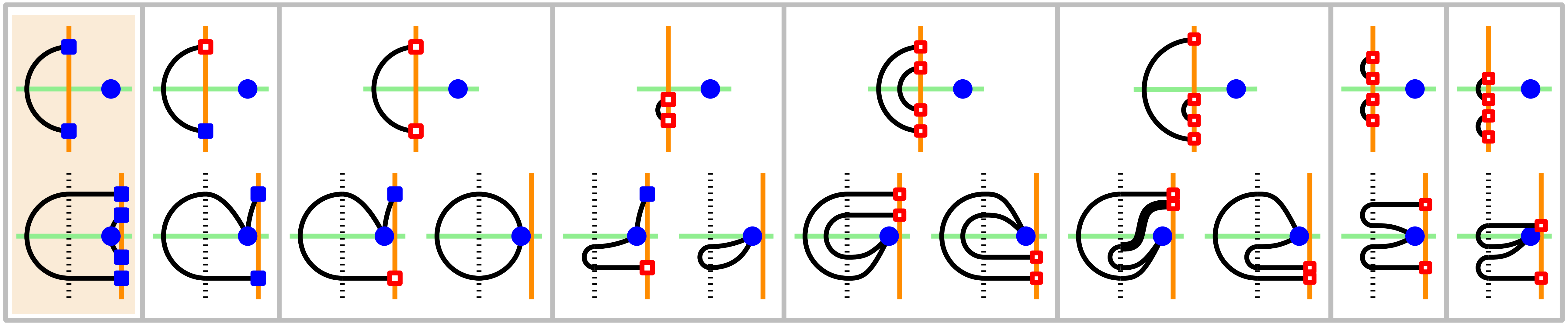

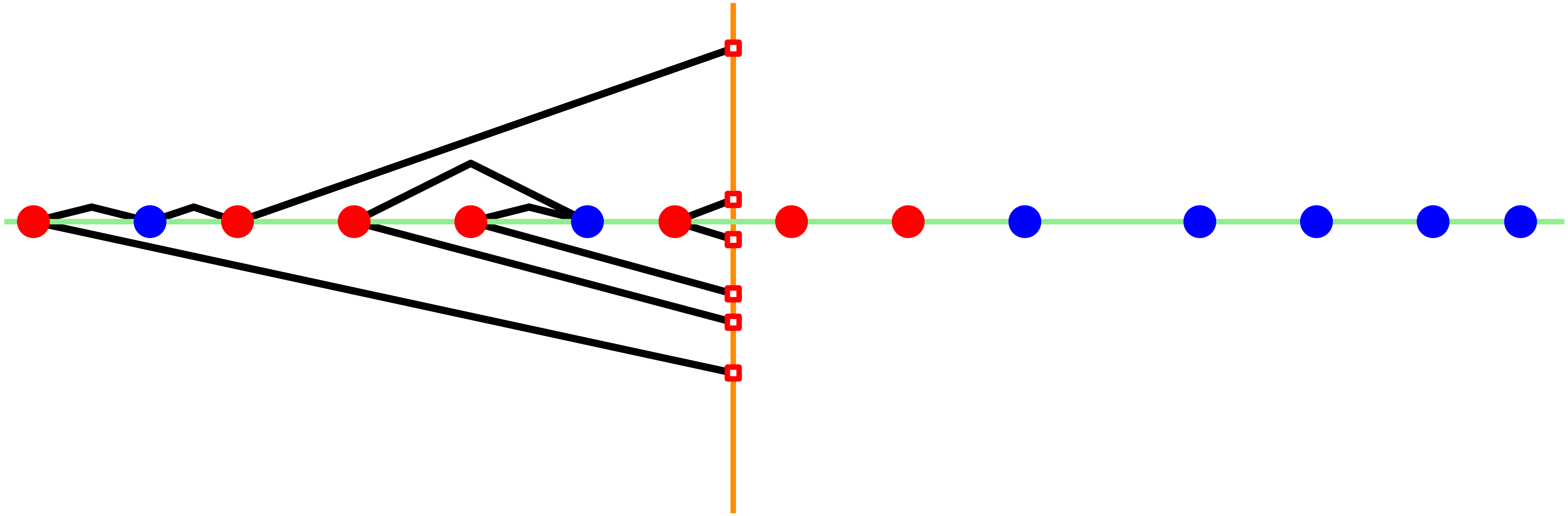

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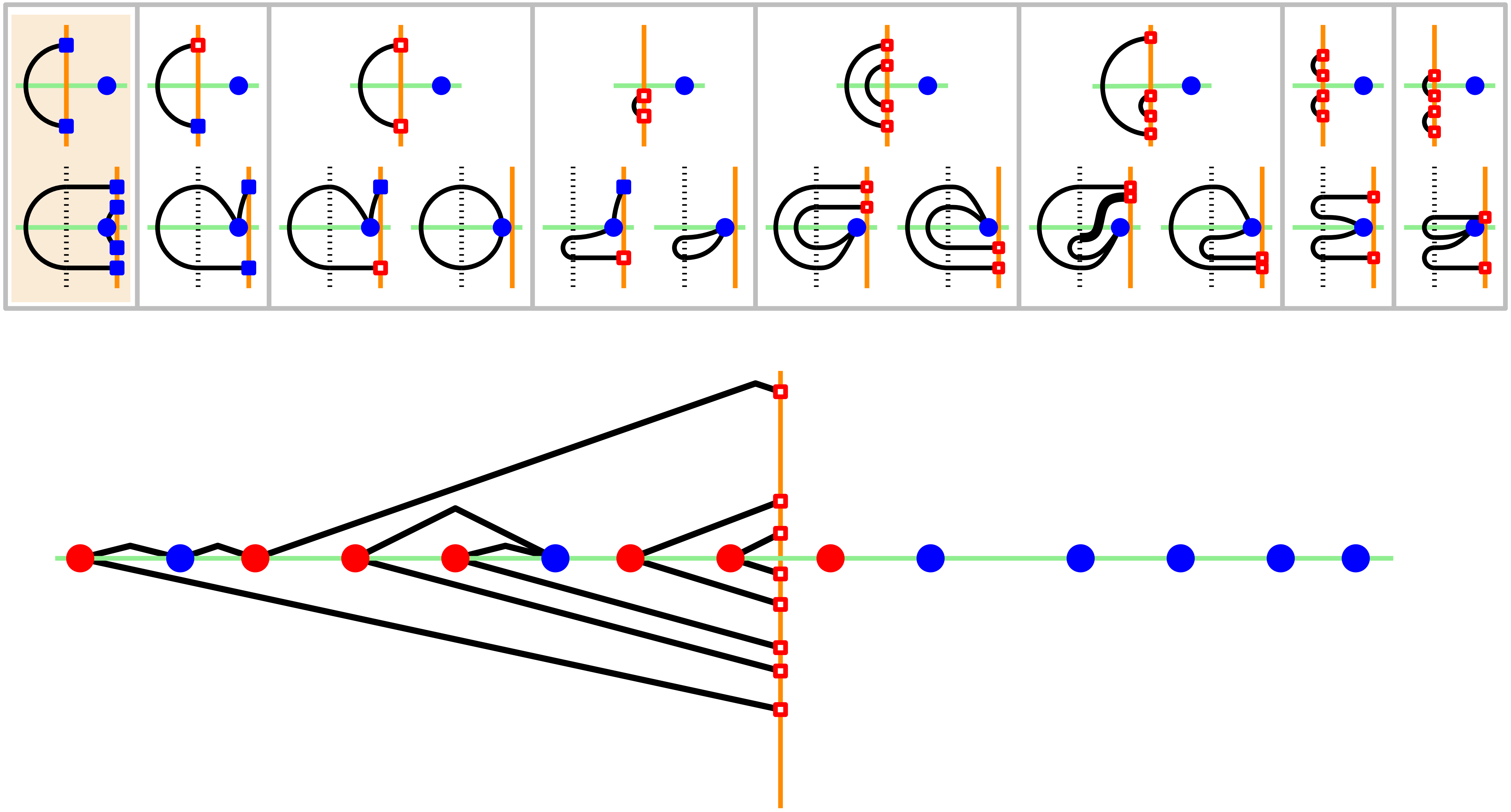

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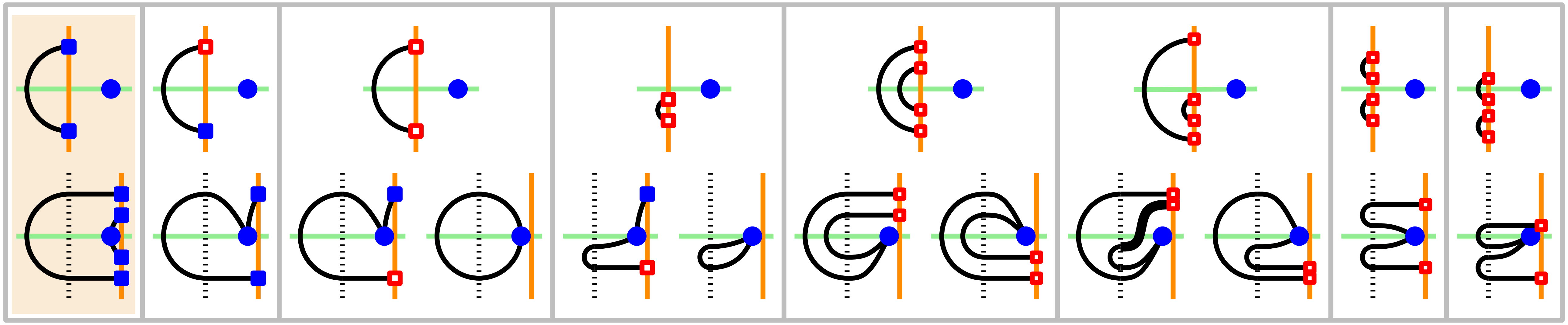

