Vertical Visibility among Parallel Polygons in Three Dimensions

GD 2015

Radoslav Fulek (IST, Austria), Radoš Radoičić (CUNY)

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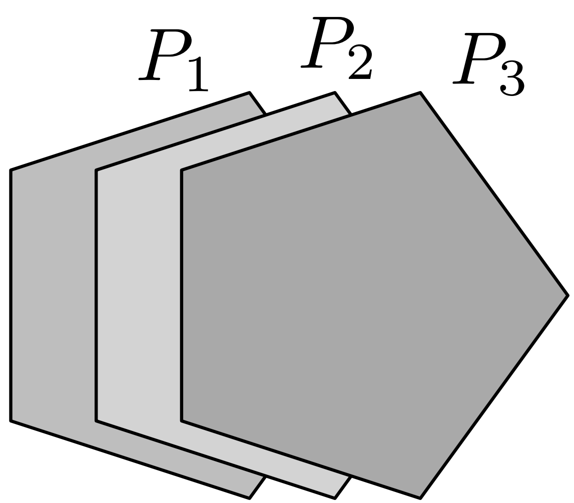

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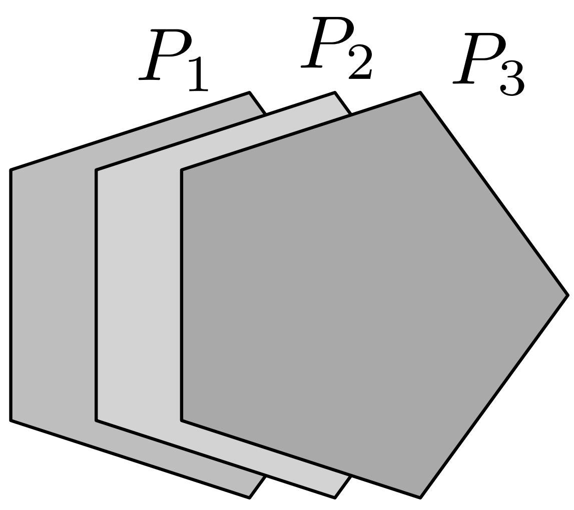

P_1 sees P_2, but P_1 does not see P_3

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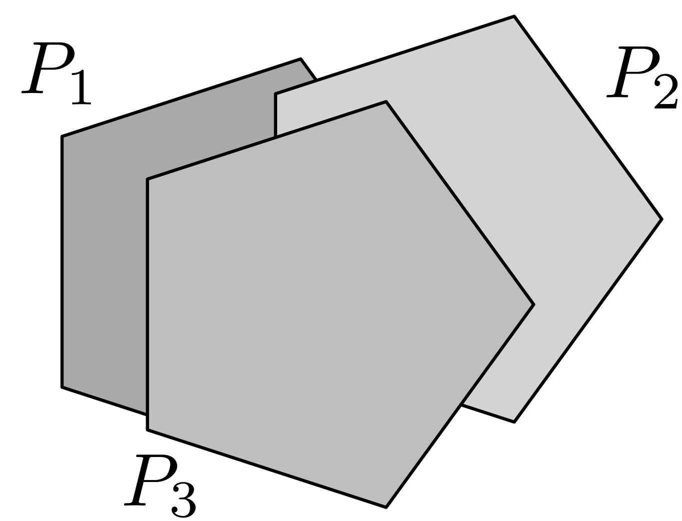

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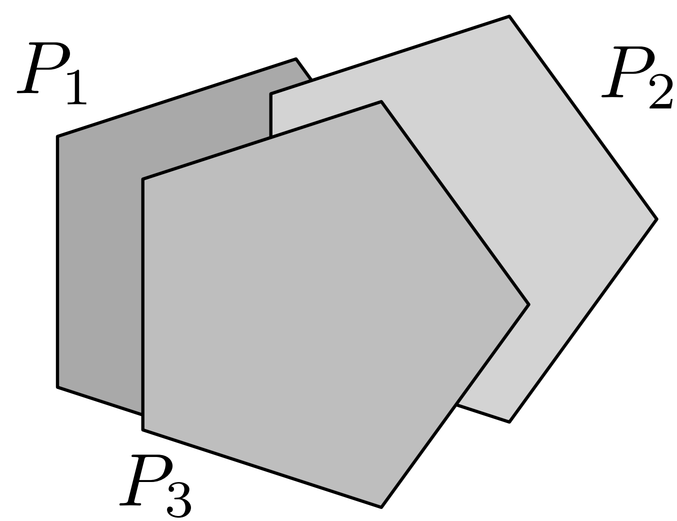
$\{P_1, P_2, P_3\}$ forms a visibility clique.



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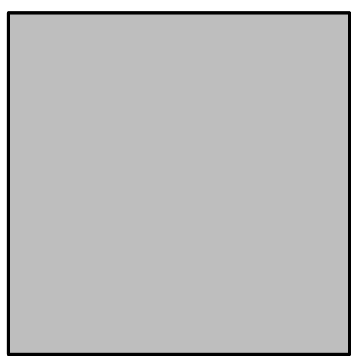

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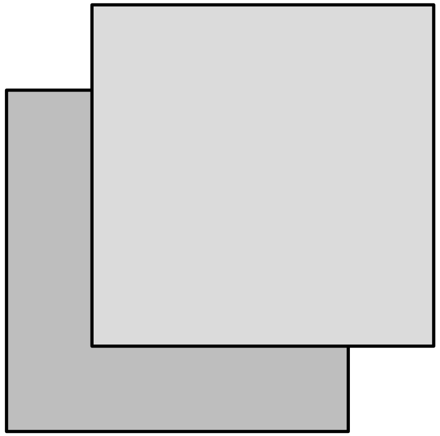

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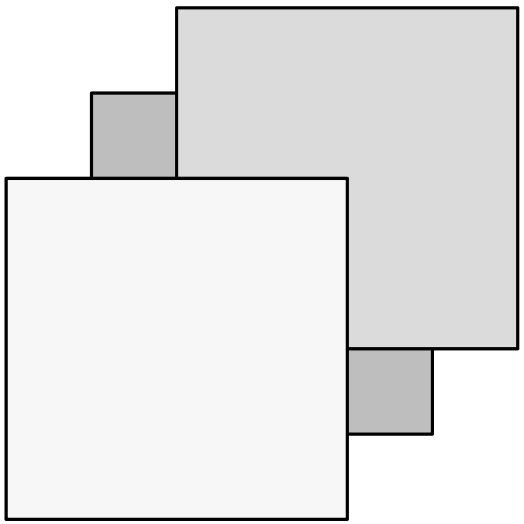

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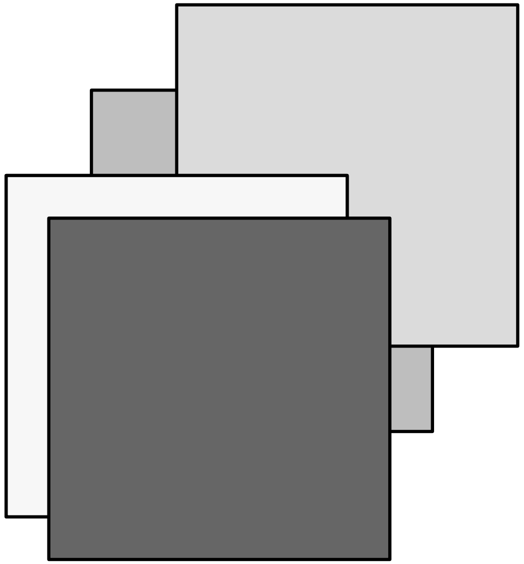

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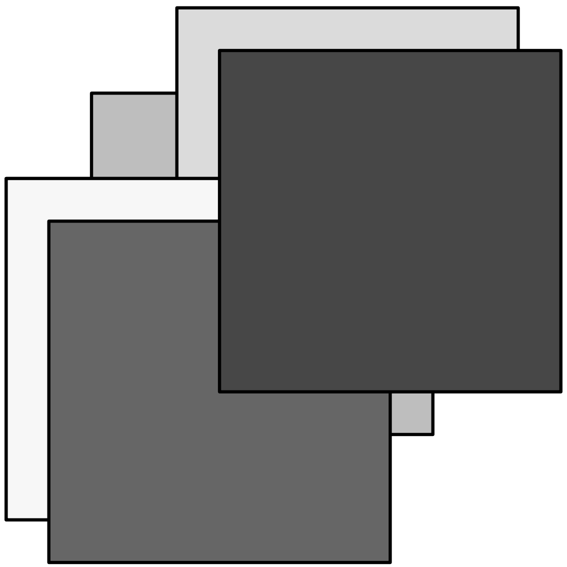

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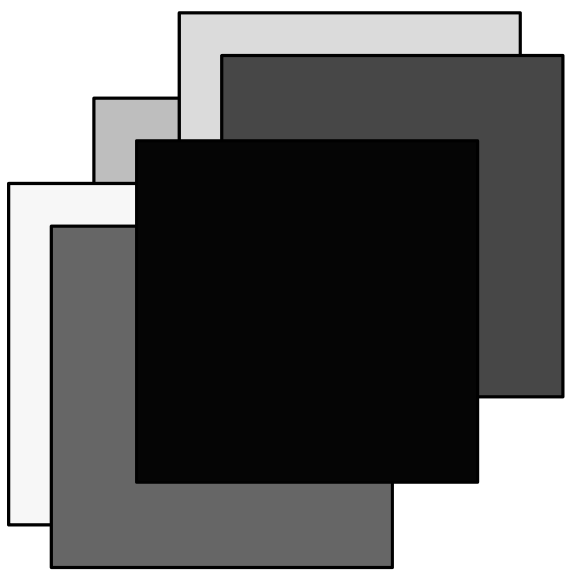

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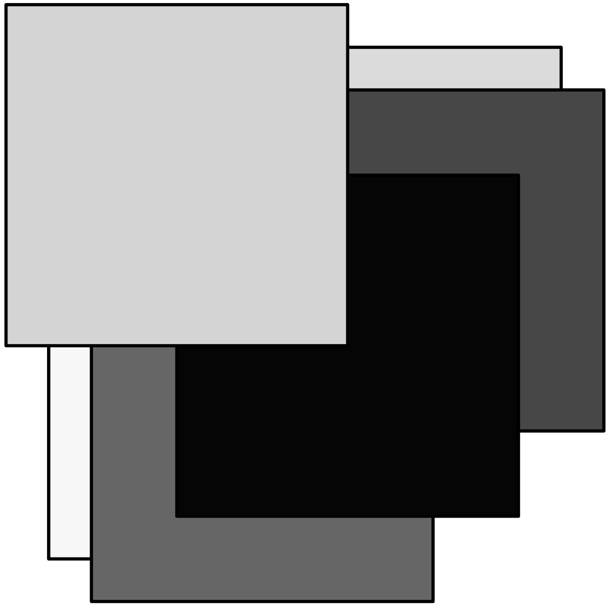

