Rook-drawing for plane graphs

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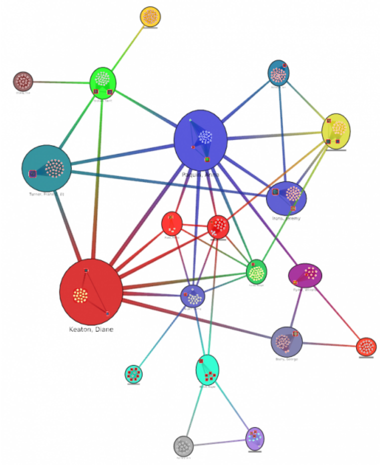
LaBRI, Bordeaux

Graph Drawing
September 25^{th}, 2015

Maybe you know them?

Drawing graphs

We want to draw **large** graphs with hierarchical view: a vertex in the drawing = a group of vertices in the graph



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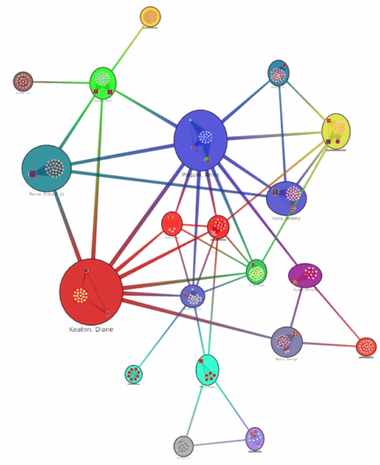

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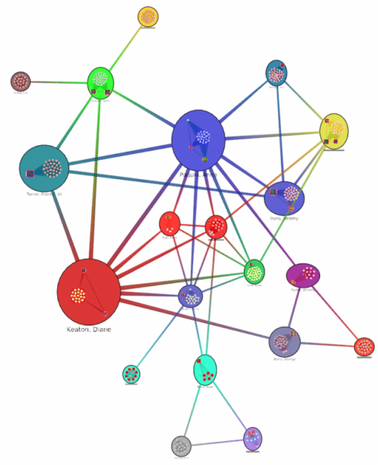

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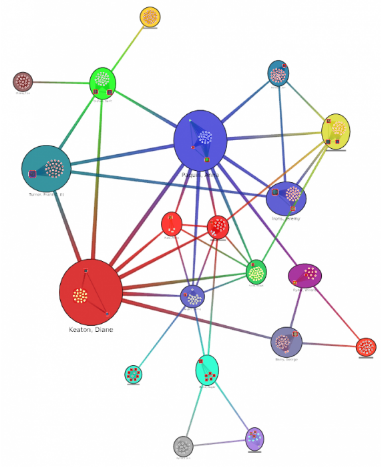

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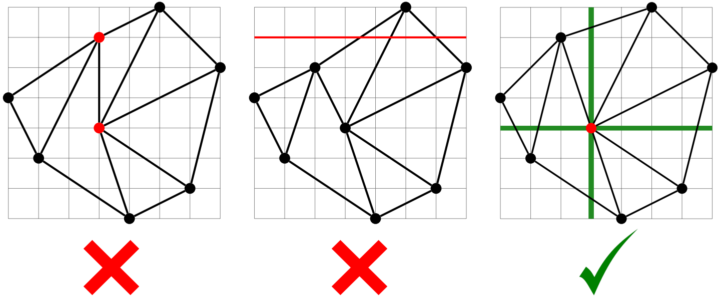
$\rightarrow$ new type of drawing with constraints: rook-drawing

Rook-drawing

A rook-drawing of a graph of n vertices:

- Straight-line edges
- Regular grid (n columns and rows)
- Exactly one vertex per row and column



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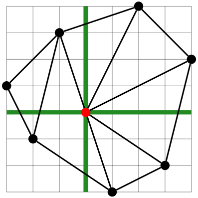

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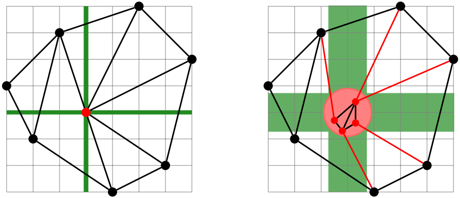