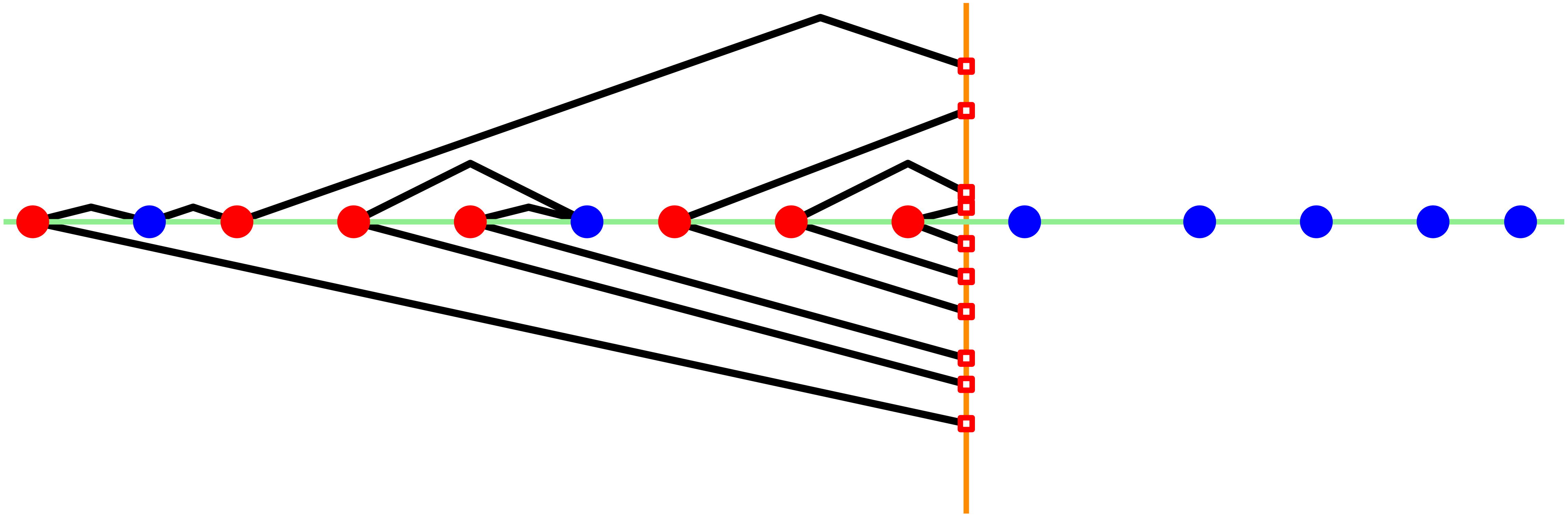

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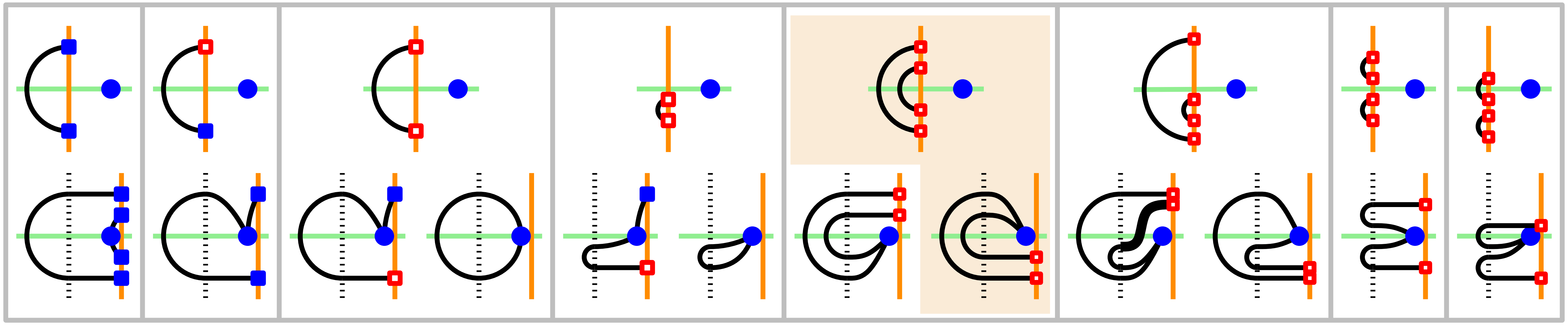

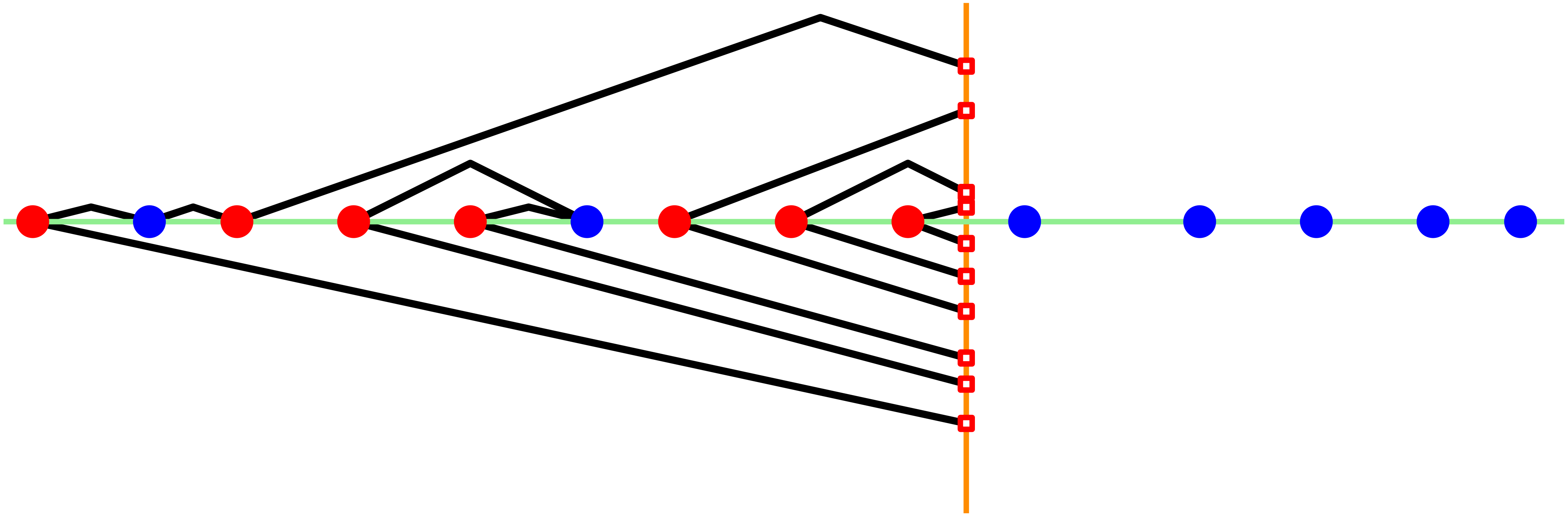

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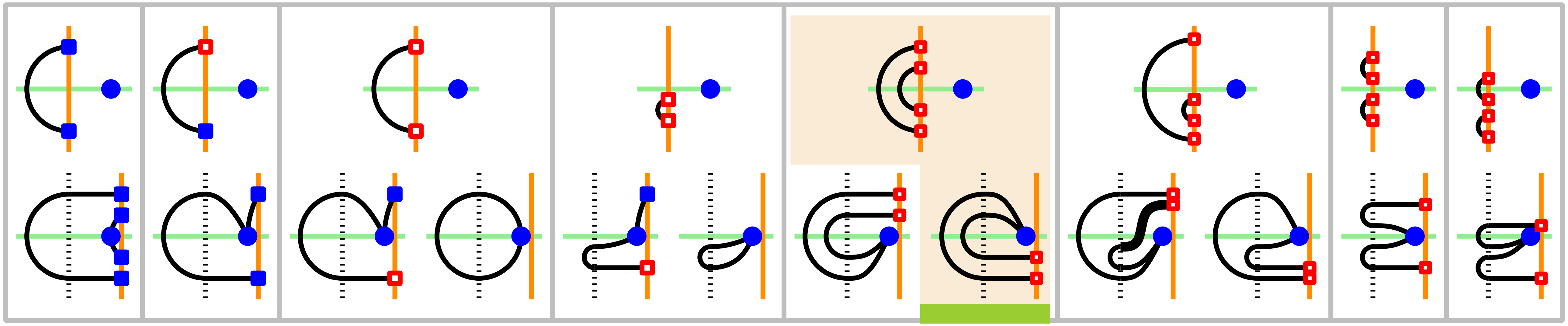