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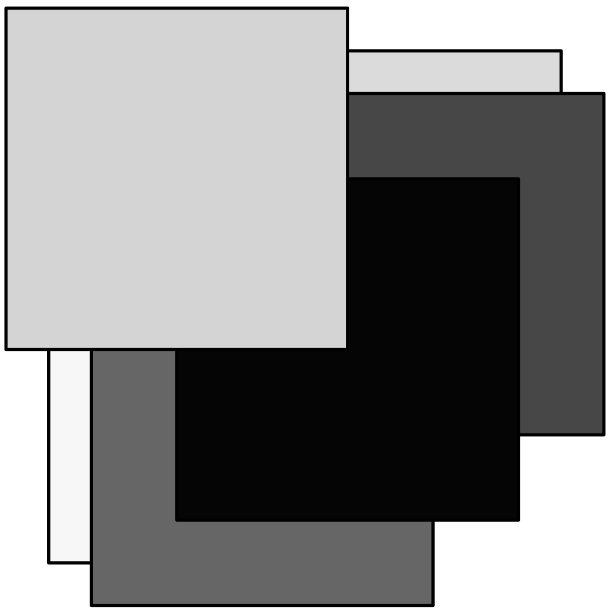

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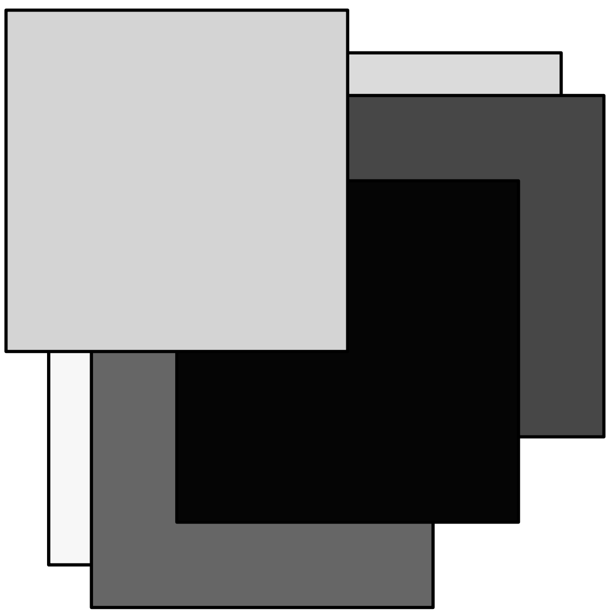

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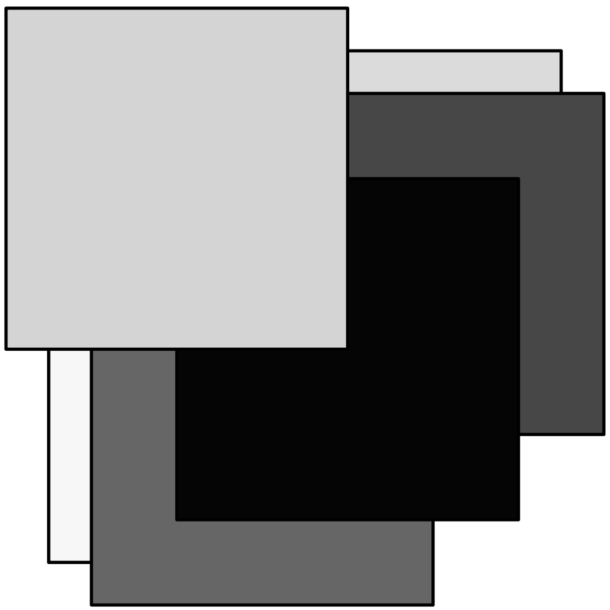

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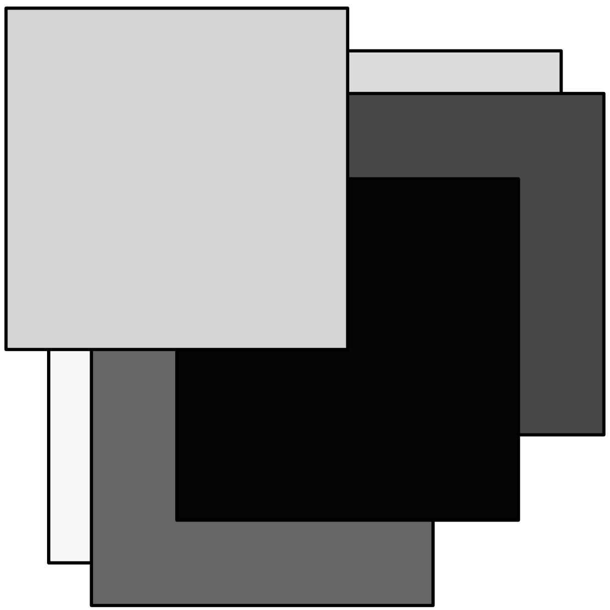

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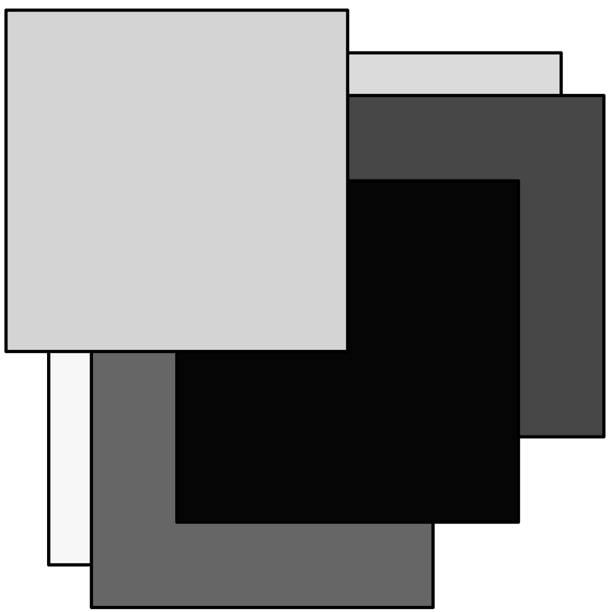

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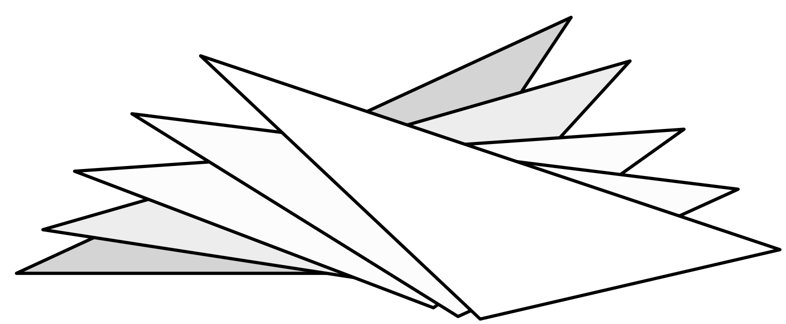

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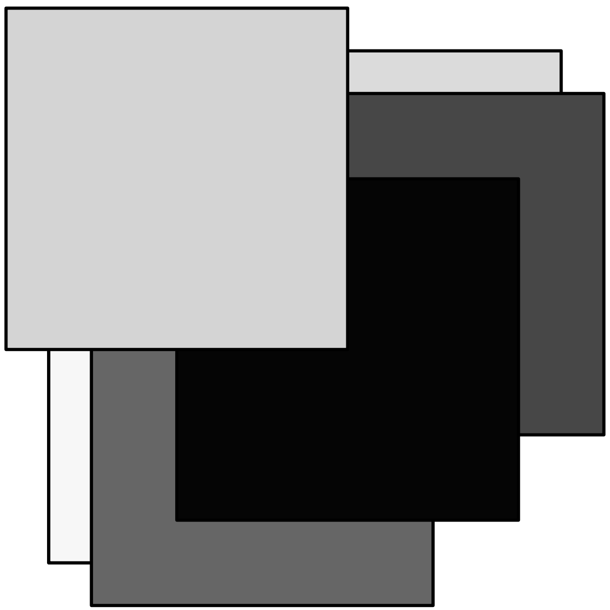

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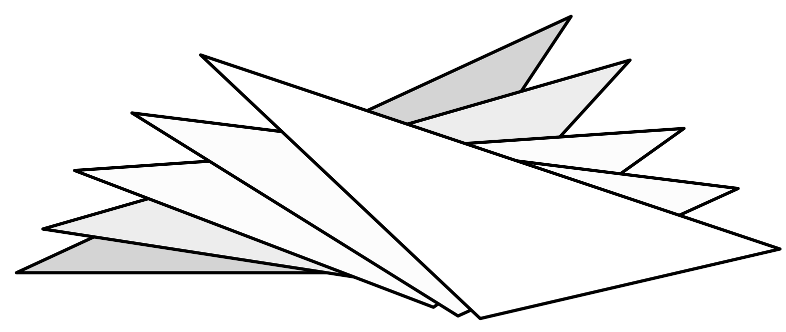

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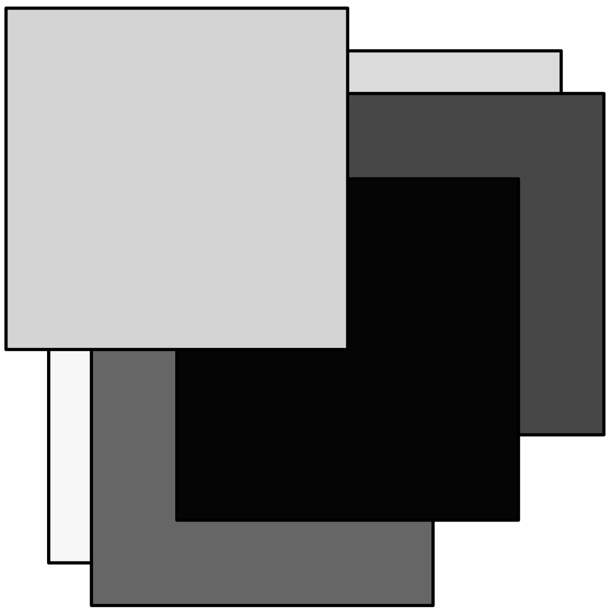