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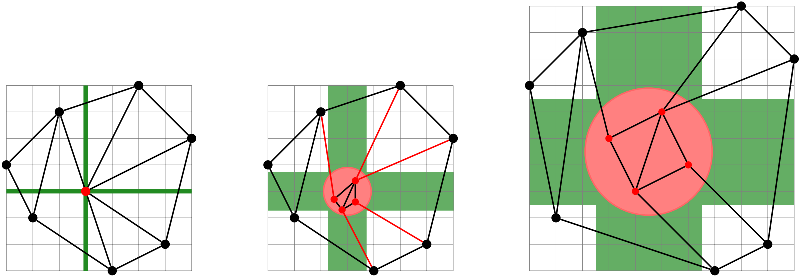

Planar rook-drawing

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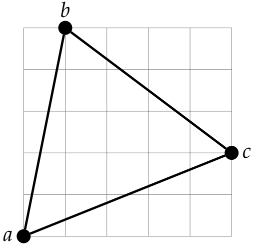
What we already know:

- Straight-lines drawing ([Fáry, 1948] : every planar graph)
- Grid drawing ([de Fraysseix, 1988], [Schnyder, 1990] : every plane graph on an $(n-2) \times (n-2)$ grid)

Outer face of degree 3



Three exterior vertices a, b and c, n vertices (here $n = 6$).

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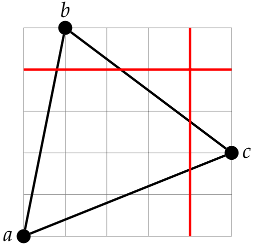

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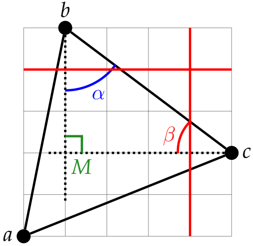

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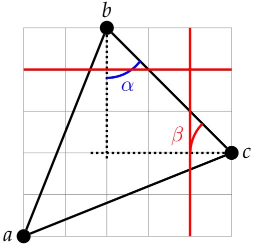

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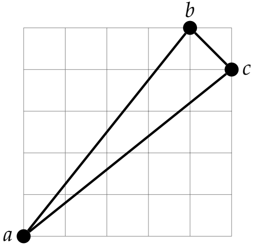

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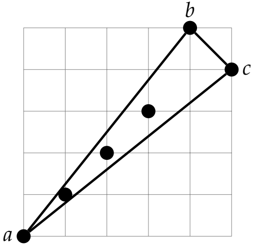

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Inner nodes: along a diagonal

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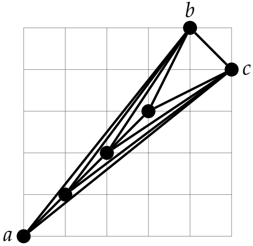

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Inner nodes: along a diagonal

Graph with a degree 3 outer face with planar rook-drawing = subgraph of the tower graph

Rook-drawing for outerplanar graphs

A graph is outerplanar if it has a planar drawing such that all its vertices are on the outer face.

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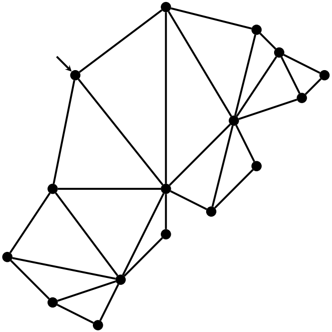

[Bonichon, Gavoille, Hanusse, 2005]

- edges of G maximal rooted outerplane graph $\rightarrow T_b, T_r$

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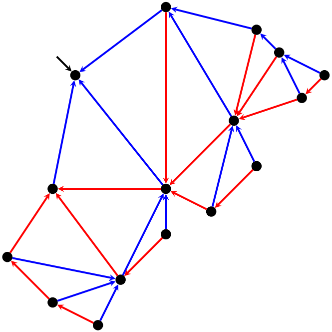

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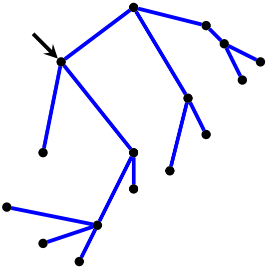

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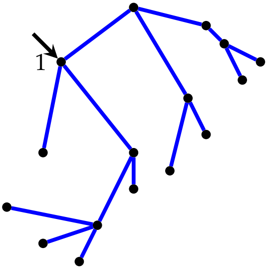

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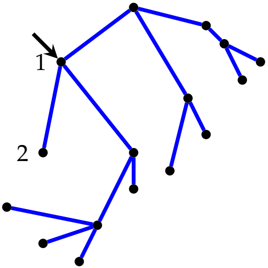

