Simple realizability of complete abstract topological graphs simplified

Jan Kynčl

Charles University, Prague

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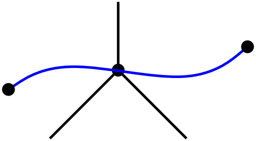

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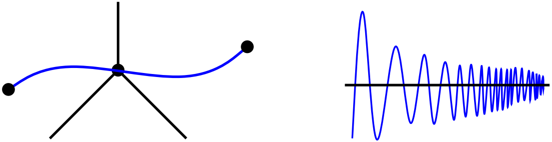

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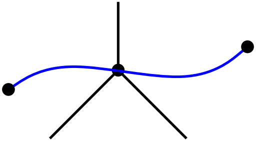

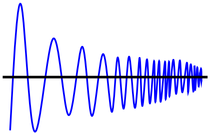

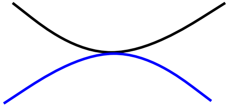

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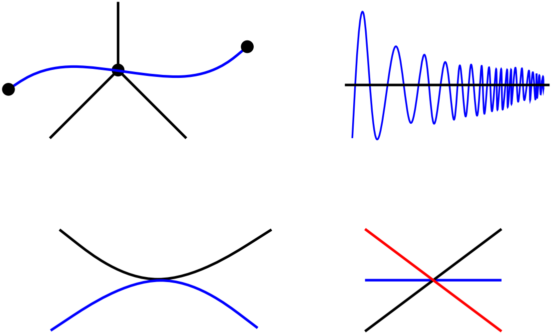

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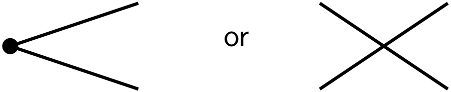

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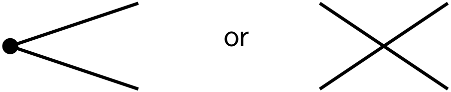

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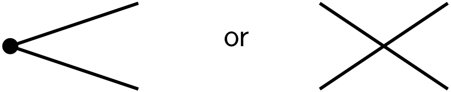

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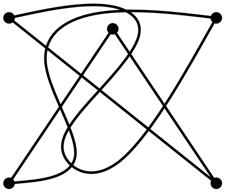

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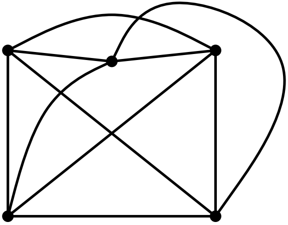

simple complete topological graph

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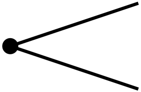

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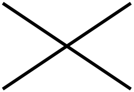

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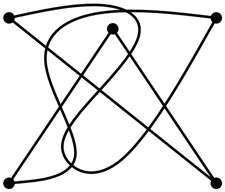

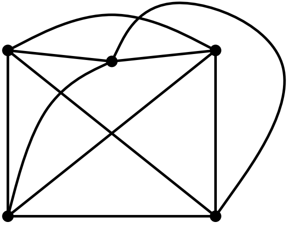

topological graph

drawing

simple complete topological graph

simple drawing of K_5

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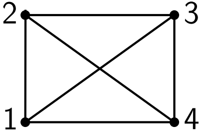

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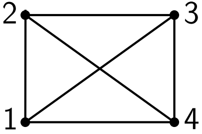

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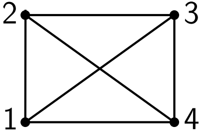

$A = (K_5, \emptyset)$ is not simply realizable

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instance: AT-graph A
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"The proof in [..] only gives a highly complex testing procedure, but no description in terms of forbidden minors or crossing configurations."
— M. Chimani, 2011

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- Ábrego, Aichholzer, Fernández-Merchant, Hackl, Pammer, Pilz, Ramos, Salazar and Vogtenhuber (2015) generated a list of simple drawings of K_n for $n \leq 9$

Proof of Theorem 1 (sketch)

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2) computing the homotopy classes of edges with respect to a star
3) computing the minimum crossing numbers of pairs of edges

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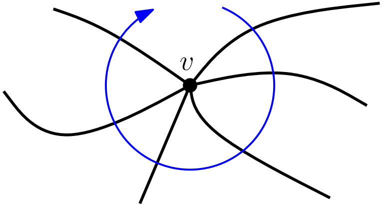

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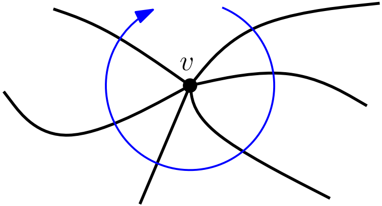

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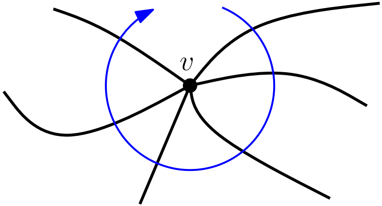