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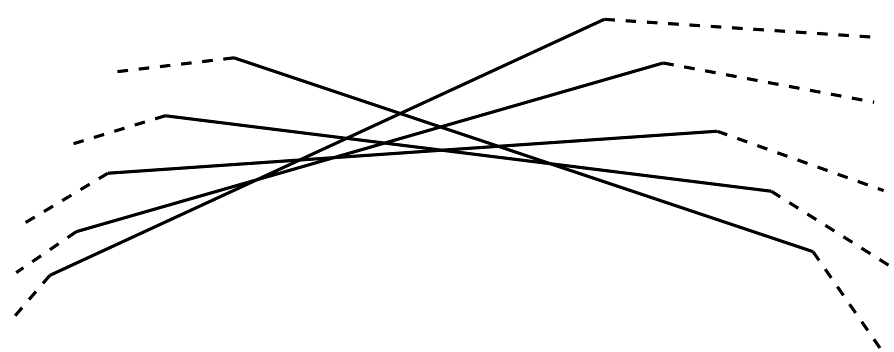

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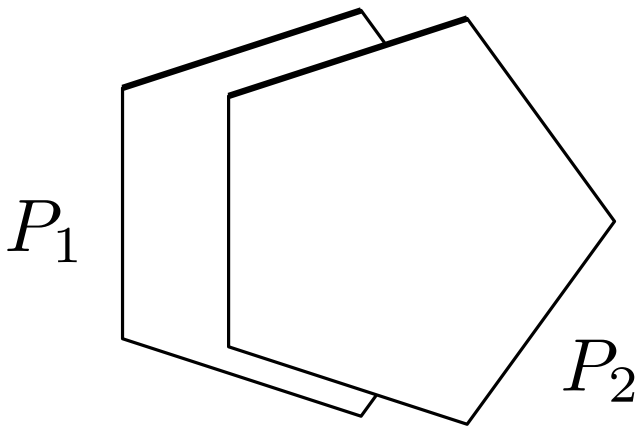

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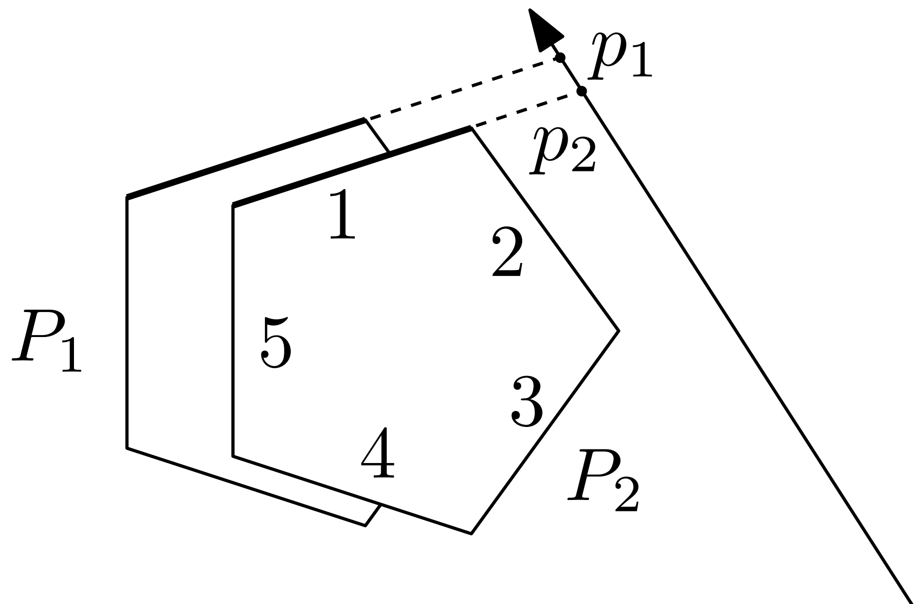

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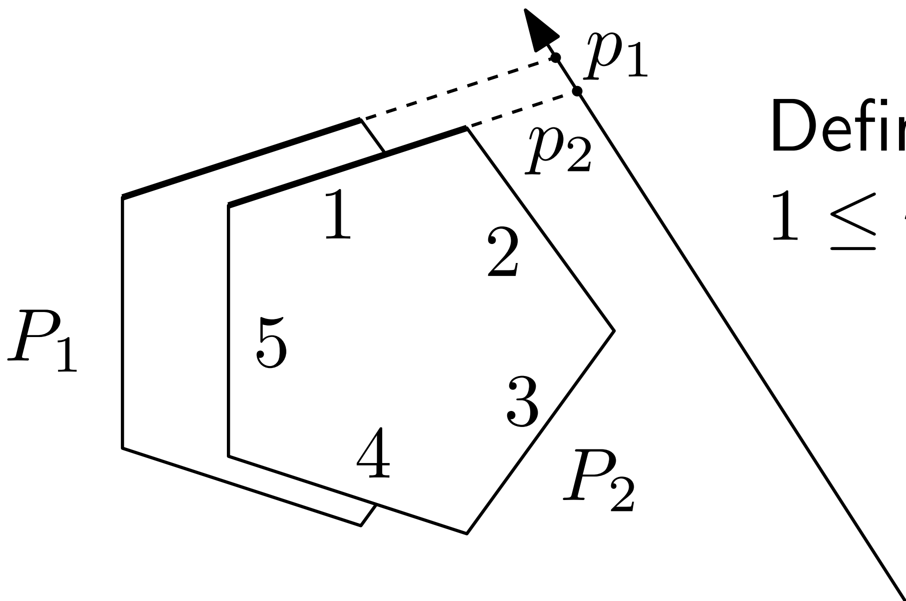

Define partial orders $<_i$ for $1 \leq i \leq k$. We have $P_1 <_1 P_2$.

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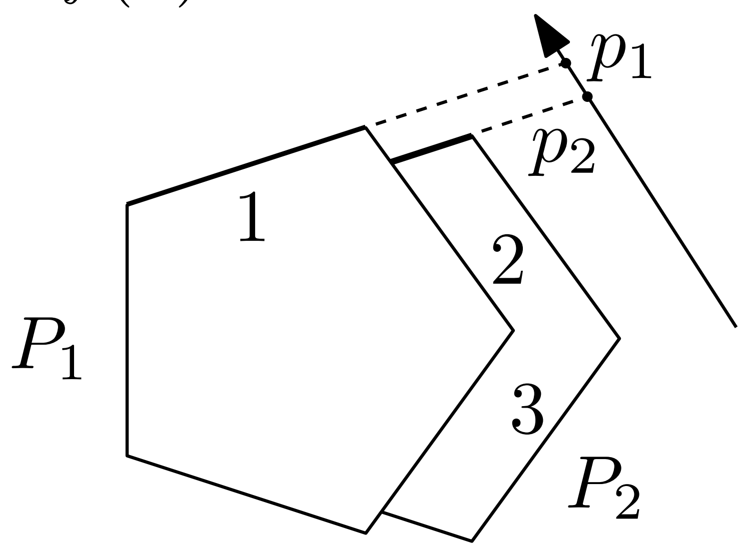

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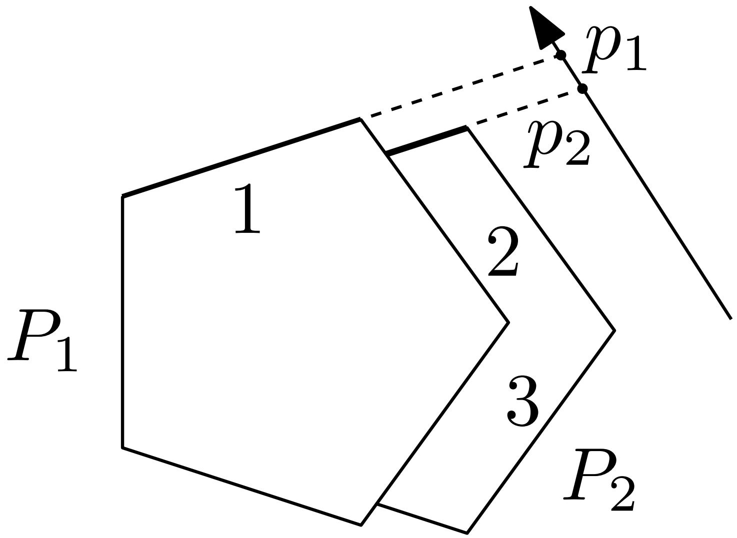

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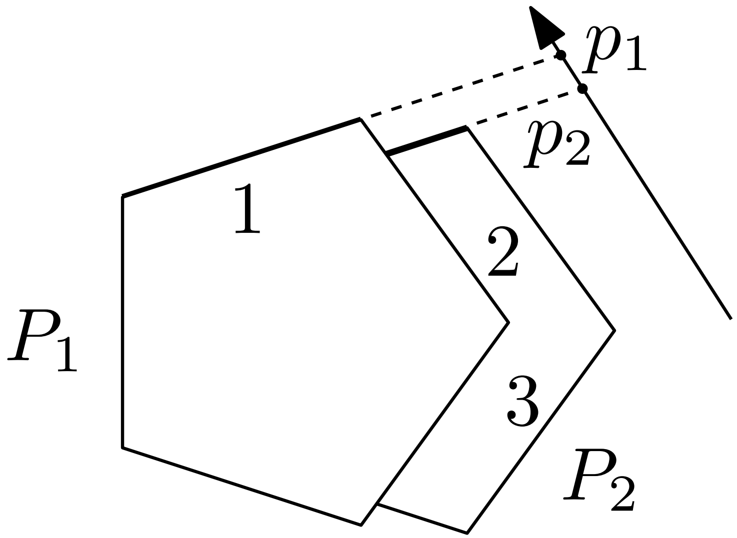

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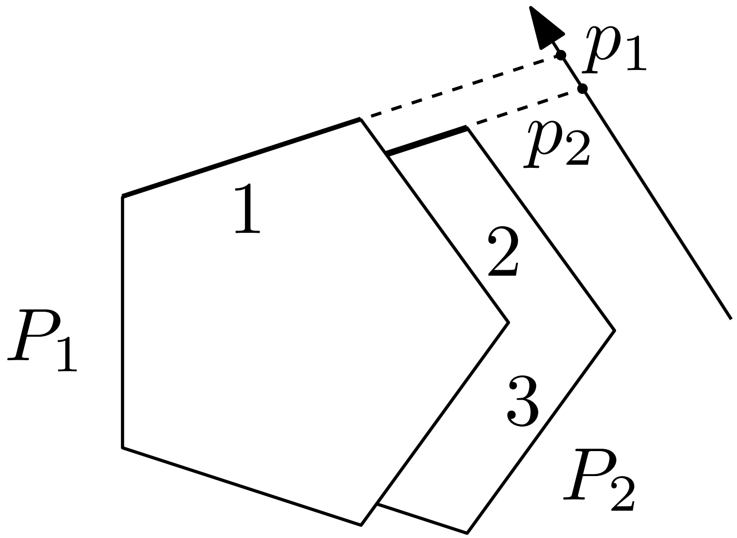

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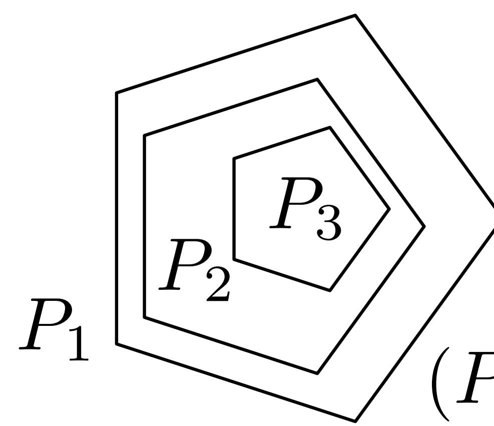

$(P_1 \cap P_3) \subset P_2$

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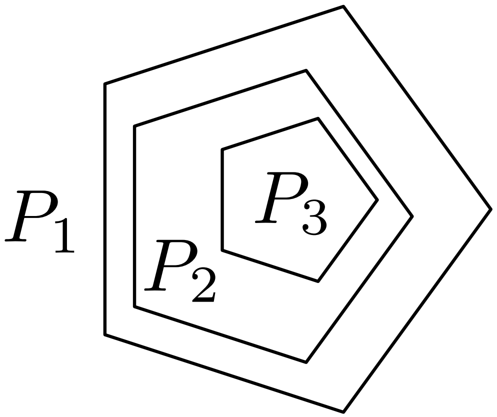

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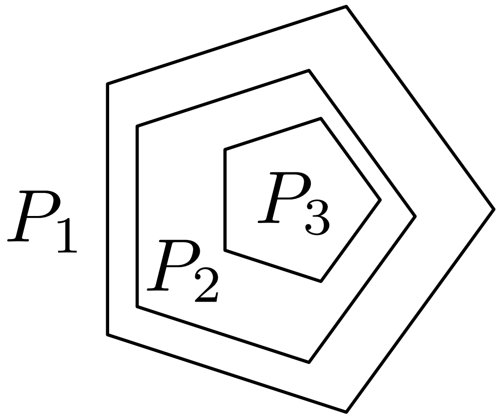
Use Dilworth theorem.



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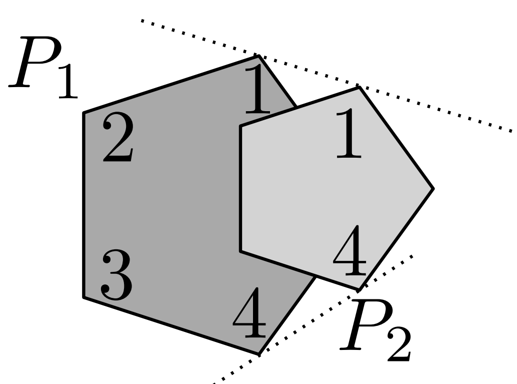
- We color each edge in the visibility clique with a pair consisting of a two element set encoding the vertices supporting the common tangents, and an indicator for its above–below relationship. We use $2\binom{k}{2}$ colors.



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- We apply a Ramsey–type theorem for ordered graphs.

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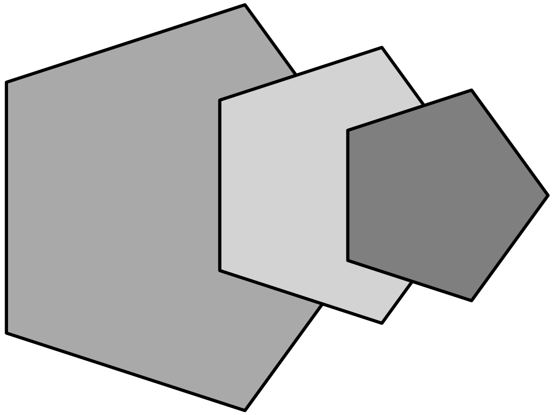

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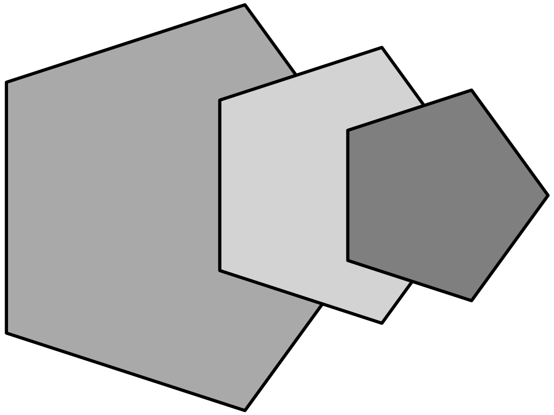

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We observe that we cannot have a monochromatic monotone (with respect to our order) path of length three.

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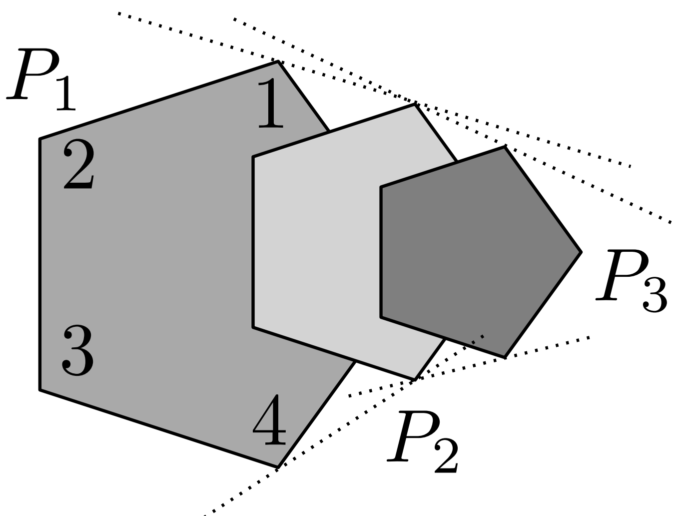

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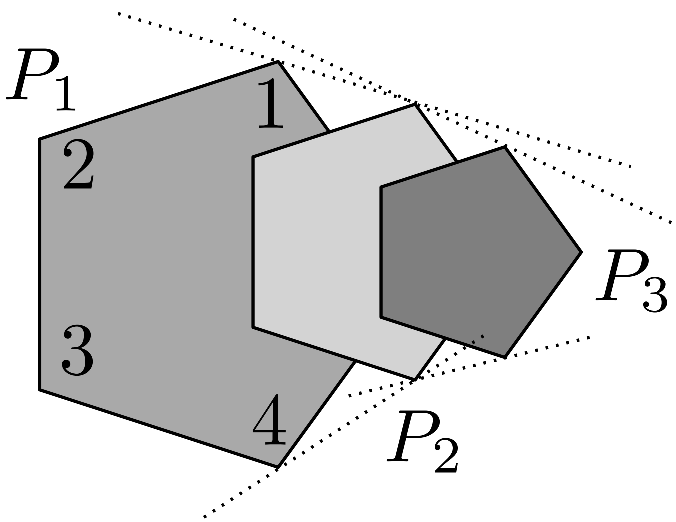

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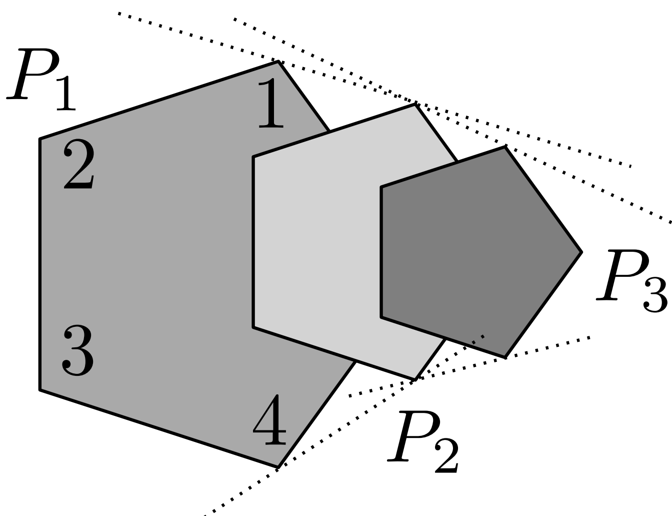

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By a result of **Milans et al. (2012)** we can have at most 2^c vertices, where c is the number of colors.

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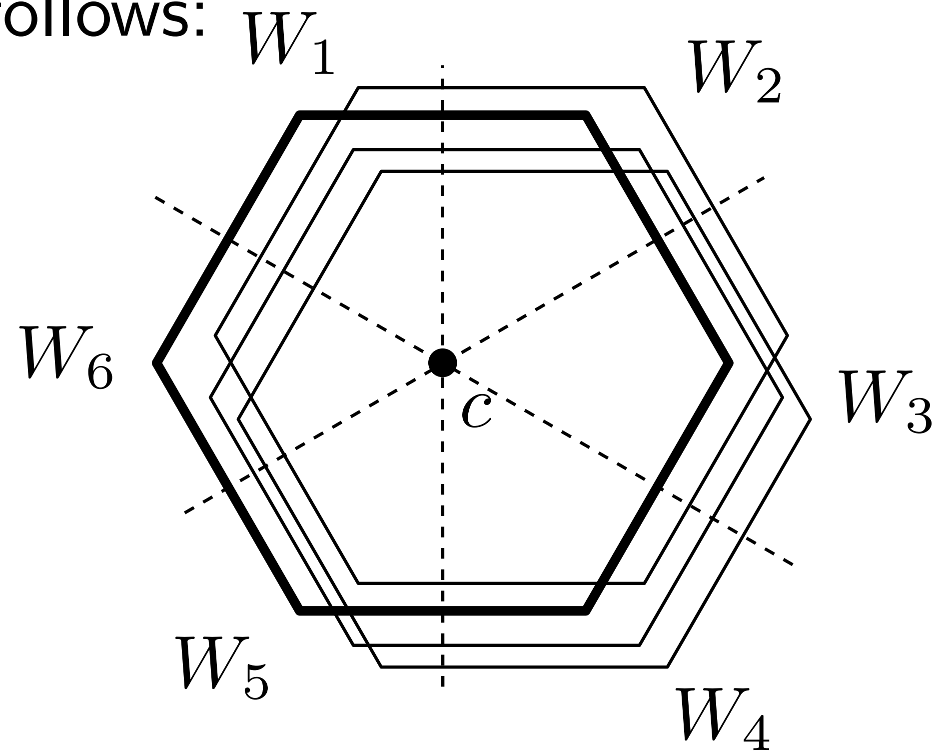

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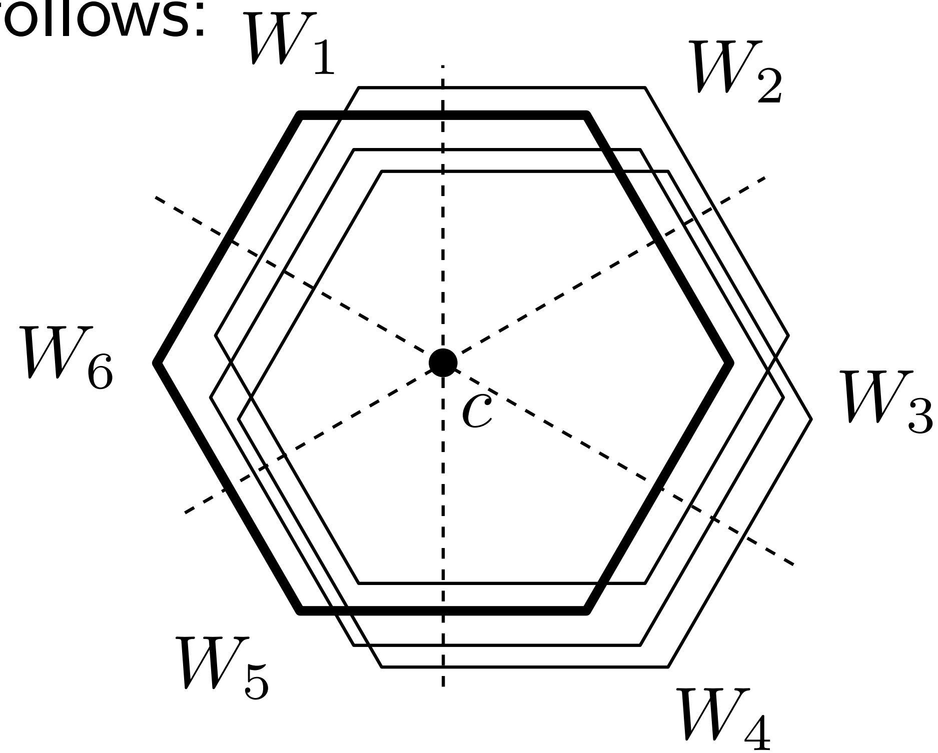

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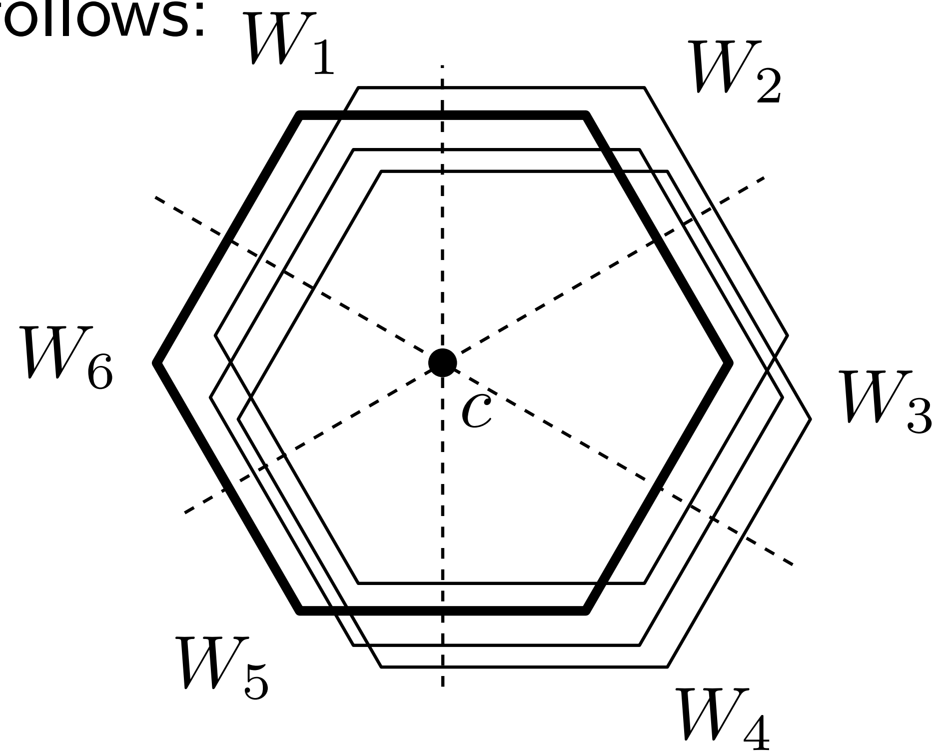

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If the radius of the circumscribed circle is unit $\sin(\frac{\pi}{k})$ would do.

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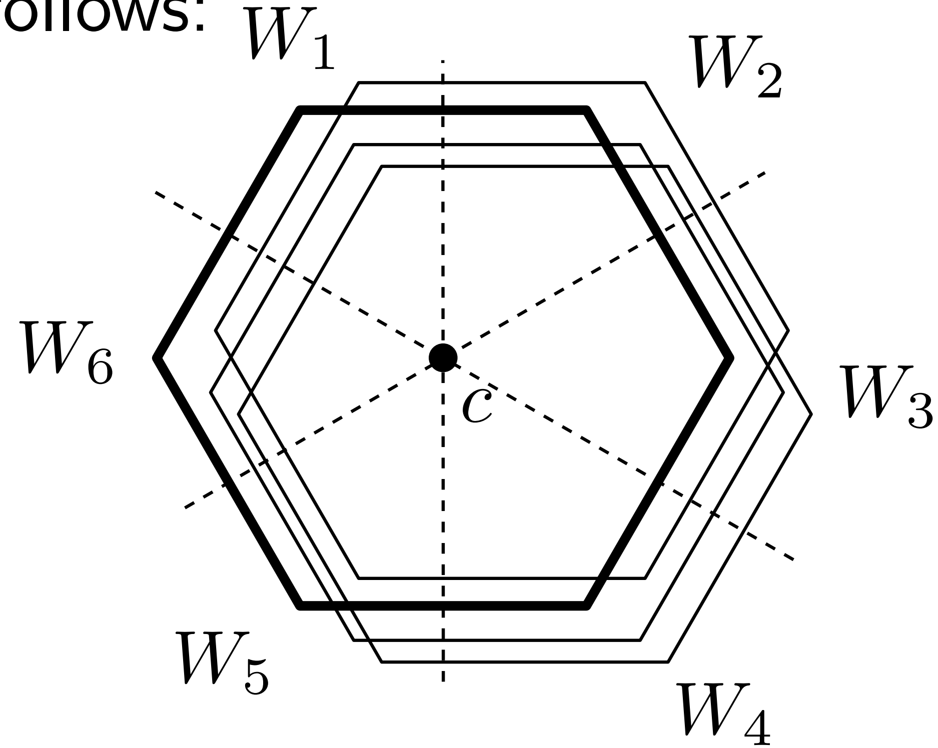

We just need to pick a subset with centers sufficiently close one to another.

If the radius of the circumscribed circle is unit $\sin(\frac{\pi}{k})$ would do.

Thus, $\frac{1}{k^2}$–fraction is still in the game.

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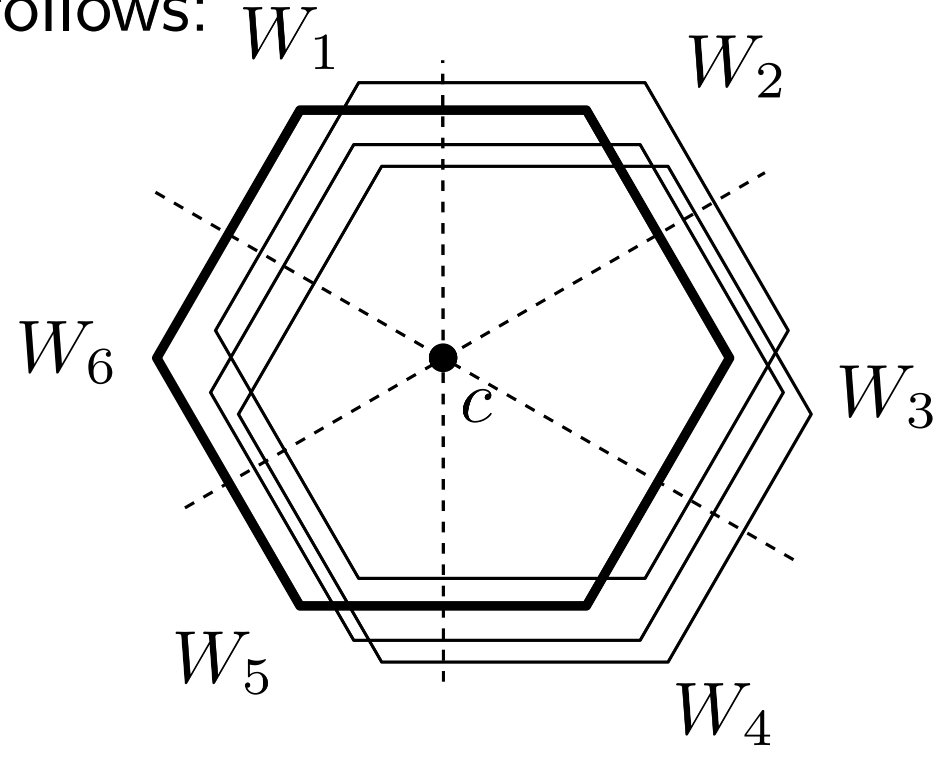

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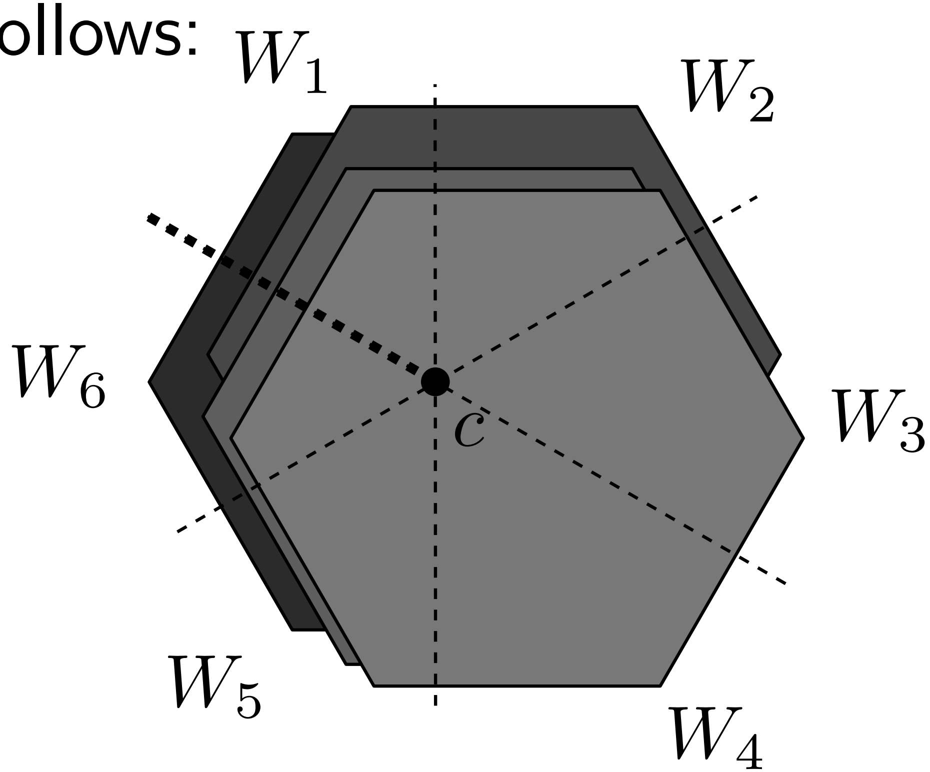

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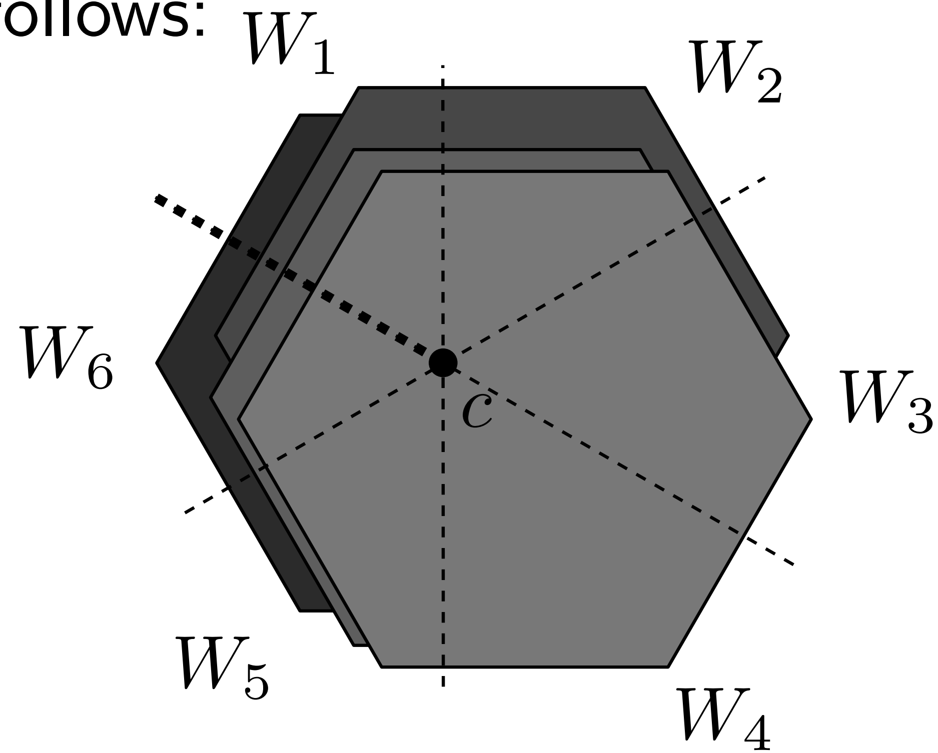

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This can be achieved by Dilworth Thm. or Erdős-Szekeres Lemma by picking $\sqrt{.}$ sets.

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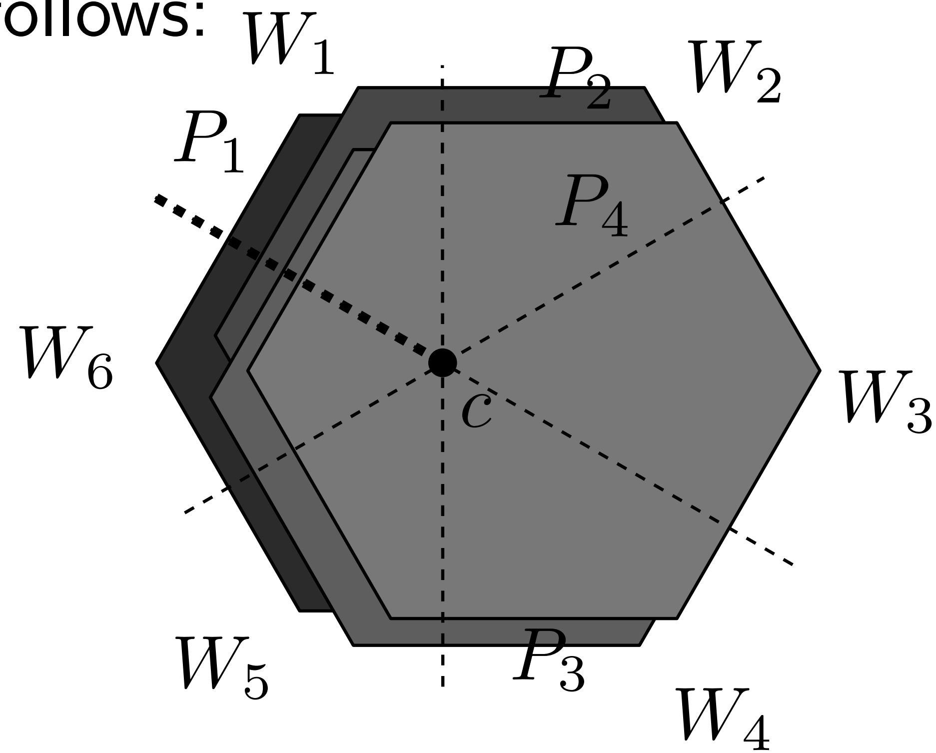

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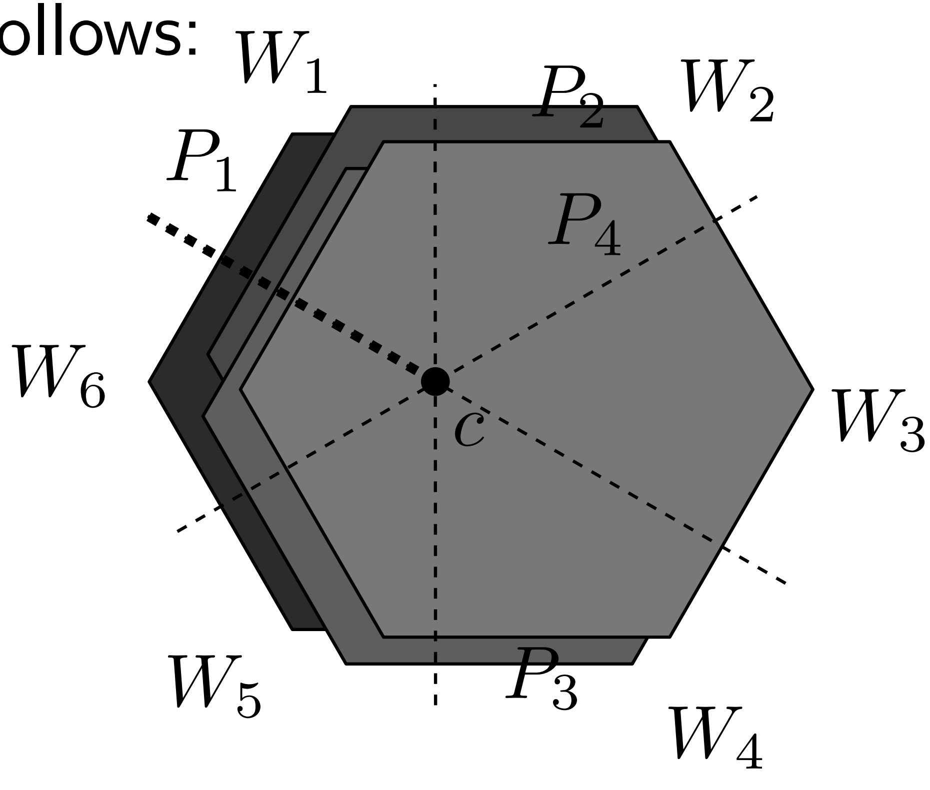

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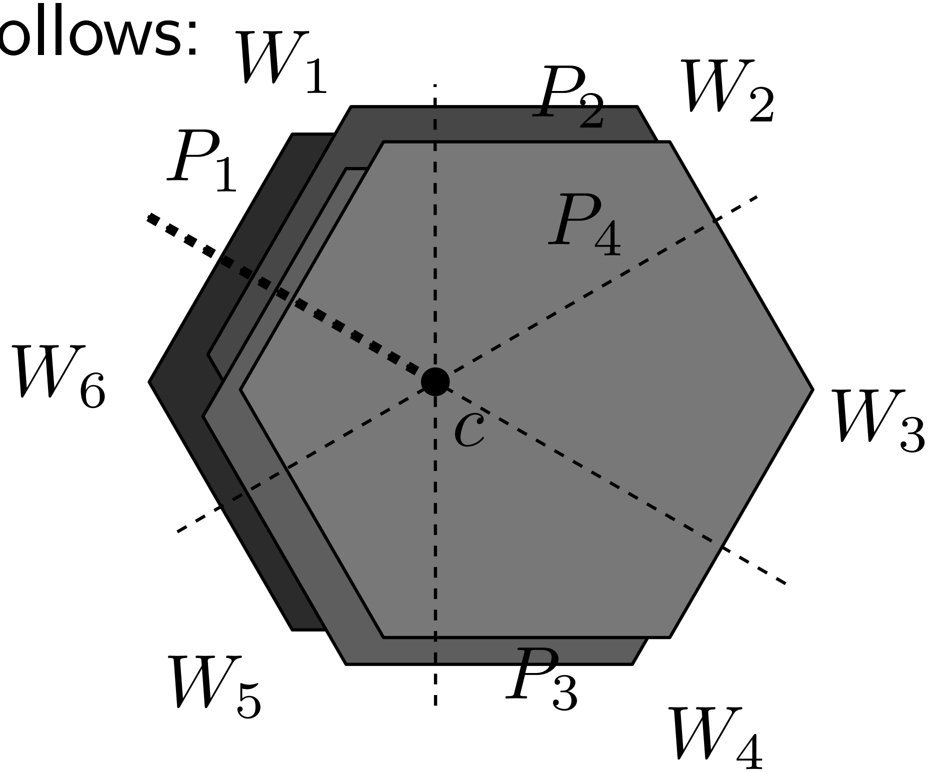

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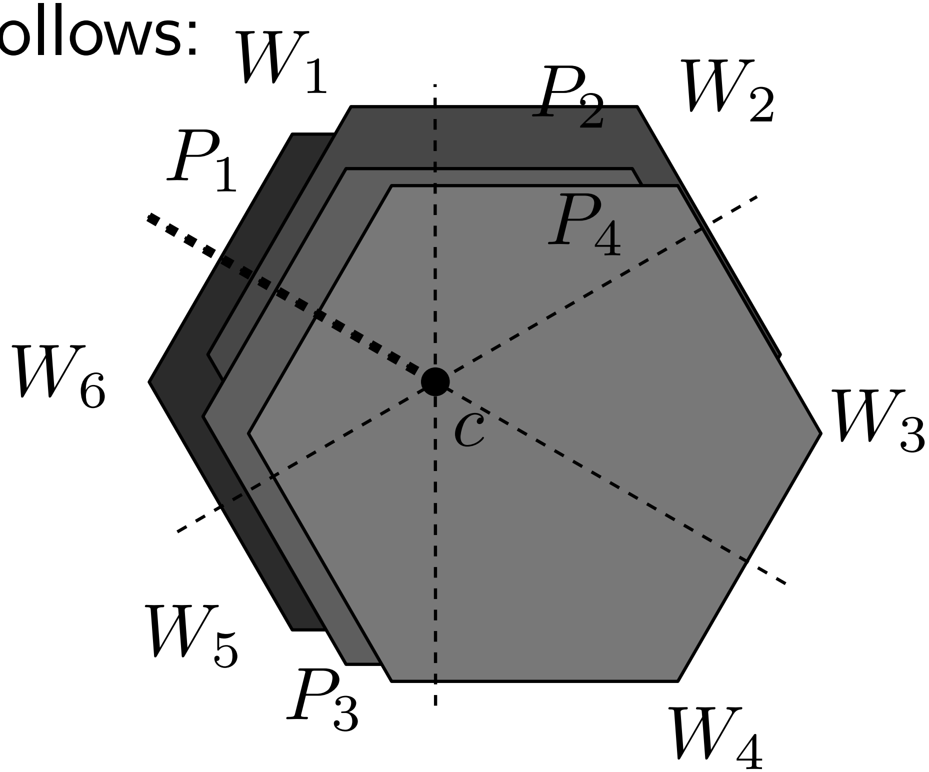

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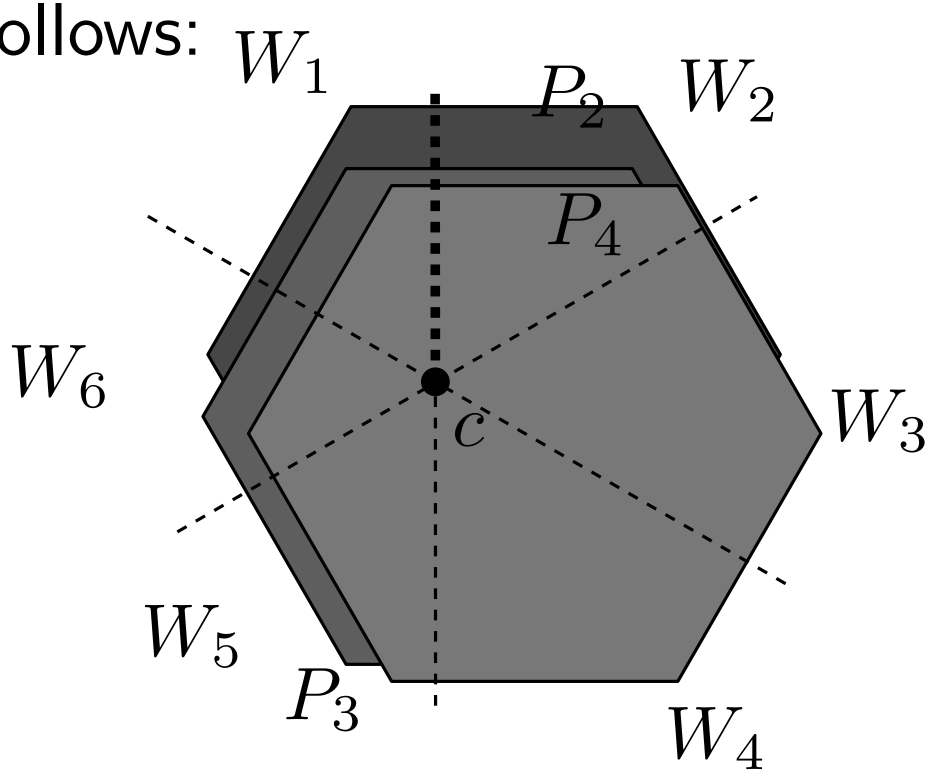

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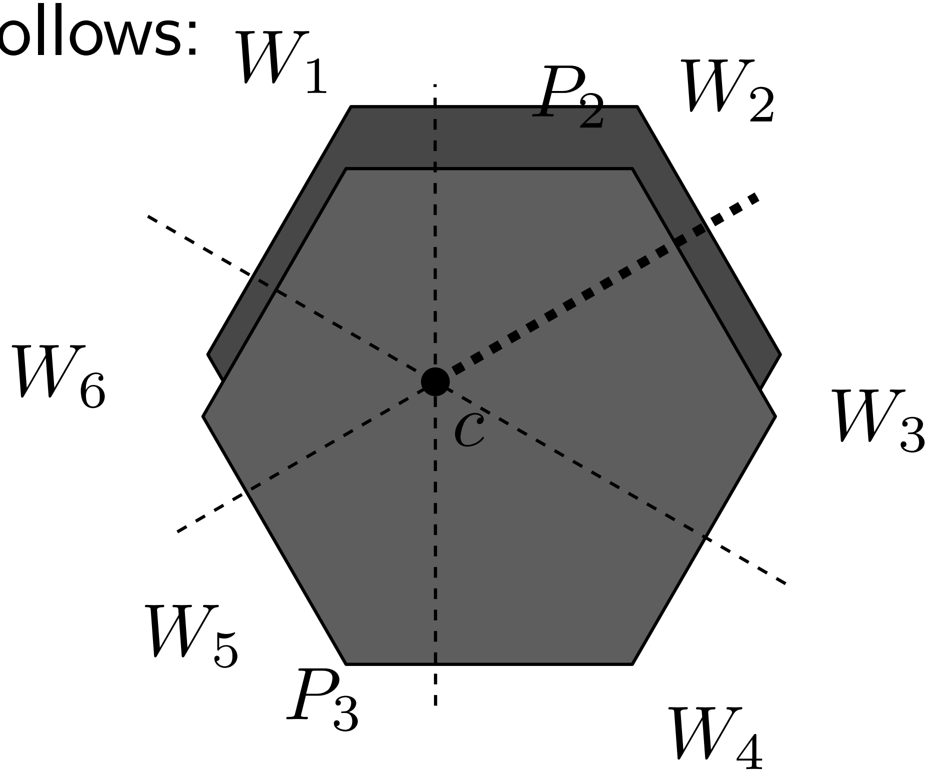

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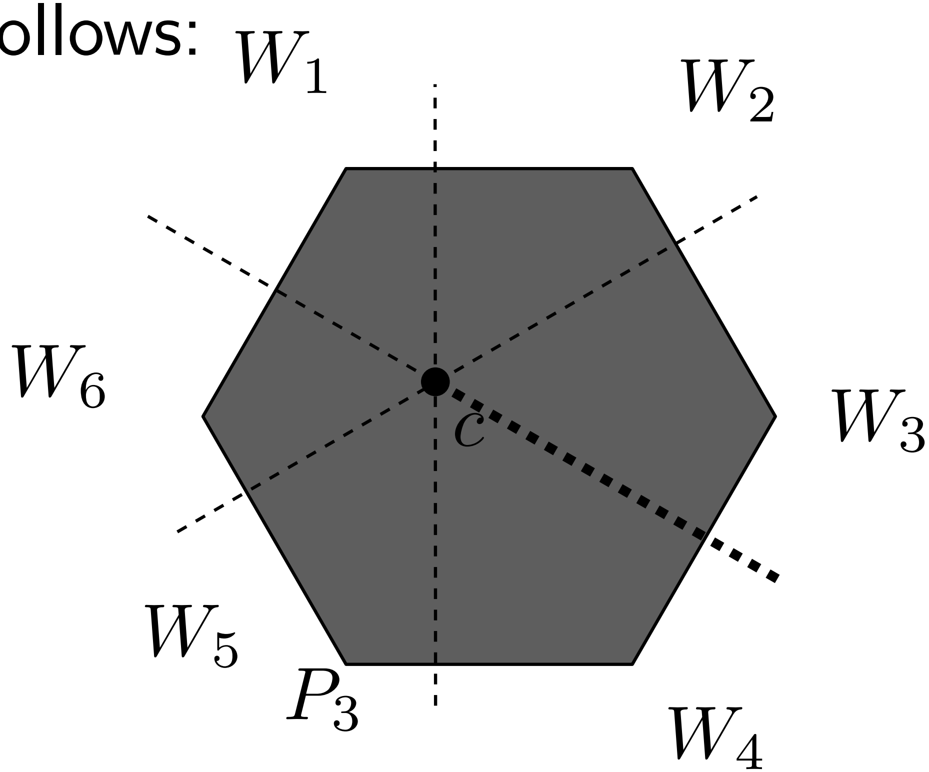

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Thus, only $k + 1$ translates remained.