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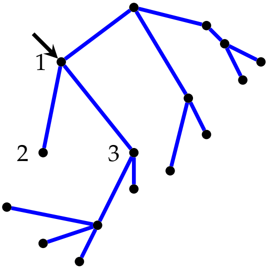

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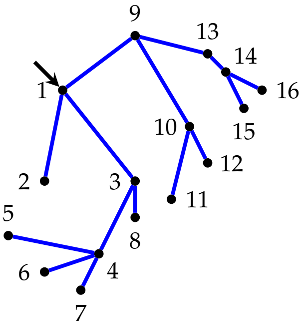

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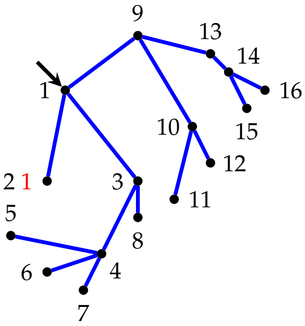

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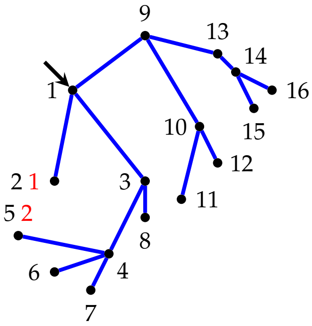

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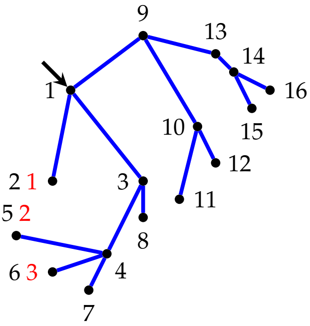

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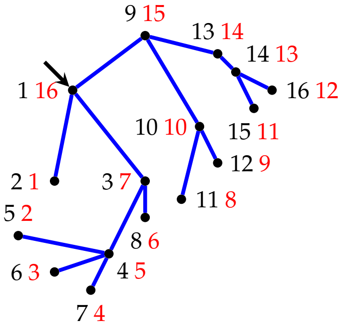

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ROOK-DRAWING FOR OUTERPLANAR GRAPHS

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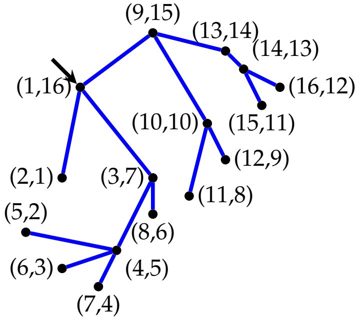

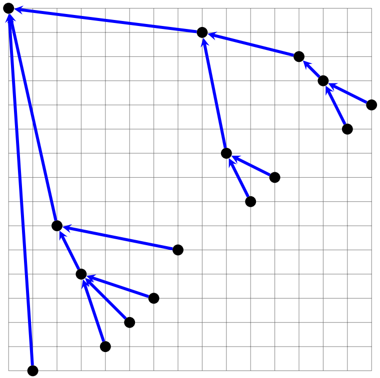

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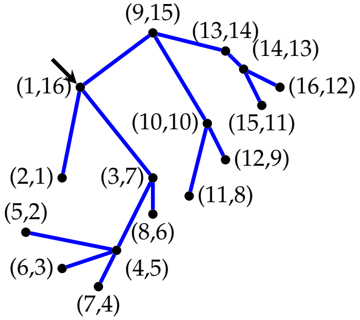

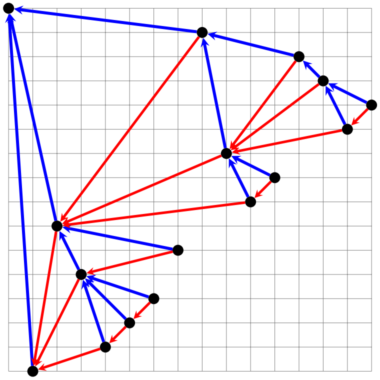

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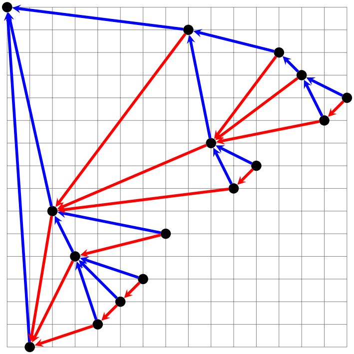