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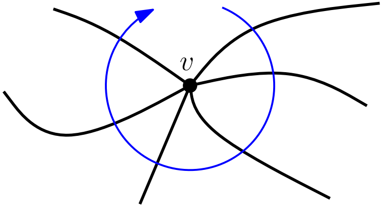

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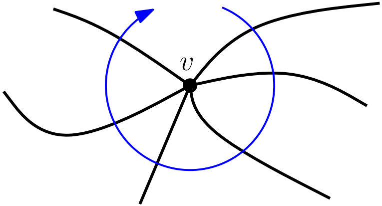

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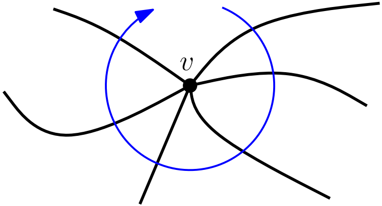

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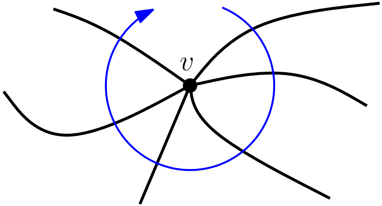

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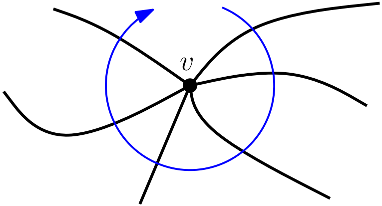

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Ábrego et al. (pers. com.) verified that an abstract rotation system (ARS) of K_9 is realizable if and only if the ARS of every 5-tuple is realizable, and conjectured that this is true for any K_n.

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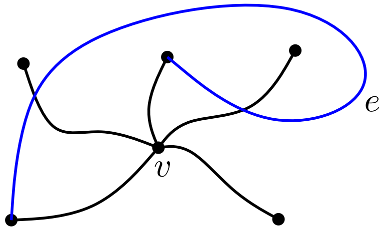

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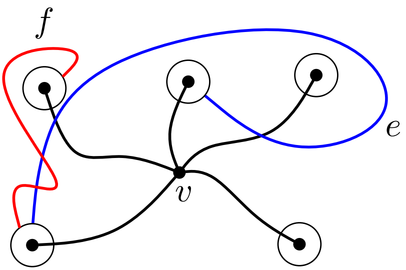
- drill small holes around the vertices, fix the endpoints of the edges on the boundaries of the holes



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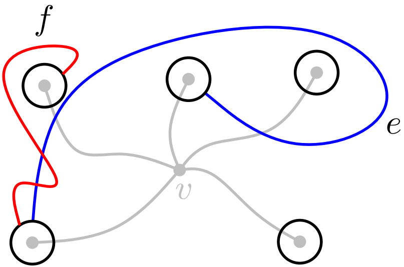

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Fact: (follows e.g. from Hass–Scott, 1985) It is possible to pick a representative from the homotopy class of every edge so that in the resulting drawing, all the crossing numbers $\mathrm{cr}(e, f)$ and $\mathrm{cr}(e)$ are realized simultaneously.

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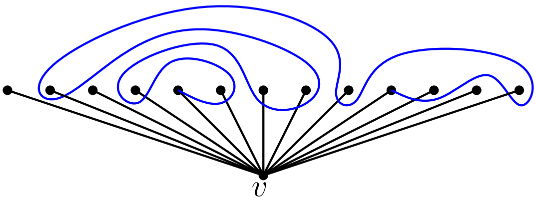
We need to verify that

- $cr(e) = 0$,
- $cr(e, f) \leq 1$, and
- $cr(e, f) = 1 \Leftrightarrow \{e, f\} \in \mathcal{X}$.

3a) characterization of the homotopy classes



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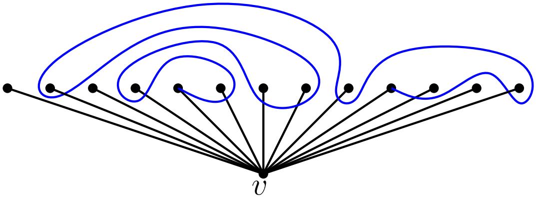

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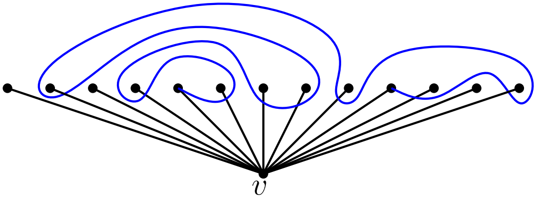

3b) parity of the crossing numbers (4- and 5-tuples)

3c) multiple crossings of adjacent edges (5-tuples)

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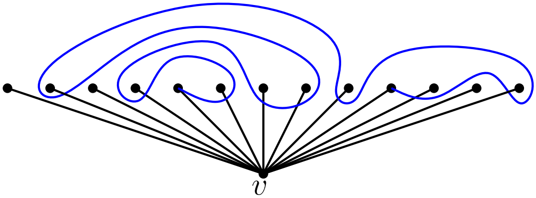

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3d) multiple crossings of independent edges (5-tuples)

Picture hanging without crossings

remove one nail: