Small-Area Orthogonal Drawings of 3-connected graphs

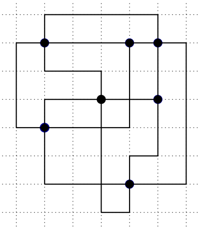
Therese Biedl [1] Jens Schmidt [2]

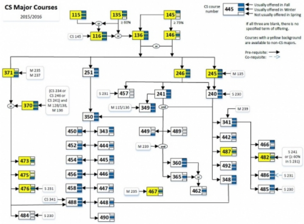
[1]University of Waterloo, Canada. *biedl@uwaterloo.ca*

[2]TU Ilmenau, Germany. *jens.schmidt@tu-ilmenau.de*

September 25, 2015

Orthogonal graph drawing





- Vertices of graph → points on an (integer) grid.
- Edges of graph → paths of the grid.
- No overlap or touching. Crossings ok.

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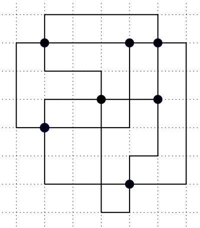

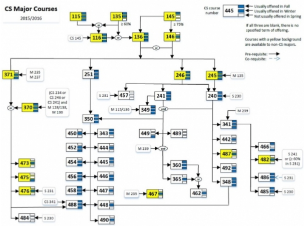

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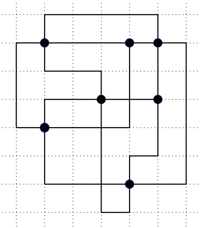
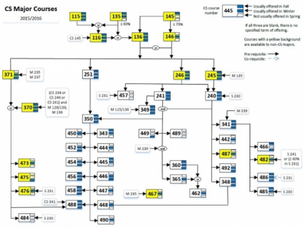
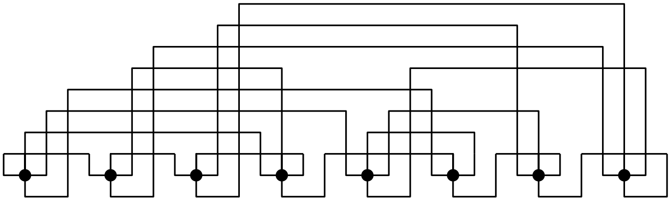




- Vertices of graph → points on an (integer) grid.
- Edges of graph → paths of the grid.
- No overlap or touching. Crossings ok.
- Requirement: At most 4 edges at each vertex (*4-graph*)
- Goal: Small area if drawn on grid.
  - width $W = \#$ columns, height $H = \#$ rows
  - area $A := H \cdot W = \#$ grid points.

Orthogonal drawing basics

- Easy to see: every 4-graph has an orthogonal drawing



- $\approx 4n \times 2n$-grid $\Rightarrow$ area $\leq 8n^2 + O(n)$.
- Can we do better?

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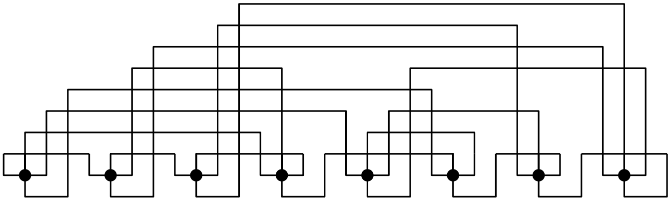

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- Can we do better?
- Valiant'81: Some 4-graphs need $\Omega(n^2)$ area.
  - Some 4-graphs need $\Omega(n^2)$ crossings.
  - Each crossing needs a grid-point.
- But constant is tiny! (area $\geq 10^{-6}n^2$ or so...)

Orthogonal drawing timeline

Orthogonal drawings with area $\leq c \cdot n^2 + o(n^2)$:

$\downarrow$ $8n^2$ area

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n^2 area (Let's review this, we'll build on top)

BK-algorithm revisited

- Assume G is 2-connected 4-graph.

  (Handle 2-connected components separately, then merge.)

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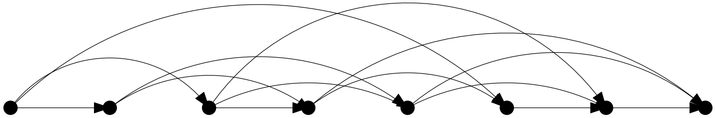
- Find a vertex order $v_1, \ldots, v_n$ that is an *st-order*:



  - $indeg(v_i) \geq 1$ for $i > 1$.
  - $outdeg(v_i) \geq 1$ for $i < n$.