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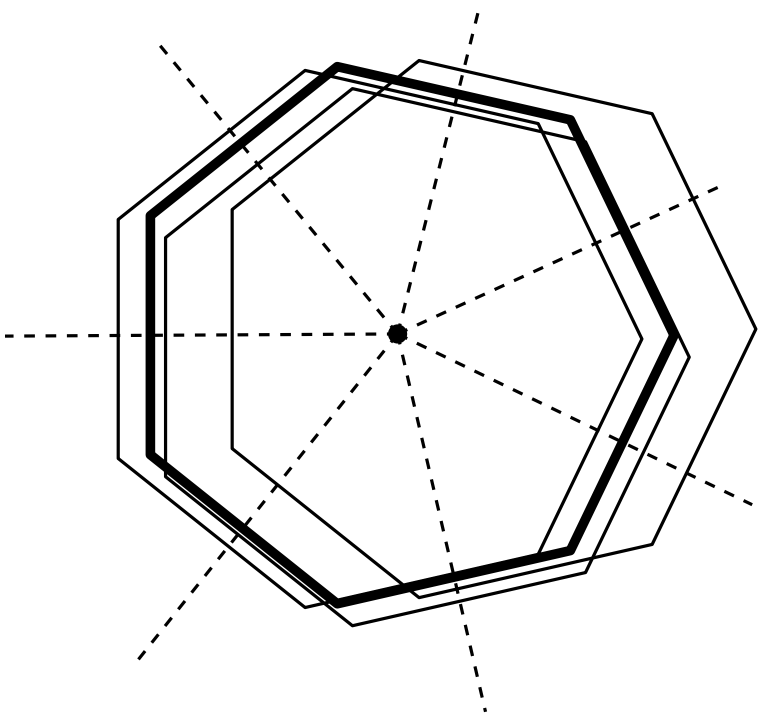

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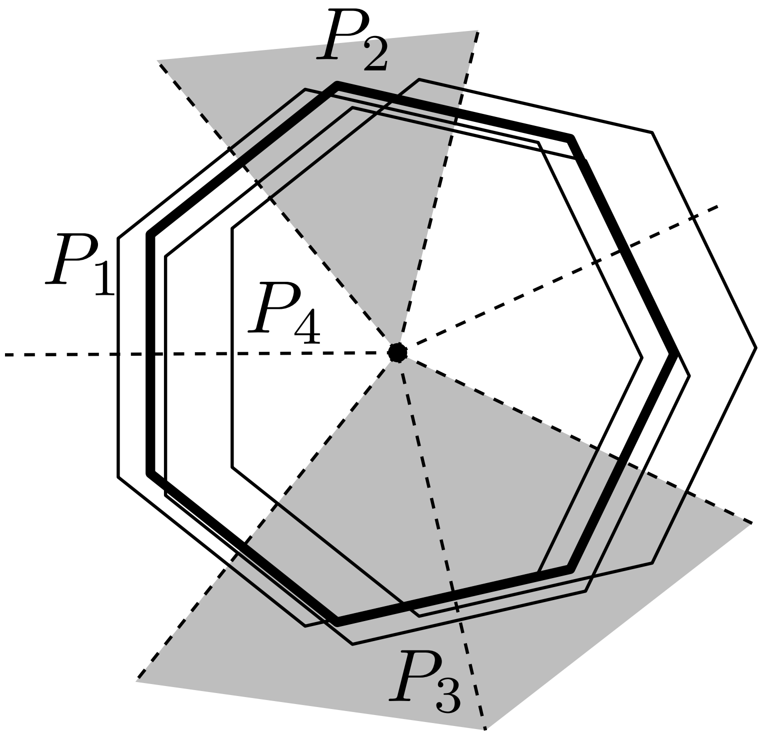

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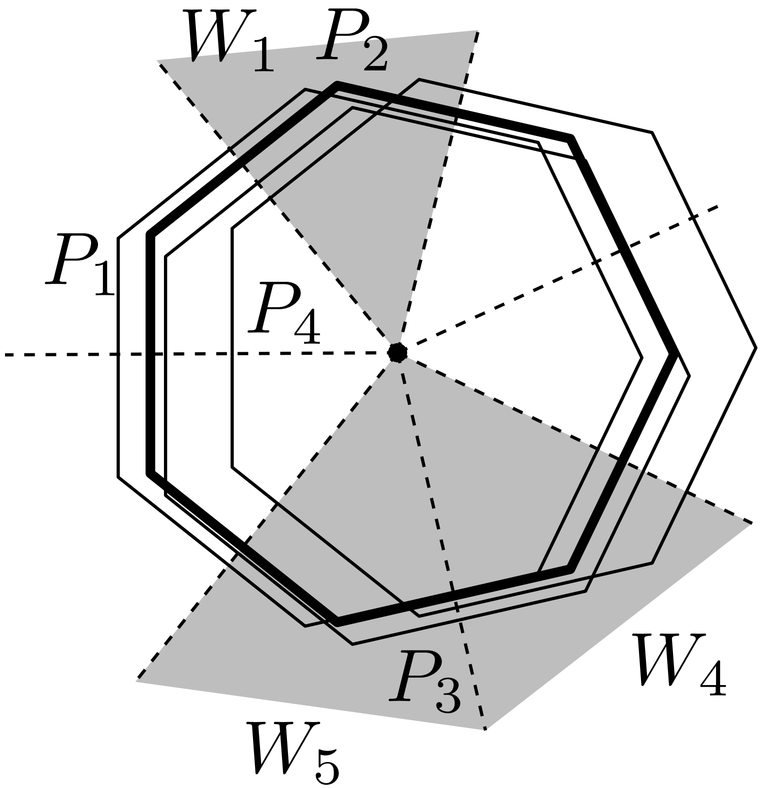

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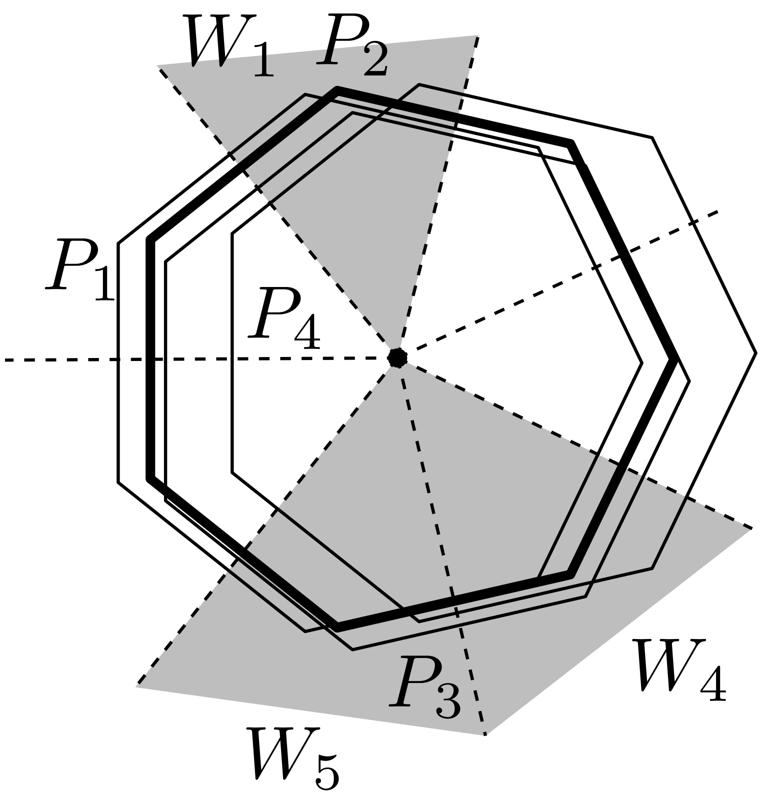

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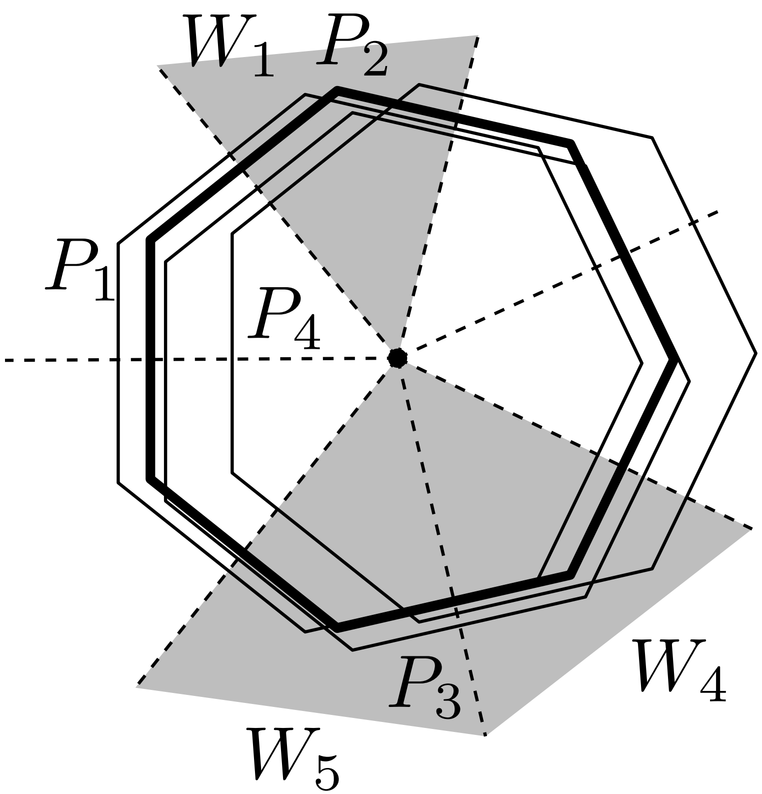

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If G_i contains c pairwise disjoint edges by a Ramsey argument we find an induced subgraph G of $G_{i+k/2 \mod k}$ or $G_{i-k/2 \mod k}$ with two disjoint edges forming a staircase such that G_{i+1} or G_{i-1} contains the same subgraph.

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