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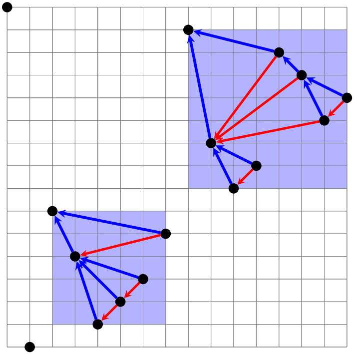

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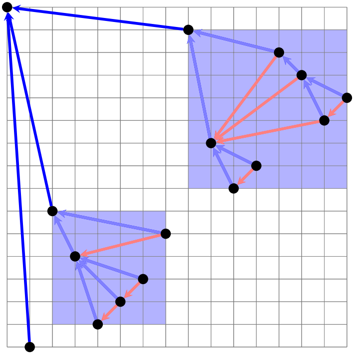

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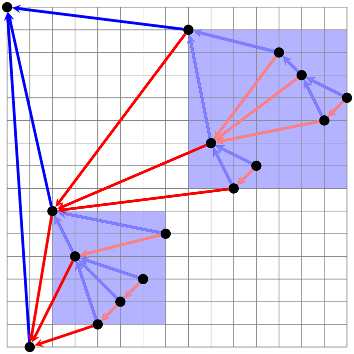

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Red edges: between u and the first vertex below u unrelated to it.
$\rightarrow$ between v_{i+1} and vertices on the left of the subtree of v_i: no crossings!

Polyline rook-drawing for planar graphs

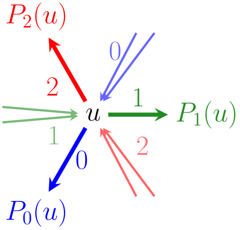
Main result

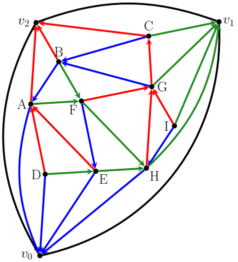
Every planar graph with n vertices admits a planar polyline rook-drawing, with at most $n-3$ bends (at most one per edge). Such a drawing can be computed in linear time.

G a triangulation (else, make it triangulated) with exterior vertices v_0, v_1 and v_2

Schnyder woods

A Schnyder wood is a partition of the internal edges of a triangulation in three trees T_0, T_1 and T_2 (directed toward the root) and with a particular configuration around each inner vertex:





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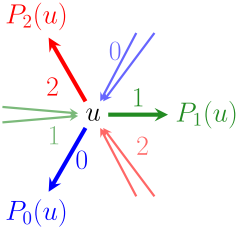
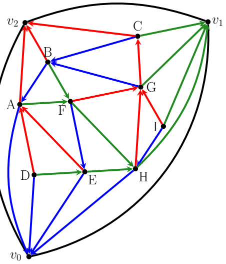


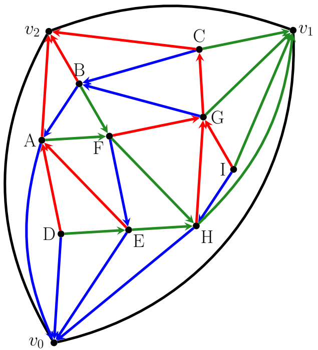


[Schnyder 1989]

Every plane triangulation admits at least one Schnyder wood, and it can be computed in linear time.

Planar polyline rook-drawing - Vertices

- (T_0, T_1, T_2): Schnyder wood of G.



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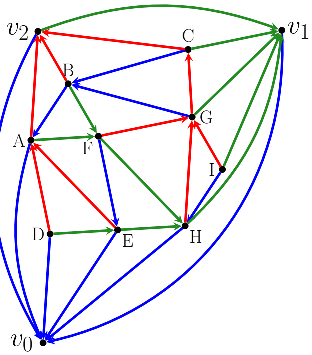

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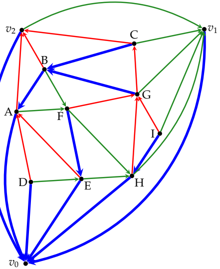

