Combinatorial Properties of Triangle-Free Rectangle Arrangements and the Squarability Problem

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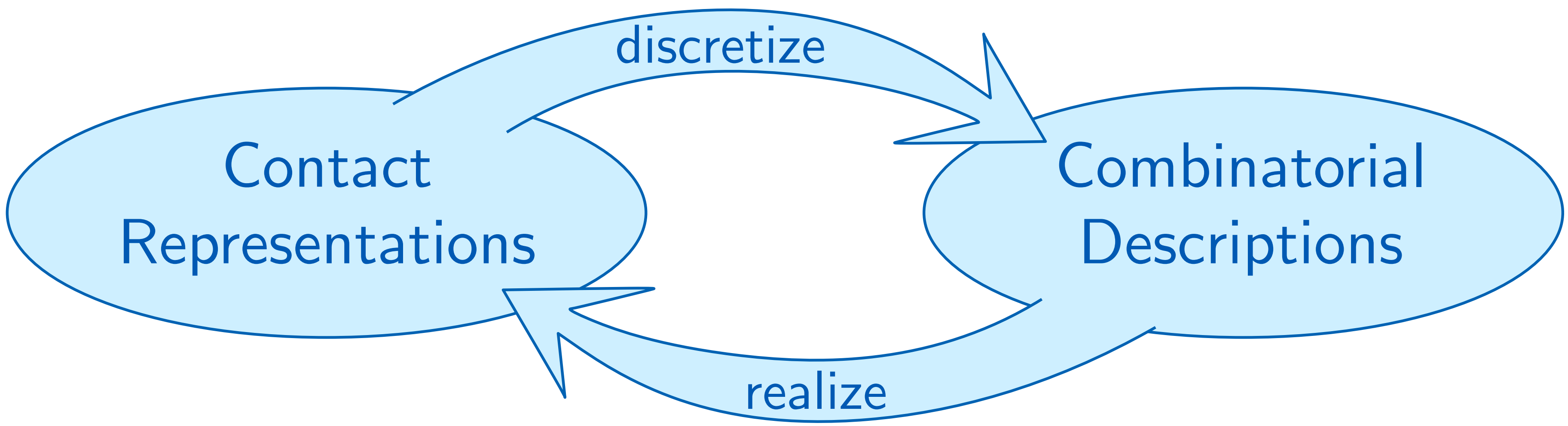
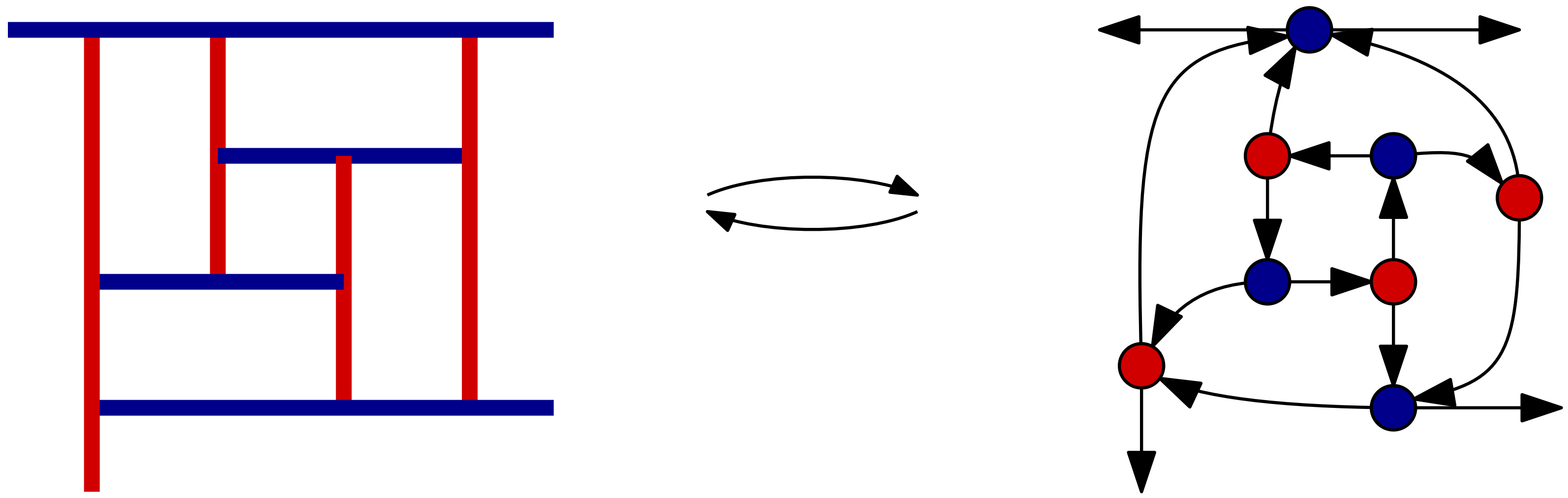
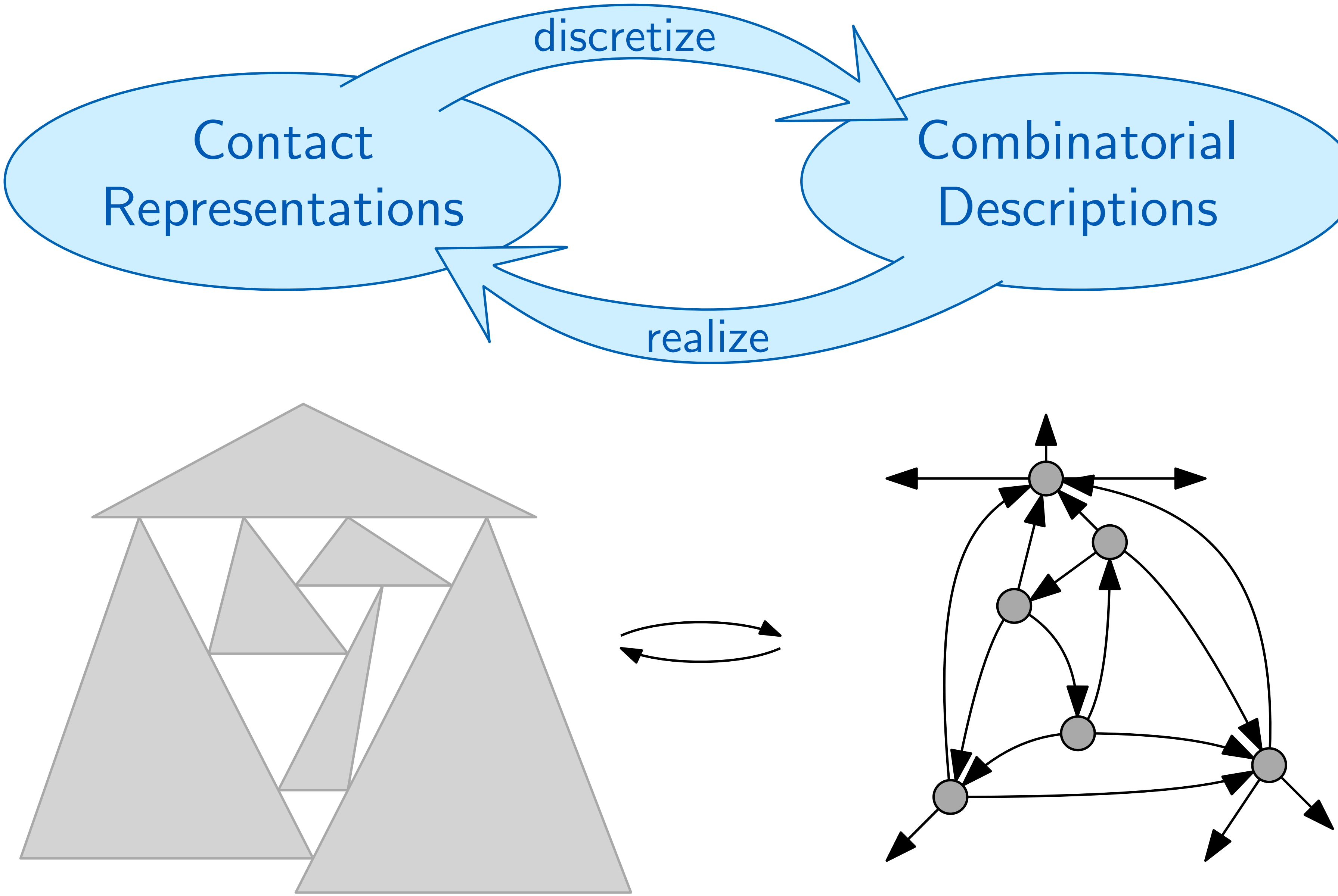
Los Angeles