(This exists [LEC'64] and can be found in linear time [ET'76].)

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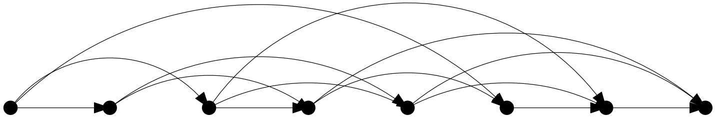

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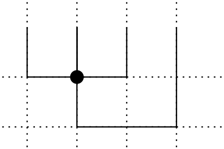
- For 4-graph: $indeg(v) \in \{1, 2, 3\}$ except for v_1, v_n.

BK-algorithm revisited

Add vertices to drawing in order.

Place v_1:



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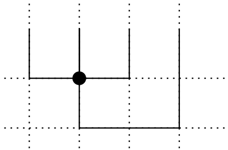

- Invariant: All unfinished edges end in upward ray.

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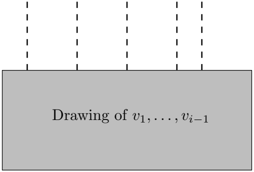
Place v_i, $indeg(v_i) = 3$:



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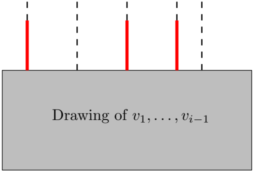

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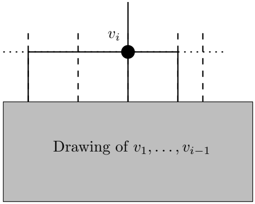
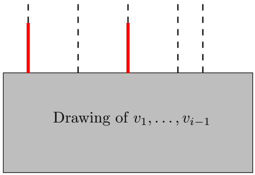


- Invariant: All unfinished edges end in upward ray.
- Rows: added 1
- Columns: added $0 \leq outdeg(v_i) - 1$

BK-algorithm revisited

Add vertices to drawing in order.

Place v_i, $indeg(v_i) = 2$:



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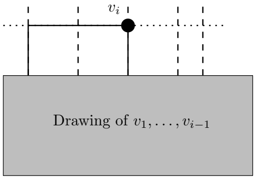

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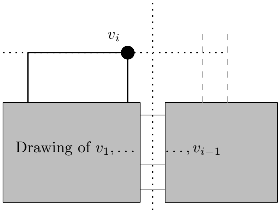

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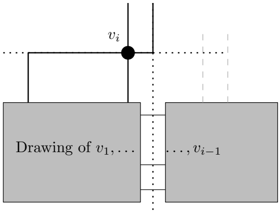
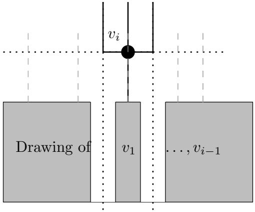


- Invariant: All unfinished edges end in upward ray.
- Rows: added 1
- Columns: added $1 \leq outdeg(v_i) - 1$

BK-algorithm revisited

Add vertices to drawing in order.

Place v_i, $indeg(v_i) = 1$:

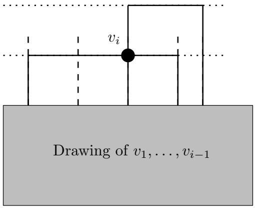


- Invariant: All unfinished edges end in upward ray.
- Rows: added 1
- Columns: added $2 \leq outdeg(v_i) - 1$

BK-algorithm revisited

Add vertices to drawing in order.

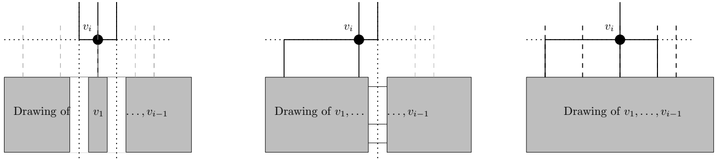
Place v_i, $indeg(v_i) = 4$:



- Invariant: All unfinished edges end in upward ray.
- Only happens for $i = n$.

BK-algorithm revisited

Add vertices to drawing in order.