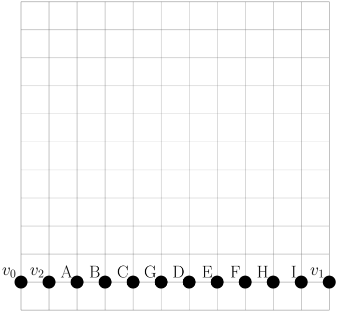

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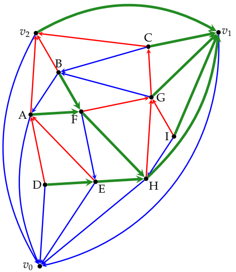
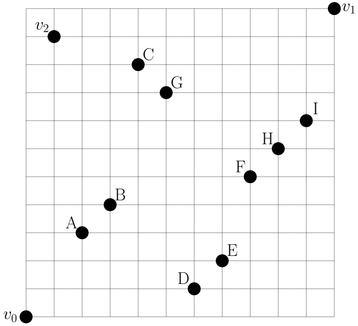
- y: clockwise postordering of $T_1 = \{DEABFHIGCv_2 v_1\}$ ($v_0 = 0$).





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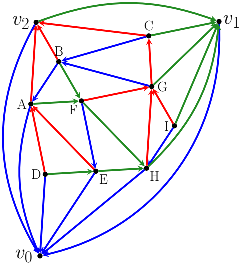
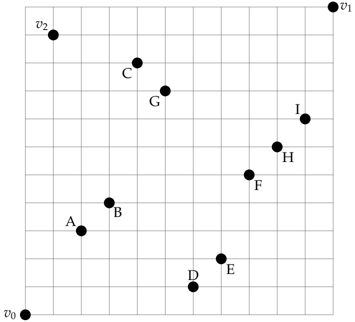
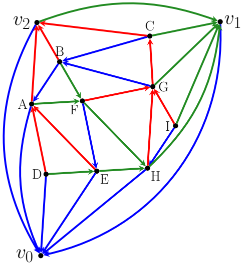
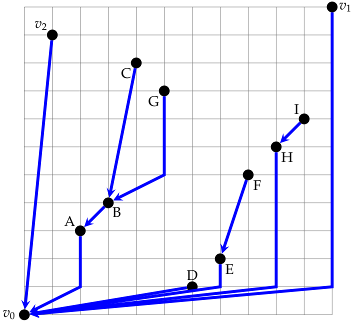




Planar polyline rook-drawing - Edges

- The edges $(u, P_0(u))$ are bent at $(x(u), y(P_0(u)) + 1)$ (except for the first child in T_0)





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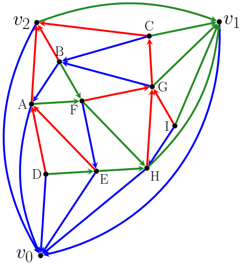
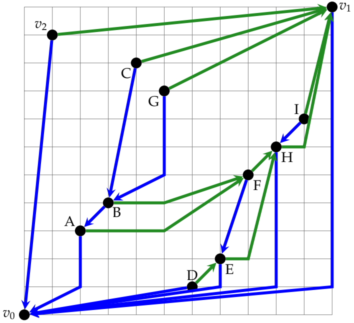
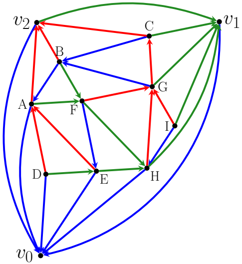
- The edges $(u, P_1(u))$ are bent at $(x(\text{last descendant}_0(u)), y(u))$ (no bend if u is a leaf of T_0)

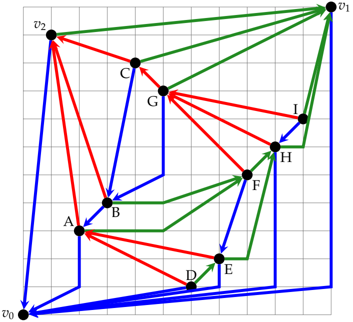




PLANAR POLYLINE ROOK-DRAWING - EDGES

- The edges $(u, P_0(u))$ are bent at $(x(u), y(P_0(u)) + 1)$ (except for the first child in T_0)
- The edges $(u, P_1(u))$ are bent at $(x(\text{last descendant}_0(u)), y(u))$ (no bend if u is a leaf of T_0)
- Edges of T_2: not bent





Conclusion

Open questions:

- Is a sublinear number of bends sufficient to draw any plane graph planarly?
- If G is a graph with no triangle outer face, what are the conditions to draw G planarly?
- What is the minimum grid size requested to draw a planar straight-lines rook-drawing for a given plane graph? Is this minimum a constant?

Thank you for your attention!