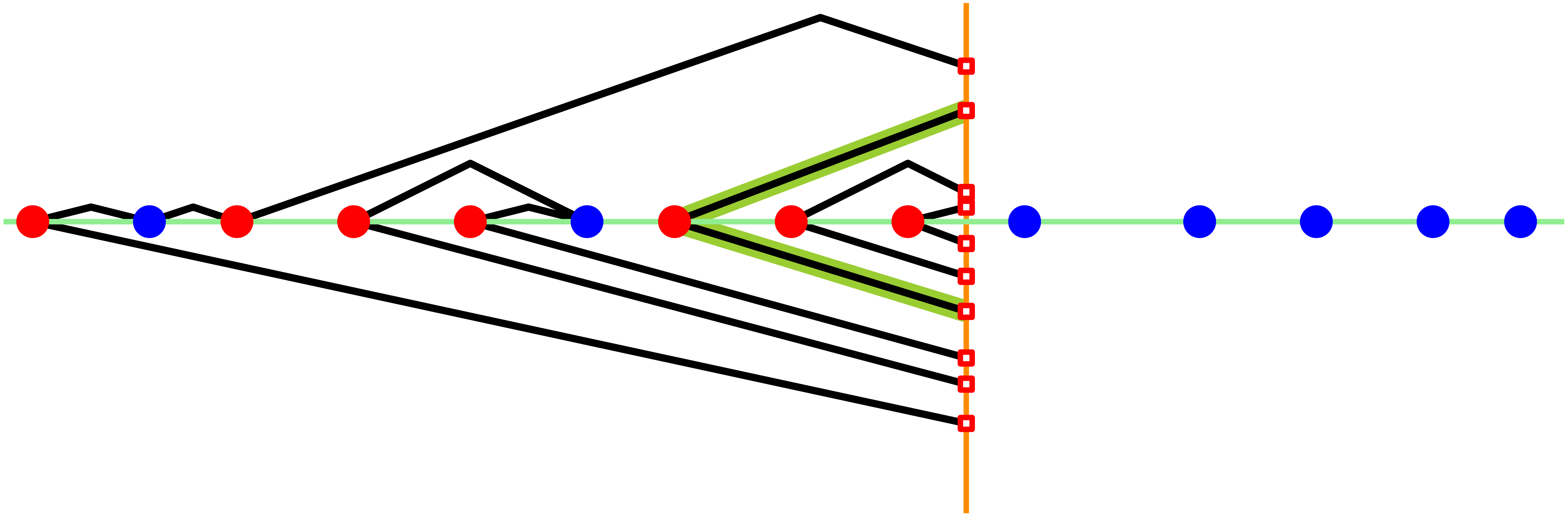

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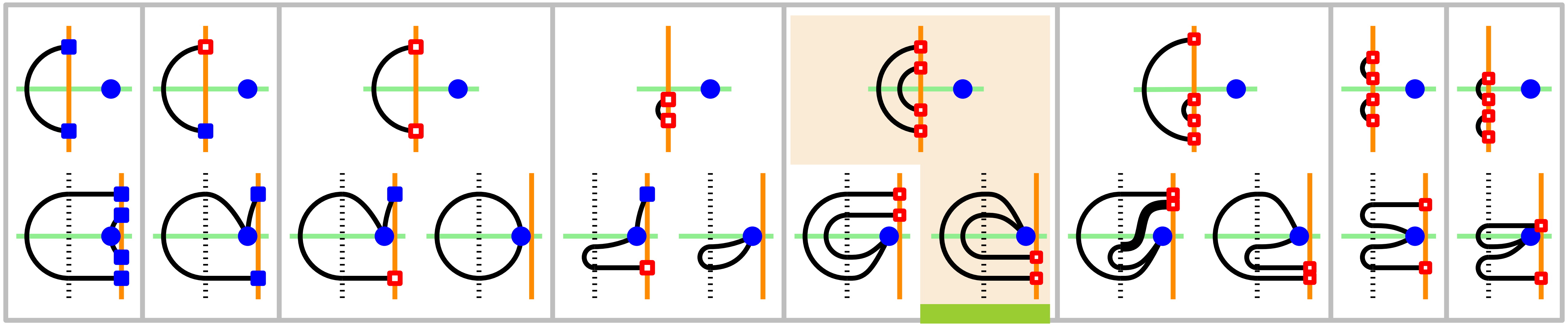

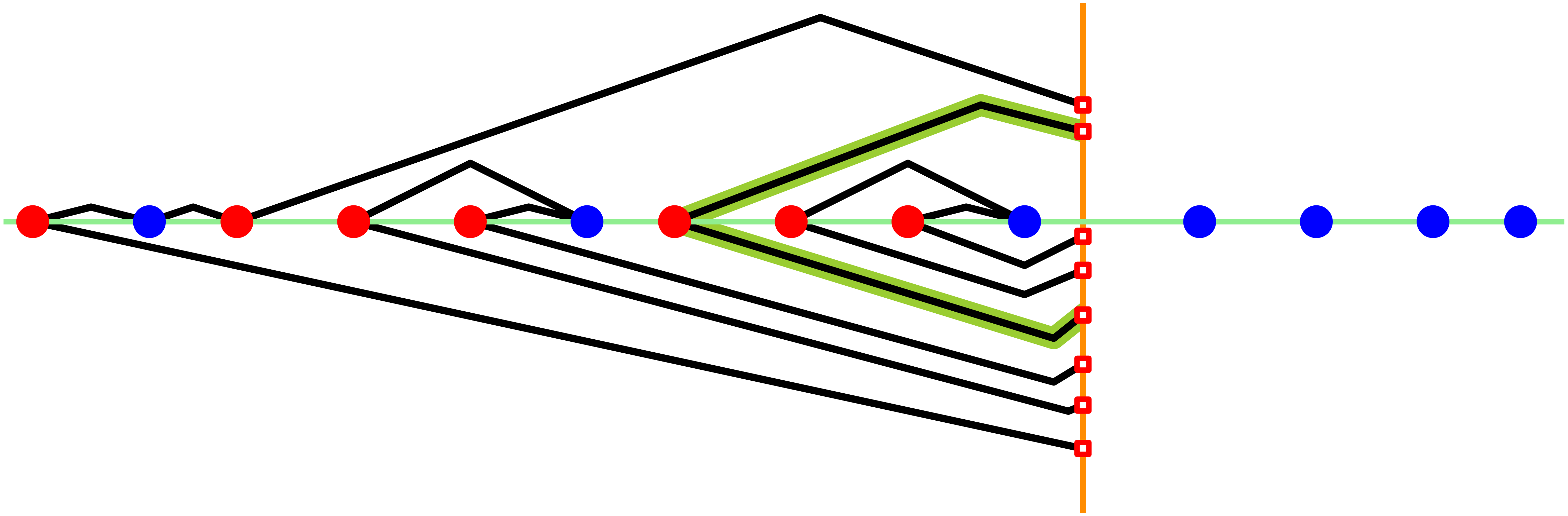

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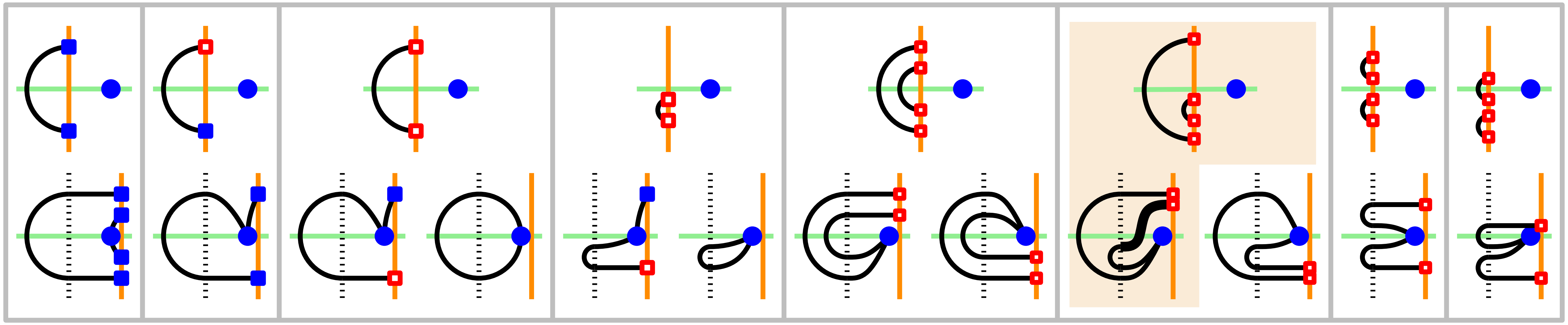