- Invariant: All unfinished edges end in upward ray.
- Rows: added 1 (if $i \neq 1, n$)
- Columns: added $outdeg(v_i) - 1$ (if $i \neq 1, n$)

Theorem (B.,Kant)

Every 4-graph has an orthogonal drawing in an $n \times n$-grid.

Orthogonal drawing timeline

Orthogonal drawings with area $c \cdot n^2 + o(n^2)$:

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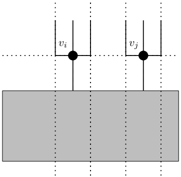
PT-algorithm: improvement by pairing

- Basic approach same as BK-algorithm.
- But: define some vertex-pairs $\{v_i, v_j\}$.
- Draw v_i and v_j together and save a row and/or column.



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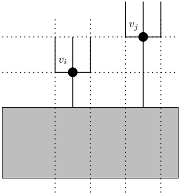

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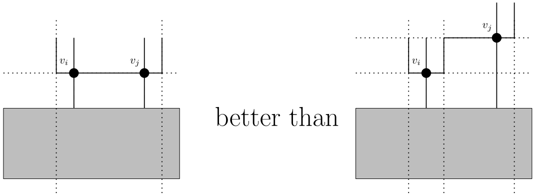

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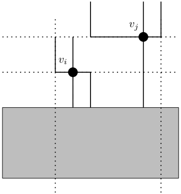

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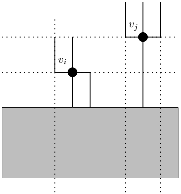

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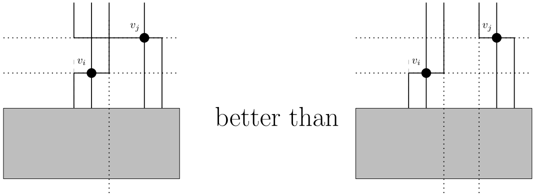

- Half-perimeter $HP = \#\text{rows} + \#\text{columns} \le 2n - \#\text{pairs}$

(ignoring lower-order terms)

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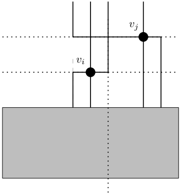

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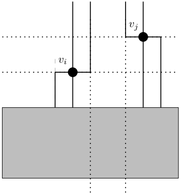

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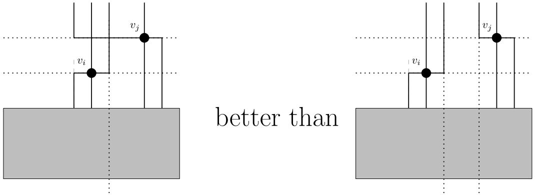

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- Show: $\#\text{pairs} \geq \frac{1}{4}n \quad \Rightarrow HP \leq \frac{7}{4}n$
- Area $\leq \left(\frac{HP}{2}\right)^2 \leq \left(\frac{7}{8}n\right)^2 \approx 0.76n^2$

Theorem (Papakostas & Tollis)

Every 4-graph has an orthogonal drawing with area at most $0.76n^2$.

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2015	This talk: **$\mathbf{0.56n^2}$ area for 3-connected graphs**

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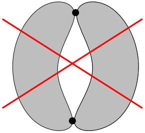

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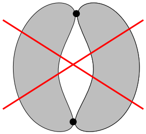
- Needed here: *canonical order*.



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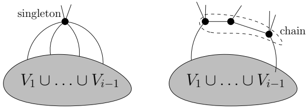
    - Straight-line convex grid-drawings
    - Orthogonal planar drawings with area $\approx 0.44n^2$
    - Lots more ...



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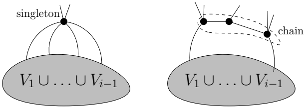

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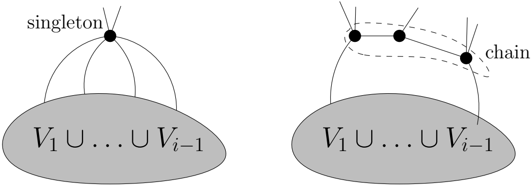
  - But does this exist for non-planar graphs?

Canonical order for non-planar 3-connected graphs



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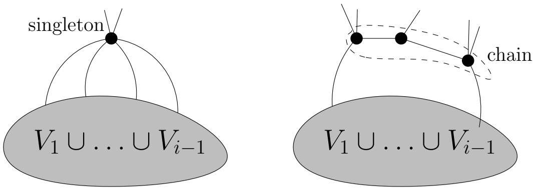

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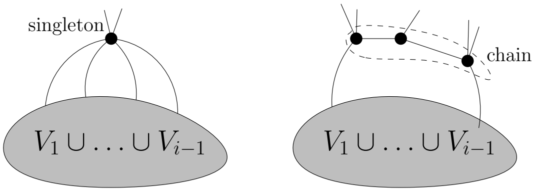

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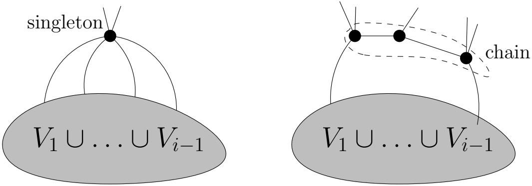

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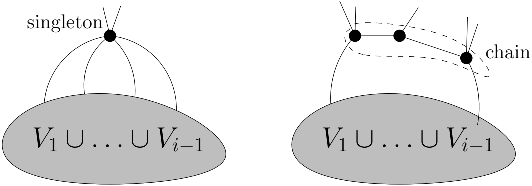

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- Schmidt, 2014: Mondshein sequence (hence canonical order) can be found in linear time.

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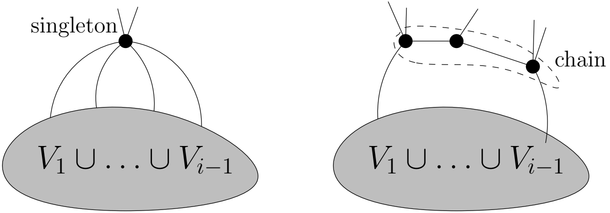

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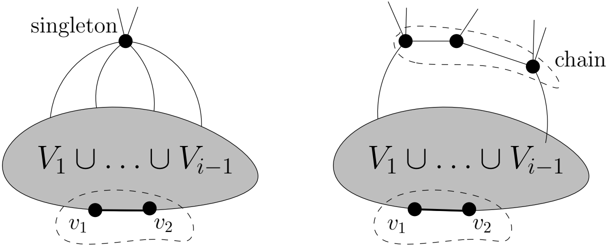

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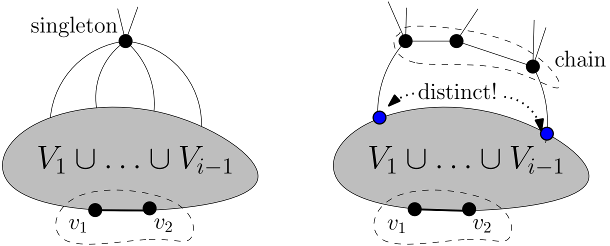

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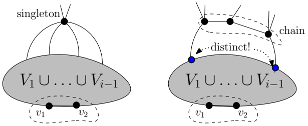

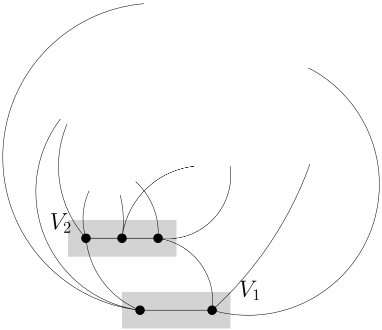

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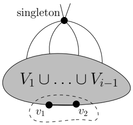

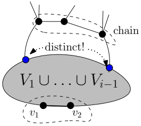

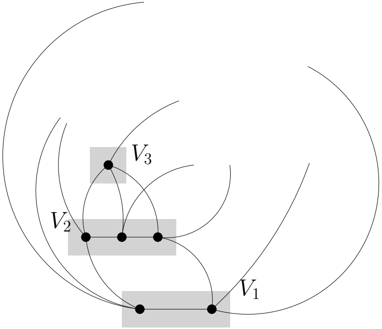

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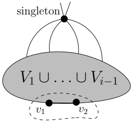

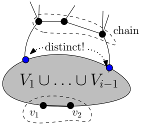

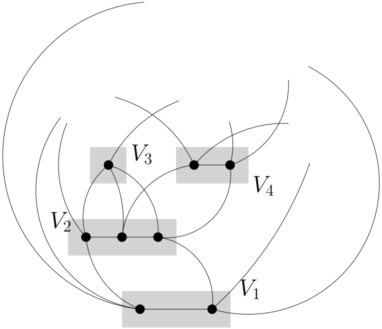

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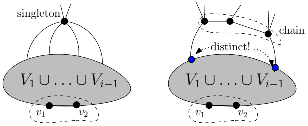

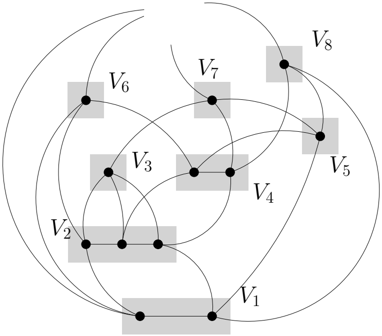

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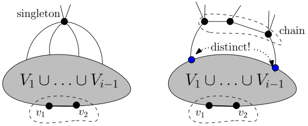

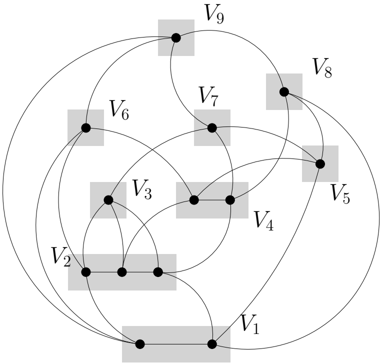

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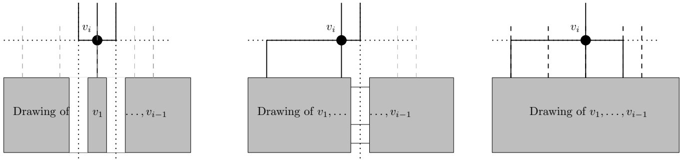

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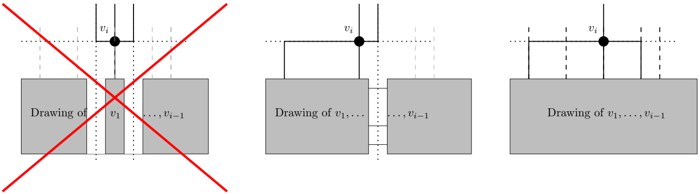

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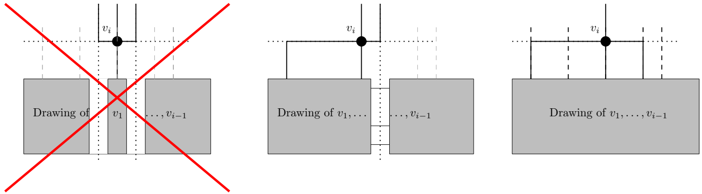

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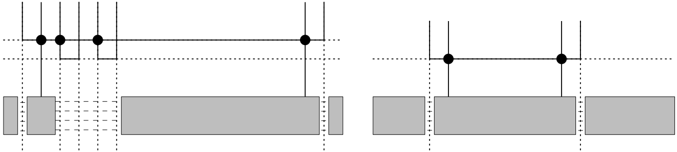

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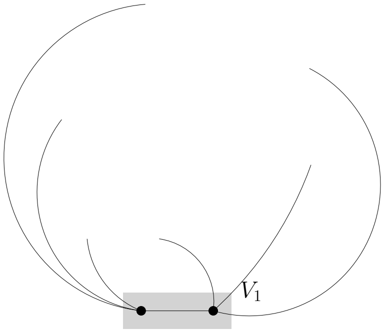

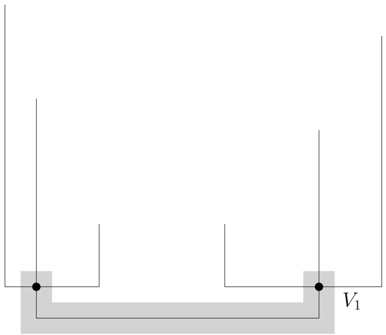

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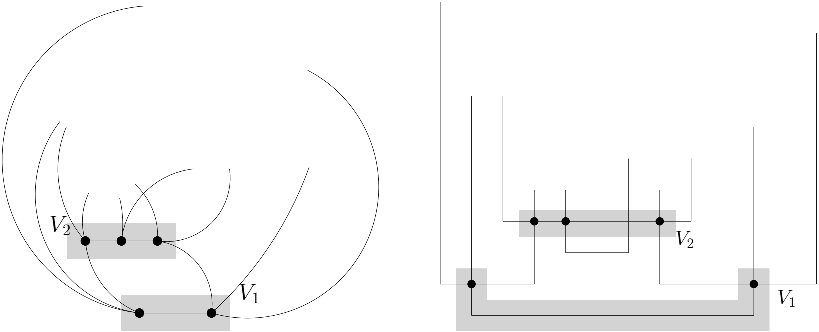

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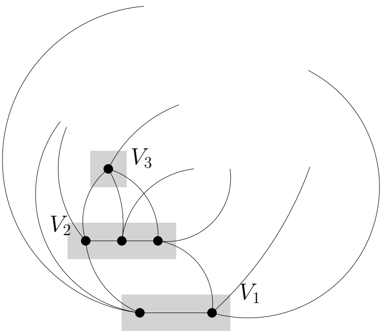

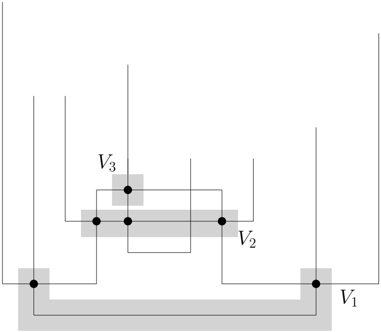

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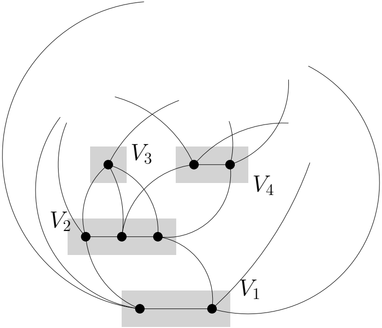

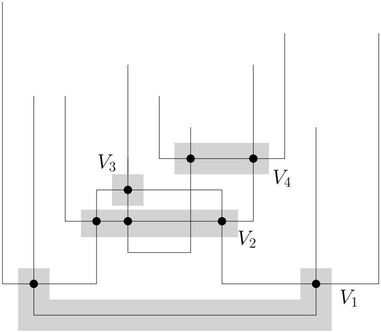

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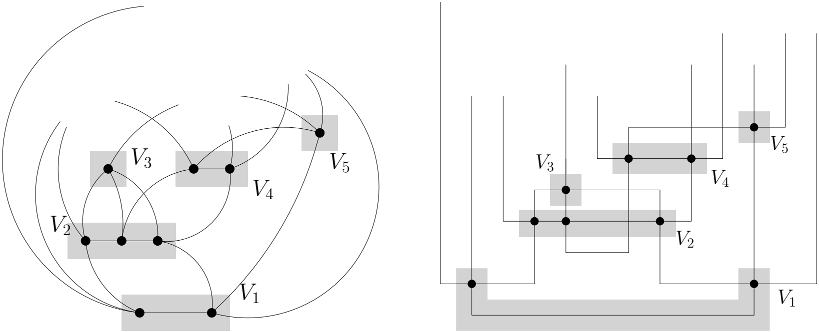

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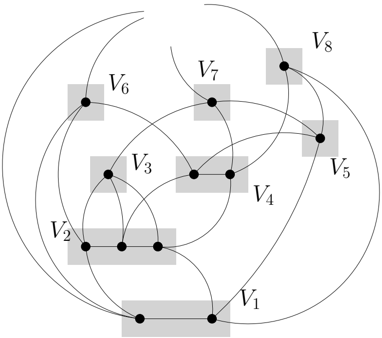

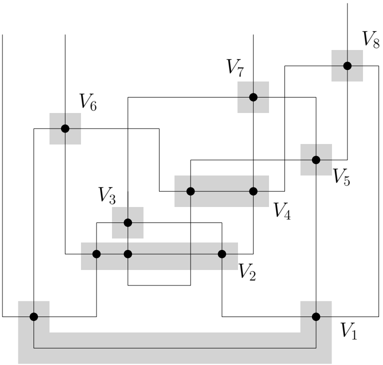

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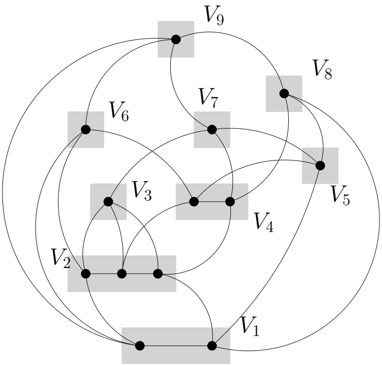

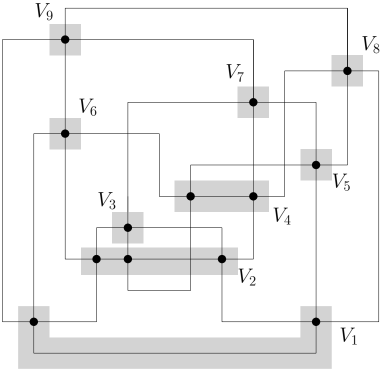

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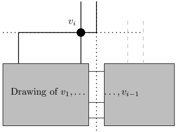

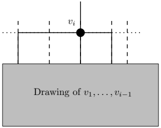

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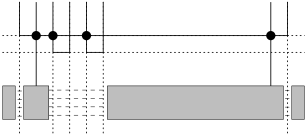

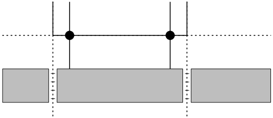

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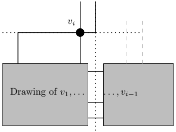

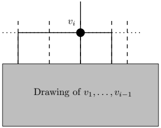

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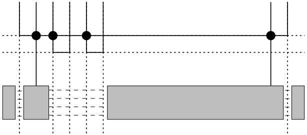

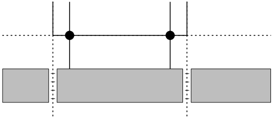

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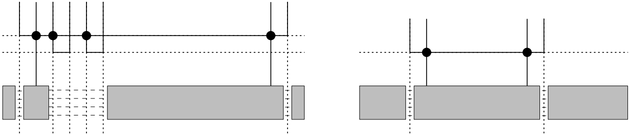

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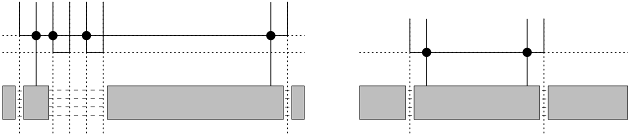

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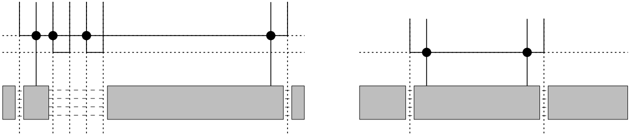

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- But that could still be $\approx 2n$.

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- Not both algorithms can be in worst-case.

Theorem

One of the above two algorithms achieves half-perimeter $\leq \frac{5}{3}n$ *(hence area* $\approx 0.69n^2$*).*

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- Now: Use pairing-idea for vertex groups.
- Define vertex-group-pairs $\{V_i, V_j\}$ in a special way
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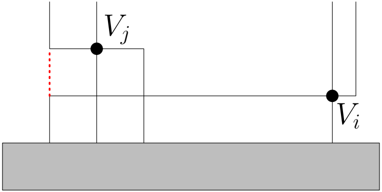

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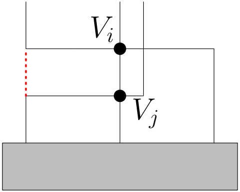

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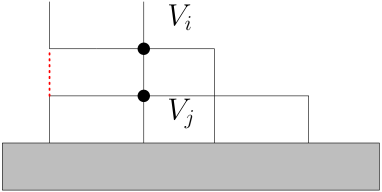

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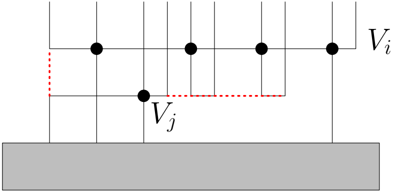

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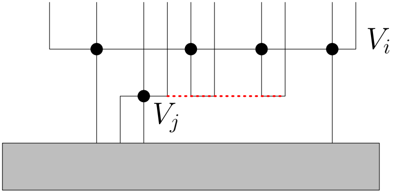

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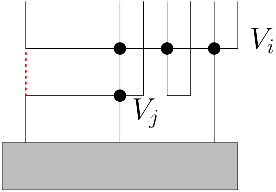

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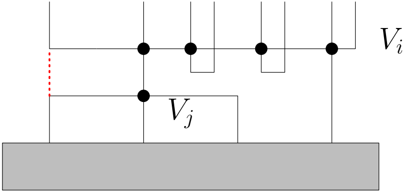

(3-singleton + long chain, one edge)

- $HP \leq \frac{3}{2}n + \frac{1}{2}\#\{indeg\text{-2-vertices}\} - \#\{\text{pairs}\}$

Orthogonal drawing with 3-canonical order

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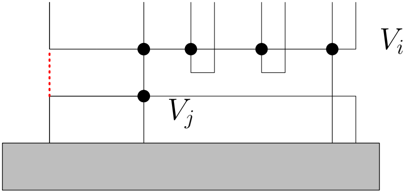

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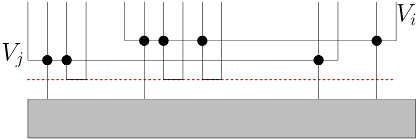

(long chain + long chain, no edge)

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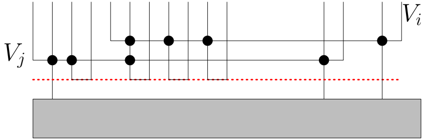

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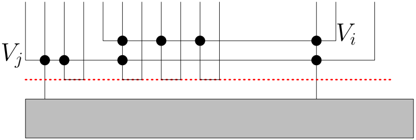

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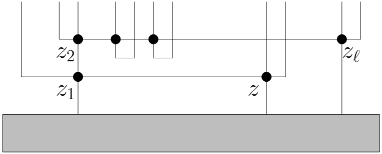

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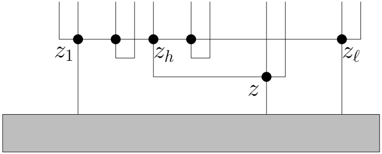

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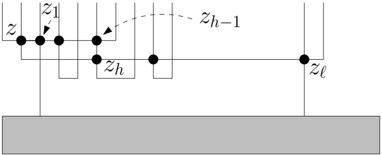

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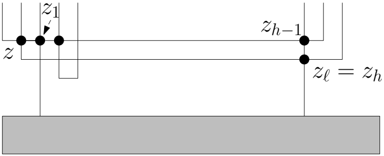

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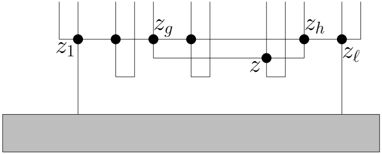

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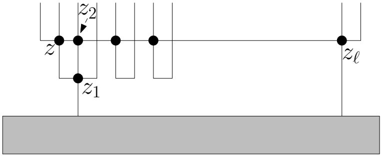

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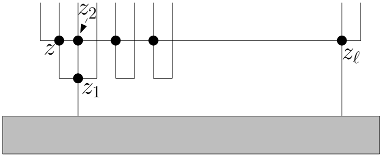

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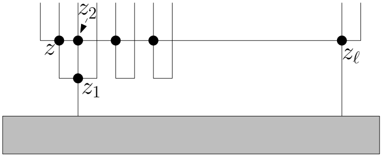

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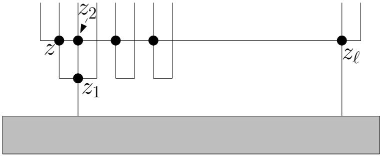

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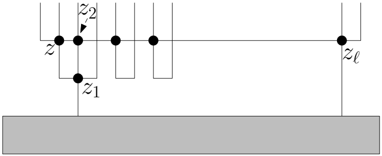

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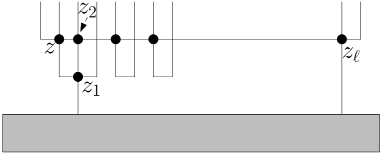

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Theorem

Every 3-connected 4-graph has an orthogonal drawing with area at most $(\frac{3}{4}n)^2 \approx 0.56n^2$.

Conclusion

↓	$8n^2$ area
1992	Schäffter (published in 1995)
↓	$4n^2$ area
1993	B. (published as B.& Kant 1994/1998)
↓	n^2 area
1994	Papakostas & Tollis, 1996/1998
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2015	This talk:
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Conclusion

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↓ $0.76n^2$ area

2015 This talk:

↓ **$\mathbf{0.56n^2}$ area for 3-connected graphs**

20??

Can you do better?

Conclusion and Open Problems

- Can we achieve $0.56n^2$ area for 2-connected 4-graphs?

  (Merge 3-connected components?)

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