Combinatorial Properties of Triangle-Free Rectangle Arrangements

The Squarability Problem

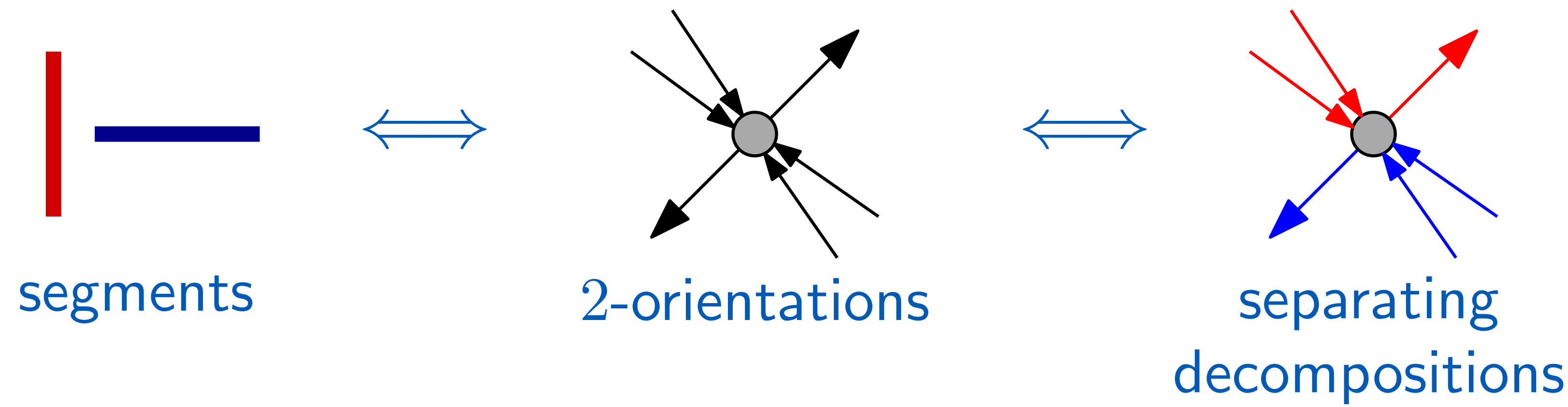

discretize
Contact
Representations
Combinatorial
Descriptions
realize




discretize
Contact
Representations
Combinatorial
Descriptions
realize

Thm. For any quadrangulation G each of the following are in bijection. [de Fraysseix-Ossona de Mendez 2001]

1. axis-aligned segment representations of G
2. 2-orientations of G
3. separating decompositions of G