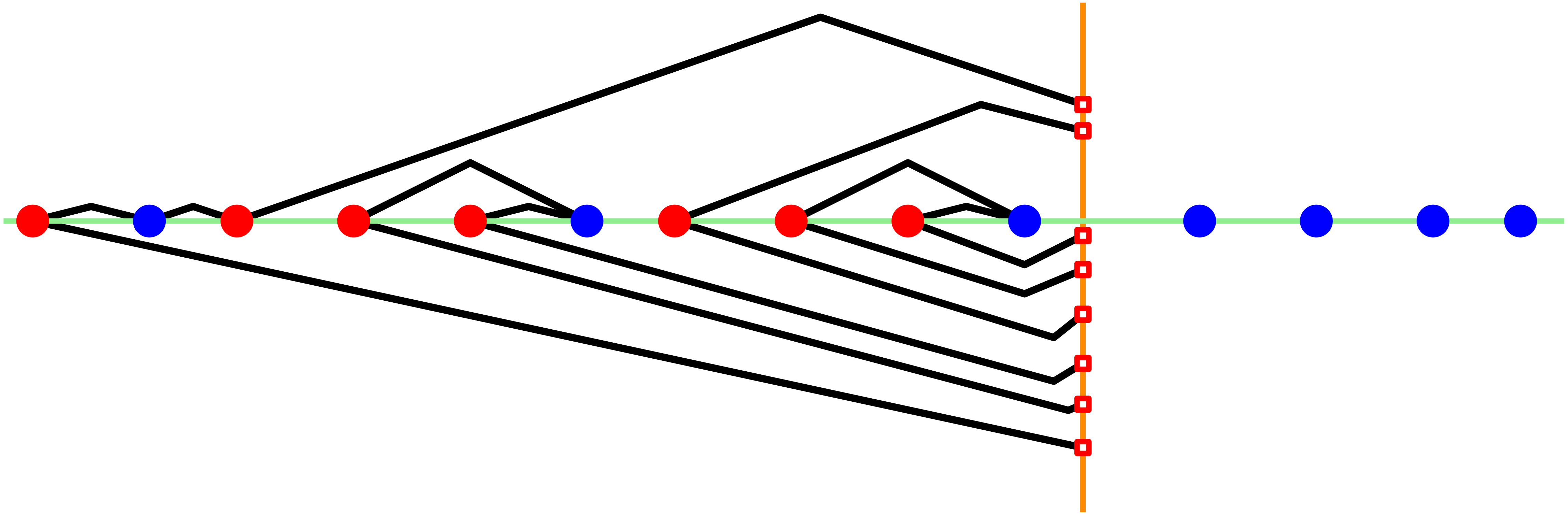

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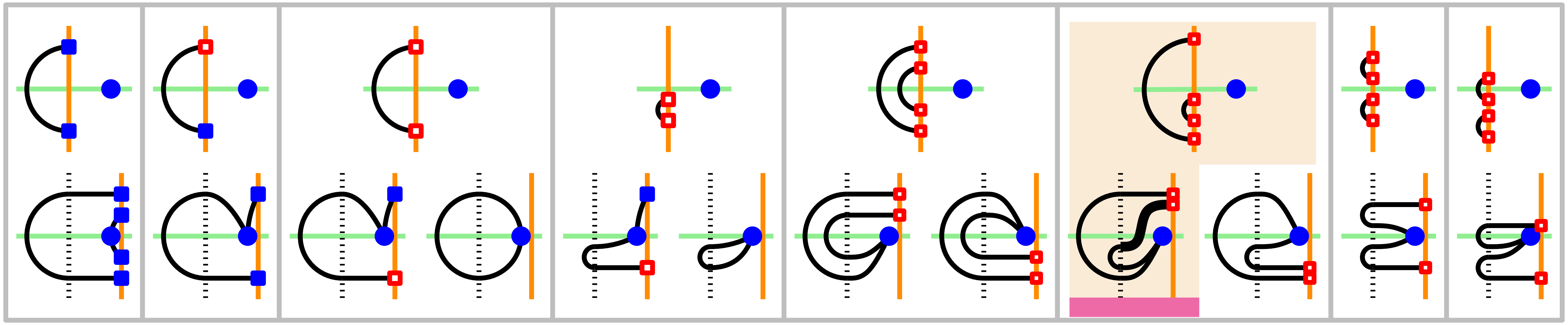

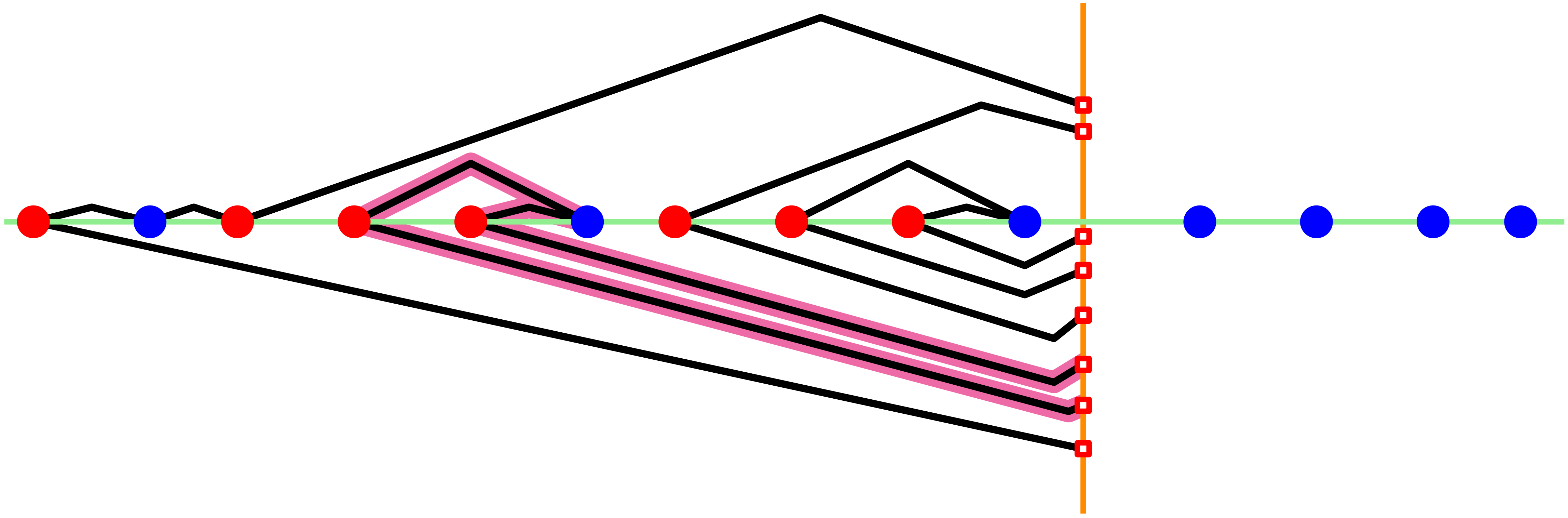

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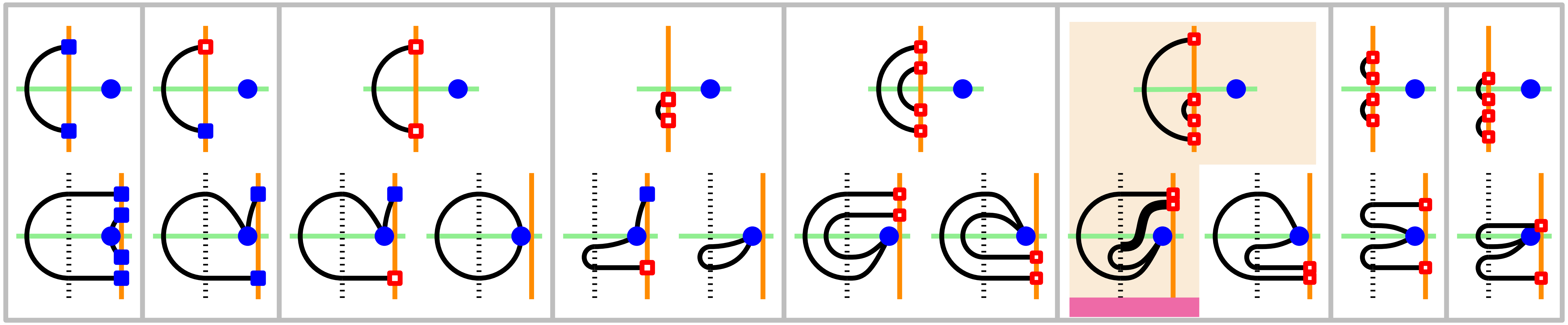

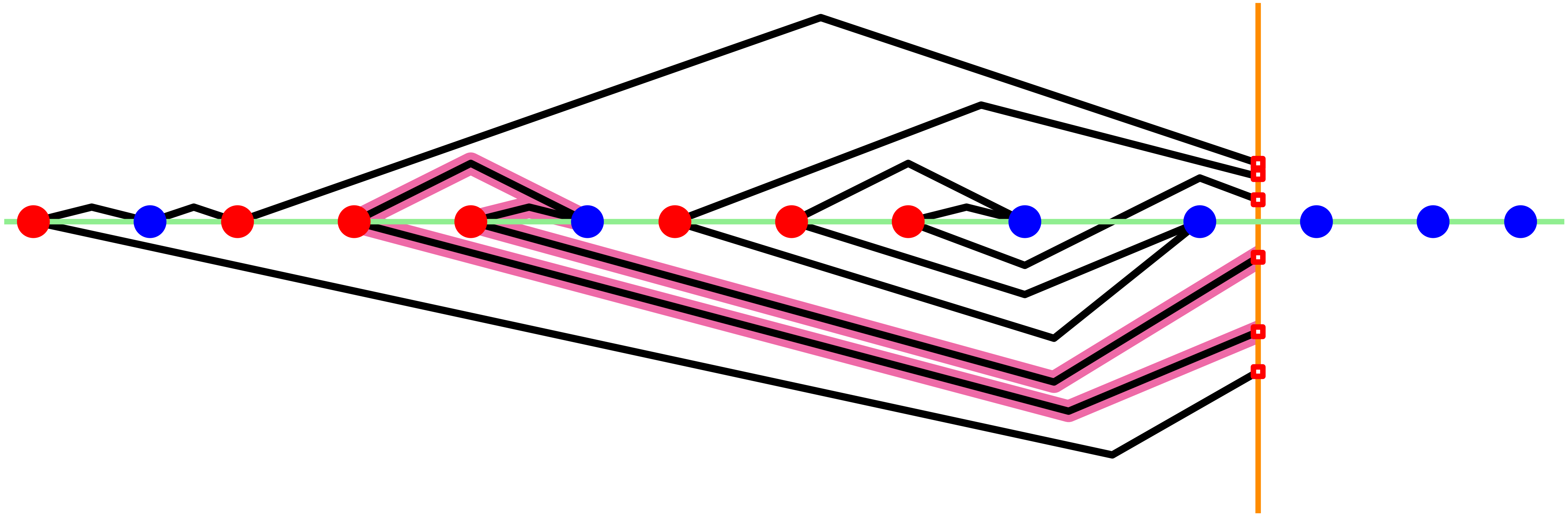

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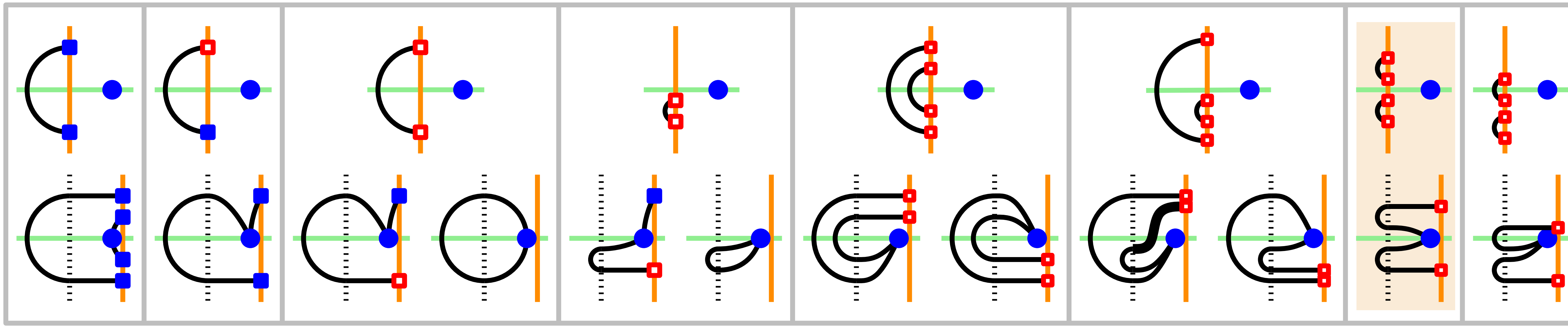
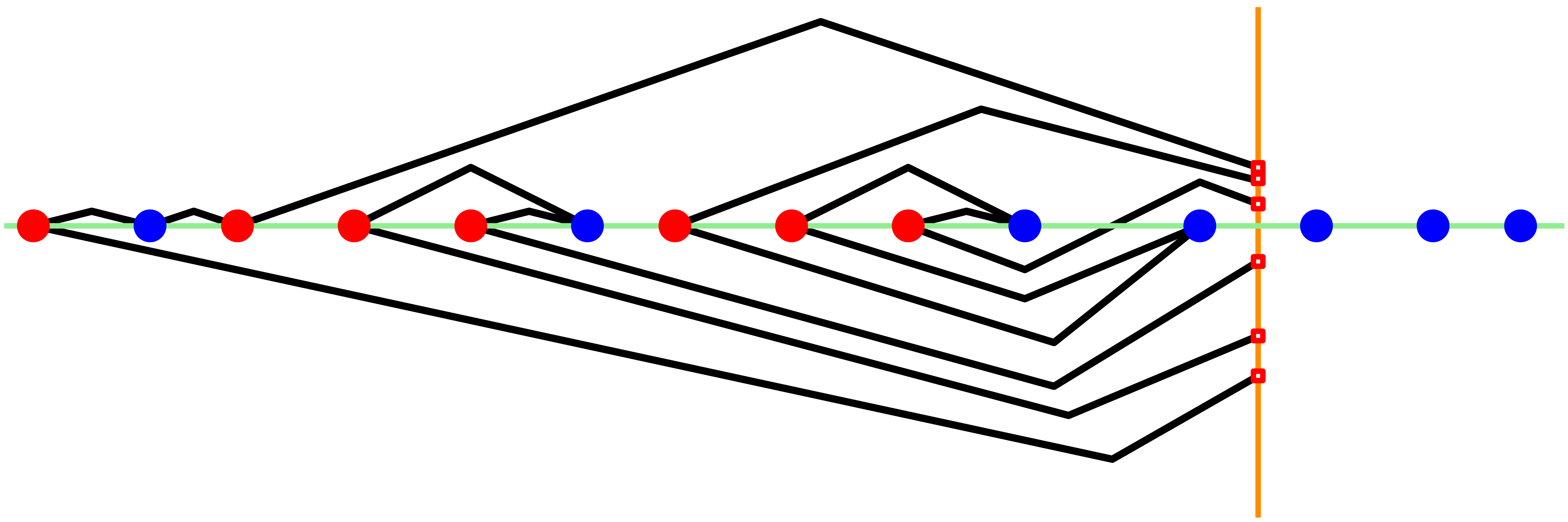
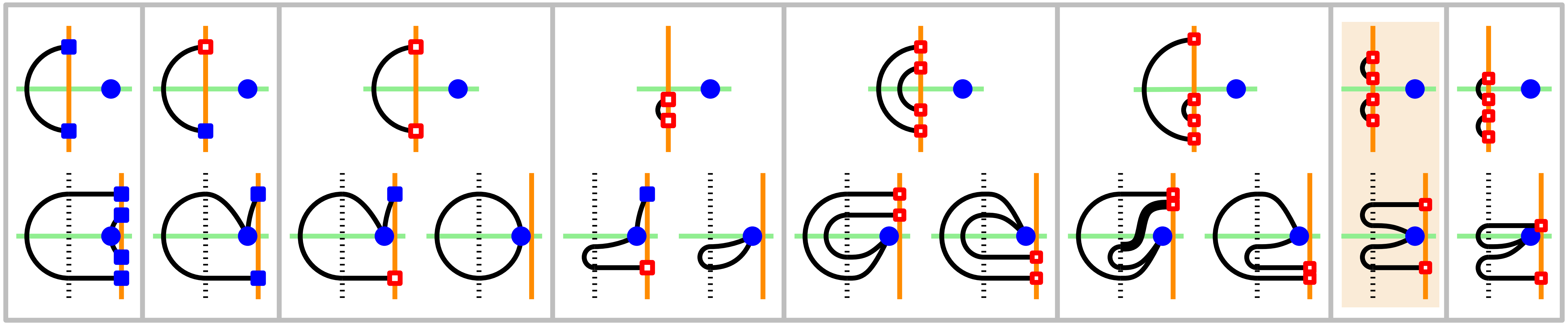
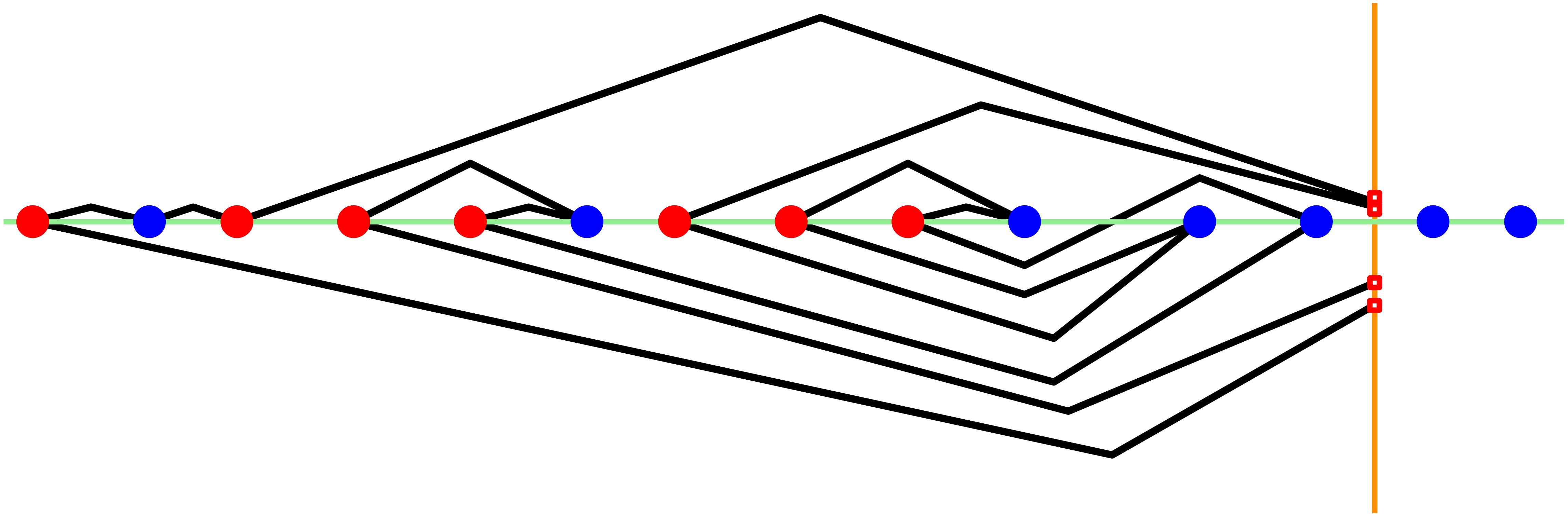




Cycles: Matching the lower bound with 2 bends

Drawing Rules





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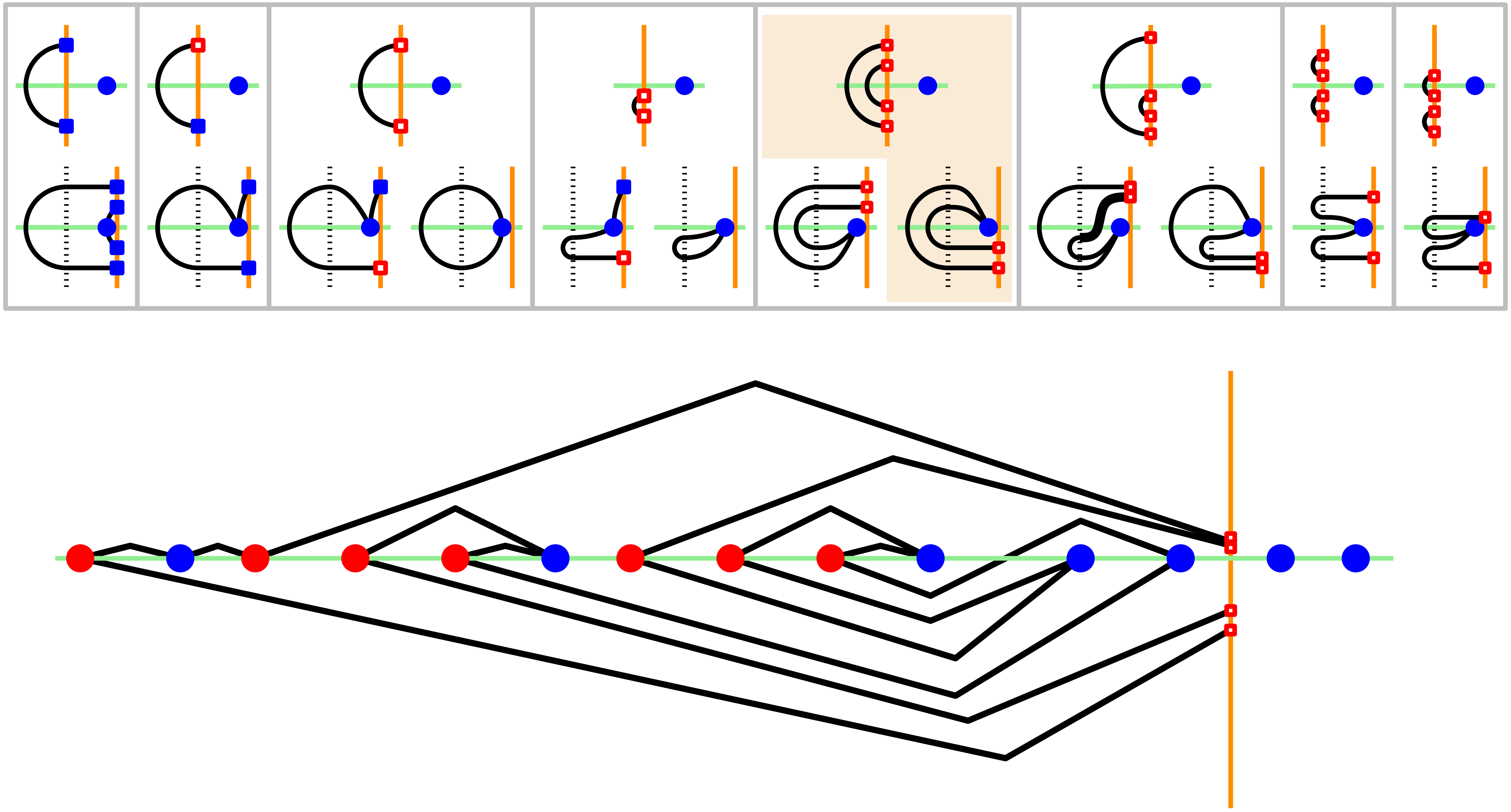

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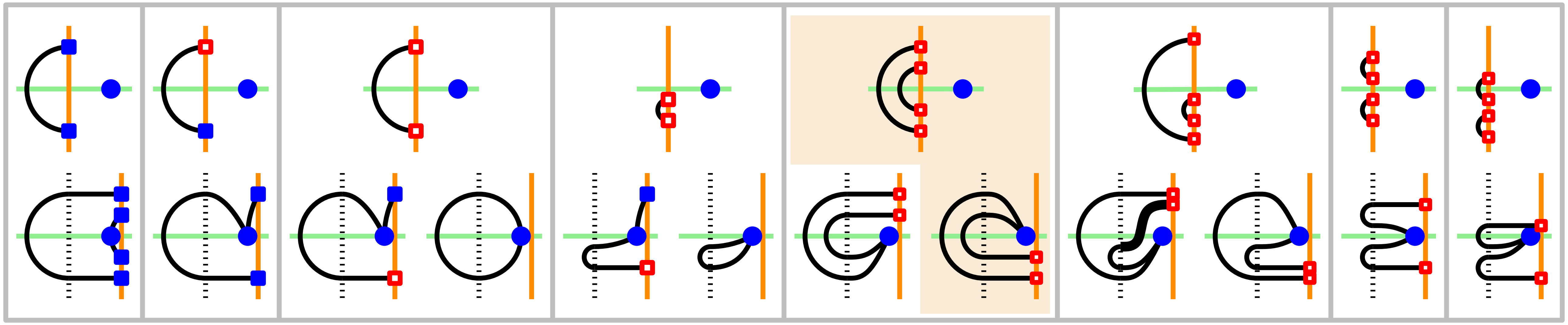

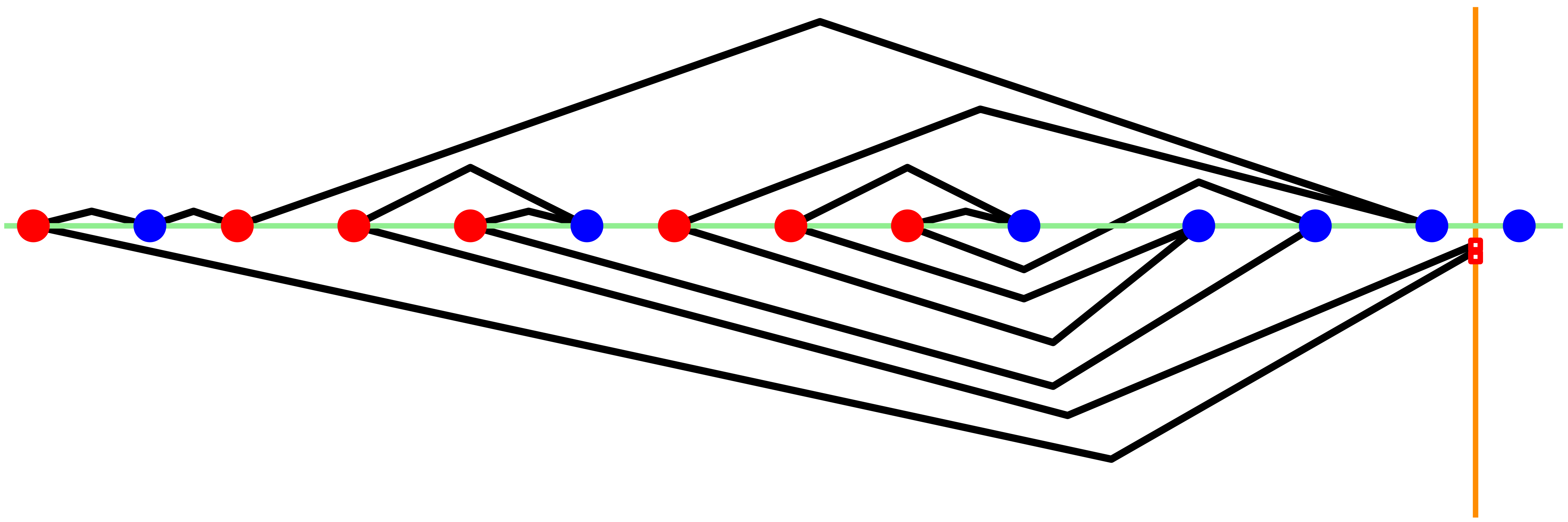

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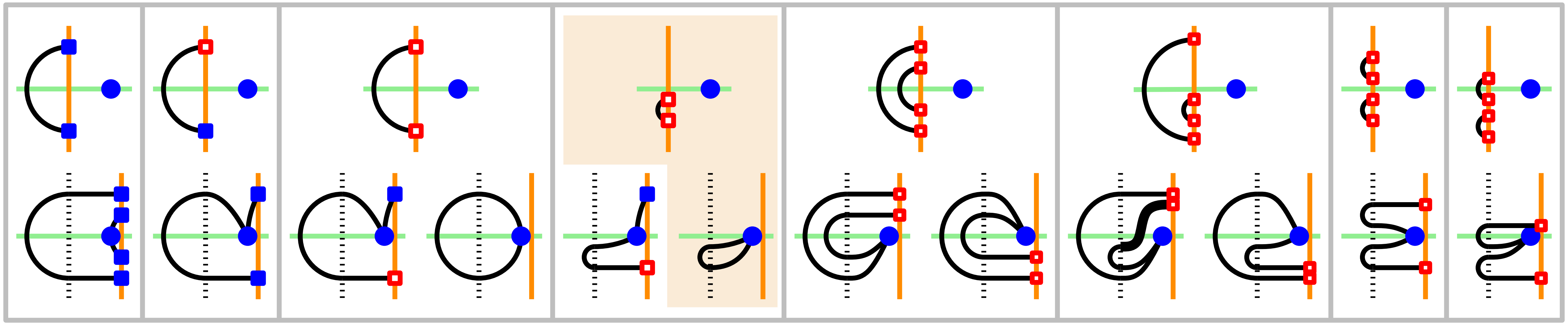

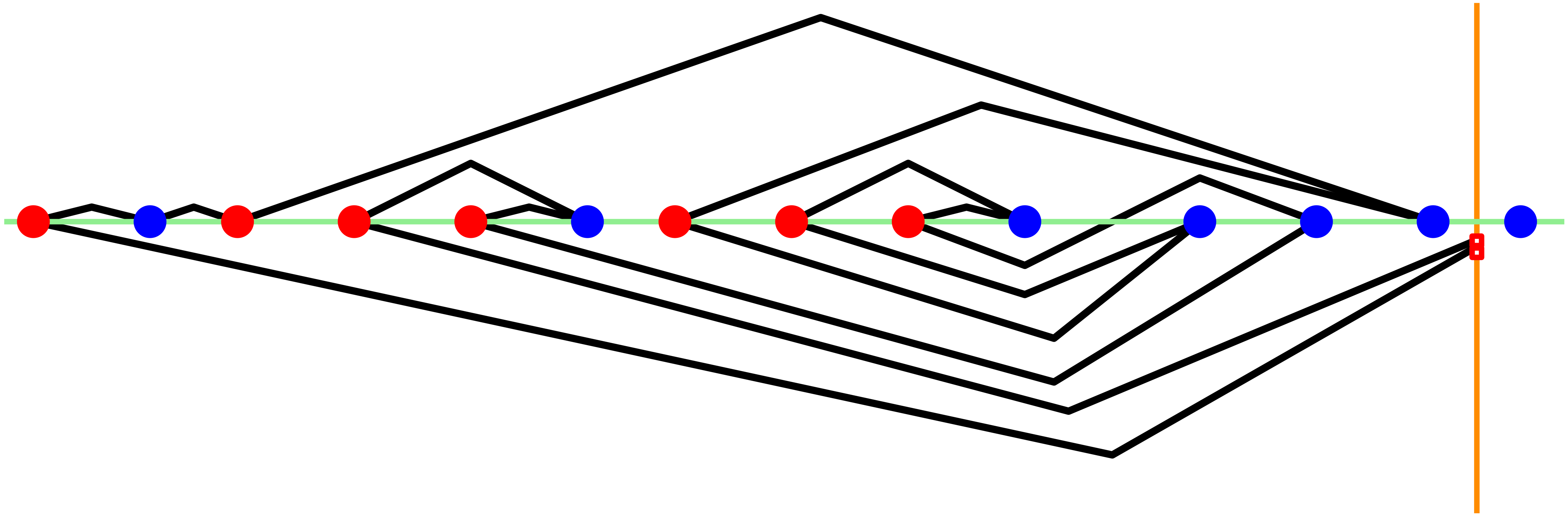

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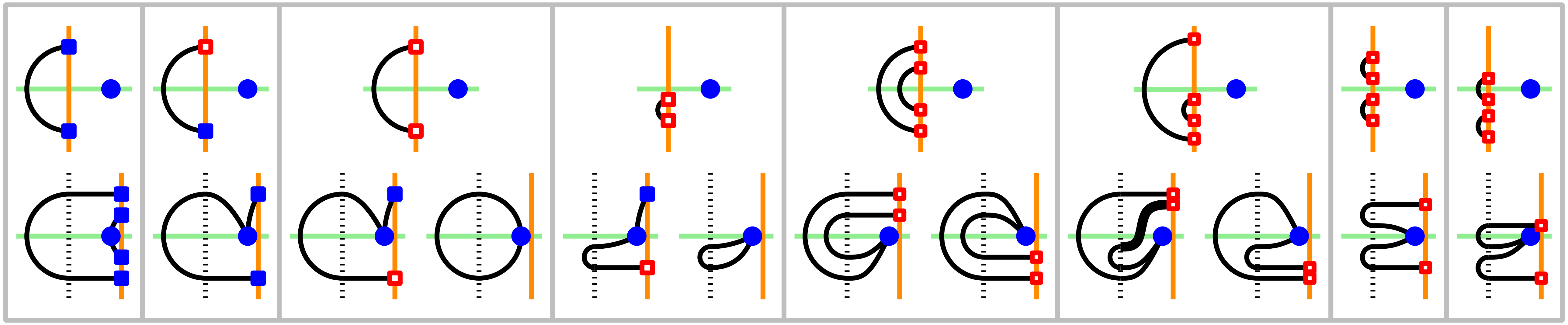

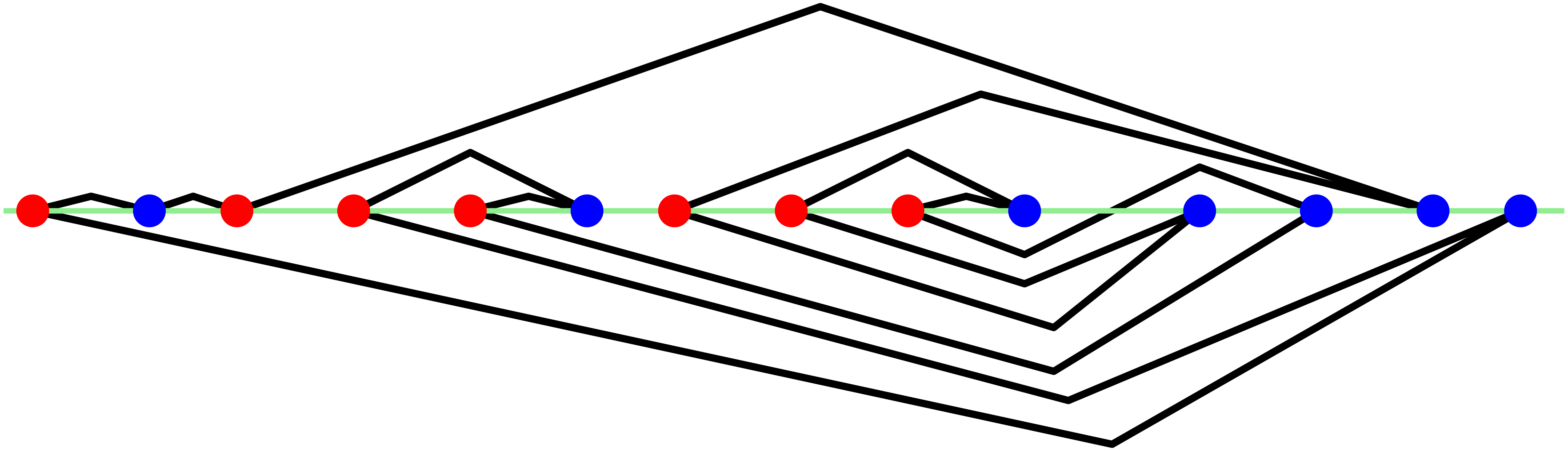

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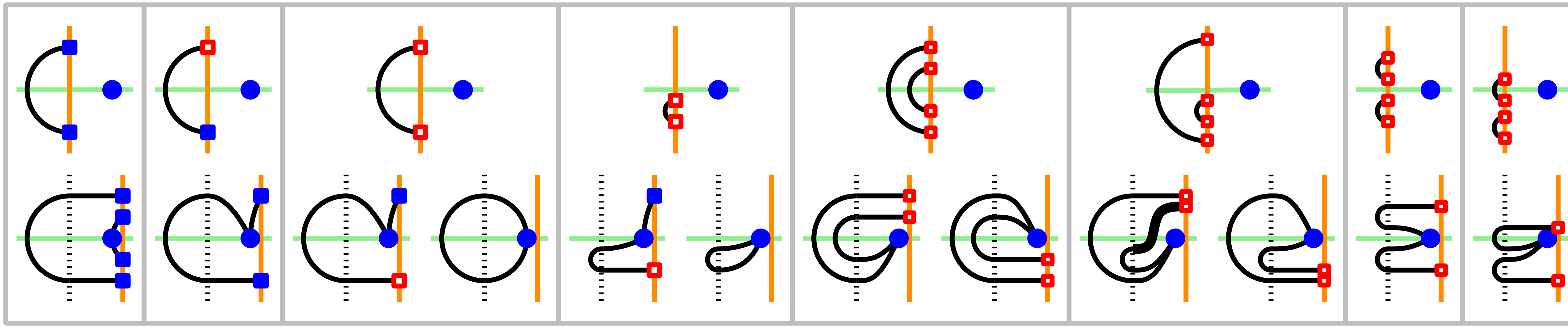
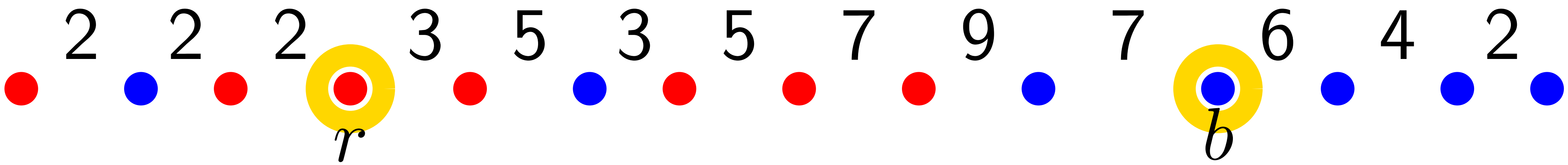


Theorem 1. Exists $O(n \log n)$-time algorithm to compute a shortest planar alternating cycle on colinear points. Each edge has ≤ 2 bends.

Paths: A lower bound

Given path endpoints r and b.



Lemma 2. Minimum number of edges crossing gap$_i$ is

$$c_i = \begin{cases} 2\max\{|r_i - b_i|, 1\} & \text{if } r, b \text{ same side} \\ 1 + 2\max\{b_i - r_i, r_i - b_i - 1\} & \text{if } r \text{ left of gap}_i \\ 1 + 2\max\{r_i - b_i, b_i - r_i - 1\} & \text{if } b \text{ left of gap}_i \end{cases}$$

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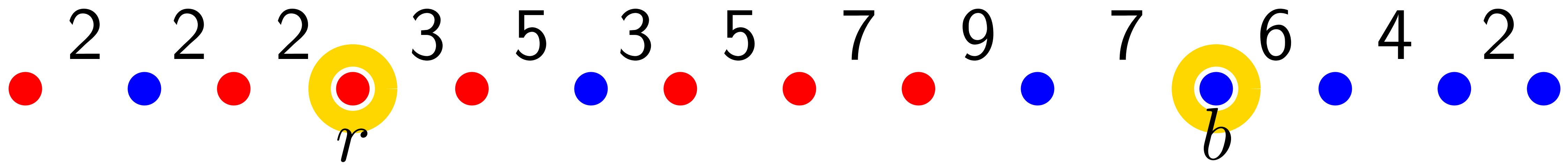

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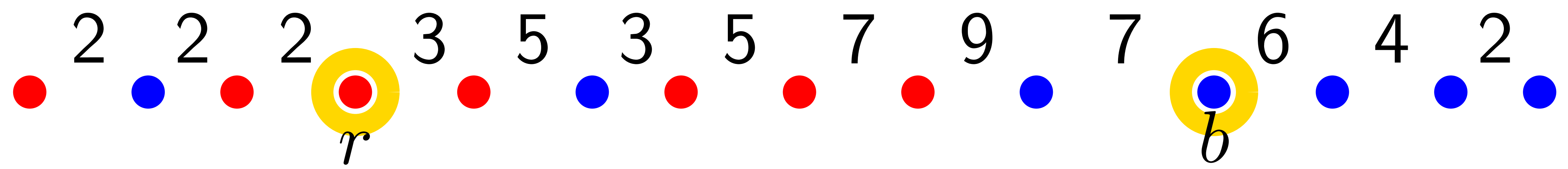
Proof: Same. Component with red endpoint can have one more red than blue points, but zero more blue than red points.

Paths: A lower bound

Given path endpoints r and b.



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r to b Path length $\geq \sum_i c_i |\text{gap}_i|$

Paths: Matching the lower bound with 2 bends

Given path endpoints r and b.

Use (almost) the same algorithm as for cycles to find a path whose length matches the lower bound.

Paths: Matching the lower bound with 2 bends

~~Given path endpoints r and b.~~

Calculate the r to b Path length lower bound for all r and b and pick the minimum. $O(n^2)$ time

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Paths: Matching the lower bound with 2 bends

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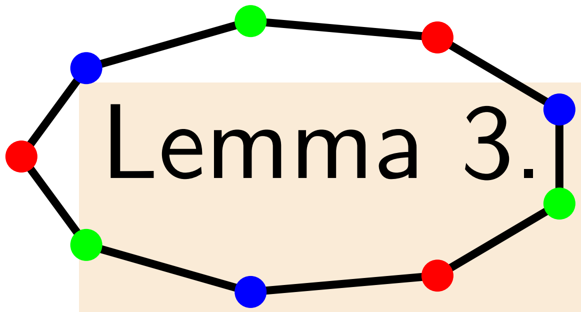
Extending to more than two colors



Lemma 3. Minimum number of edges crossing gap_i is

$$c_i = 2\max\{|r_i - g_i|, |g_i - b_i|, |b_i - r_i|, 1\}$$

$r_i =$ red, $g_i =$ green, $b_i =$ blue points before gap_i

Similar algorithm achieves lower bound. ($O(n)$ bends)

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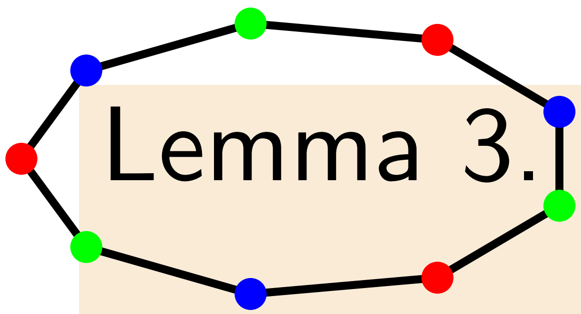

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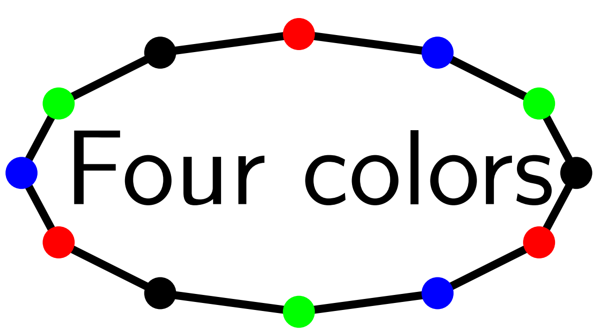
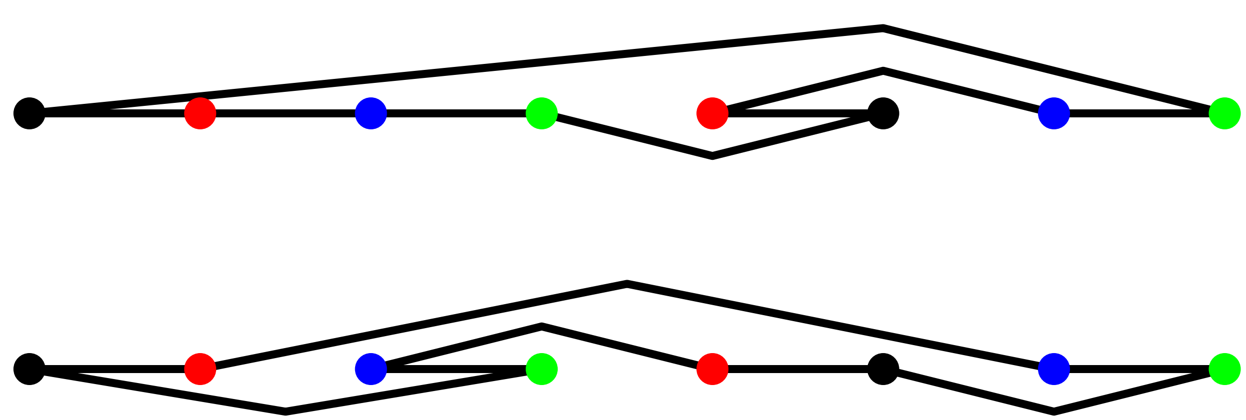


Four colors: Lower bound cannot be achieved.



Open Problems

- Shortest alternating path in $o(n^2)$ time.

- Shortest 3-color path/cycle with $o(n)$ bends.

- Shortest 4-color path/cycle.

- Shortest arbitrary (not alternating) 2-color path/cycle.

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- Shortest alternating path in $o(n^2)$ time.
- Shortest 3-color path/cycle with $o(n)$ bends.
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Thank you