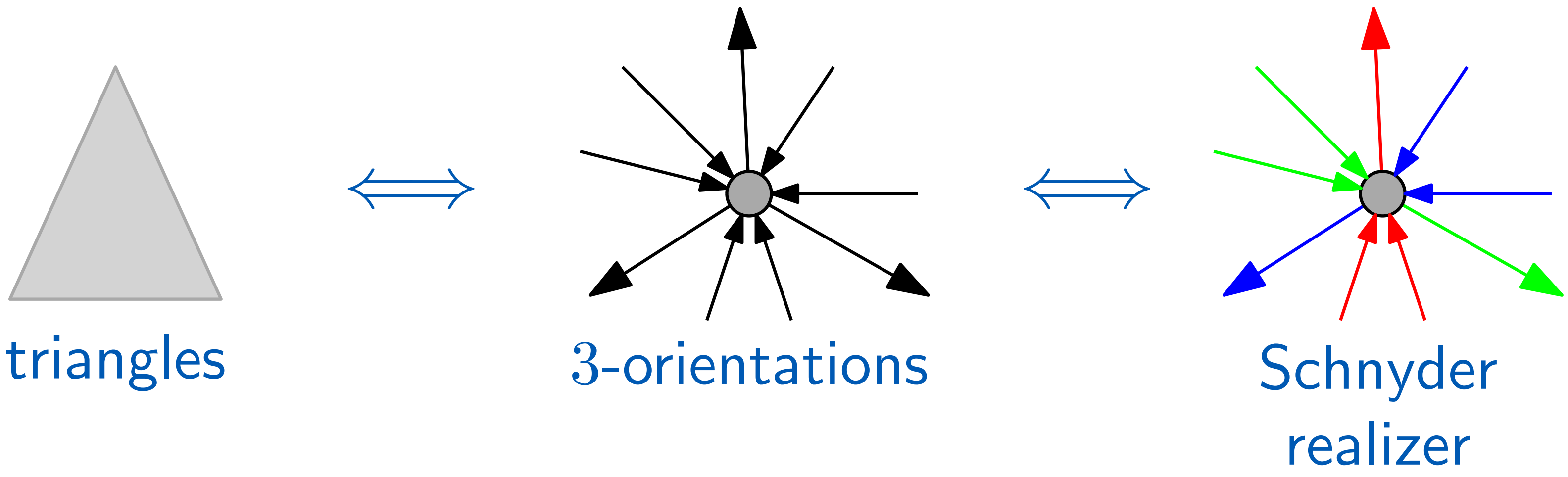


Thm. For any triangulation G each of the following are in bijection. [Schnyder 1991]

① bottom-aligned triangle representations of G

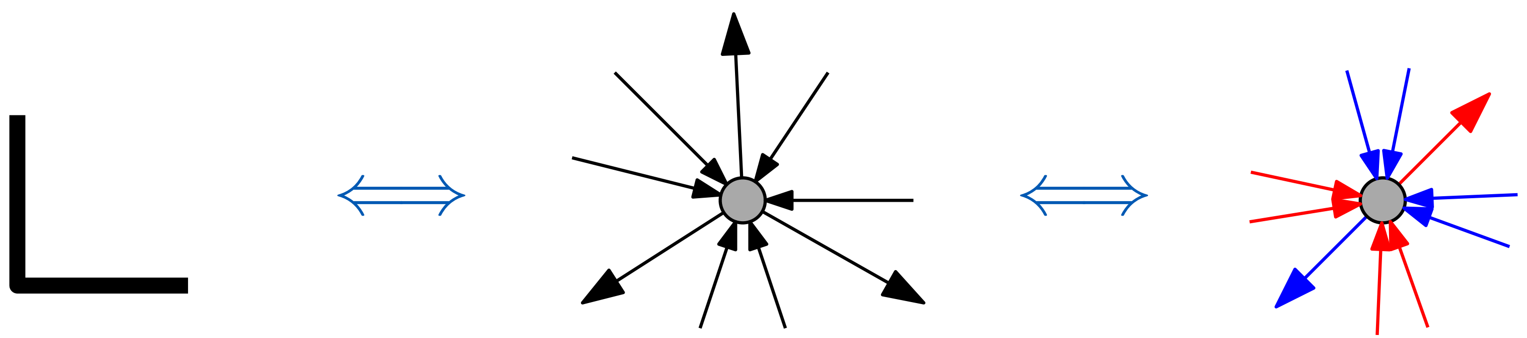
② 3-orientations of G

③ Schnyder realizer of G



Thm. For any plane Laman graph G each of the following are in bijection. [Kobourov-U-Verbeek 2013]

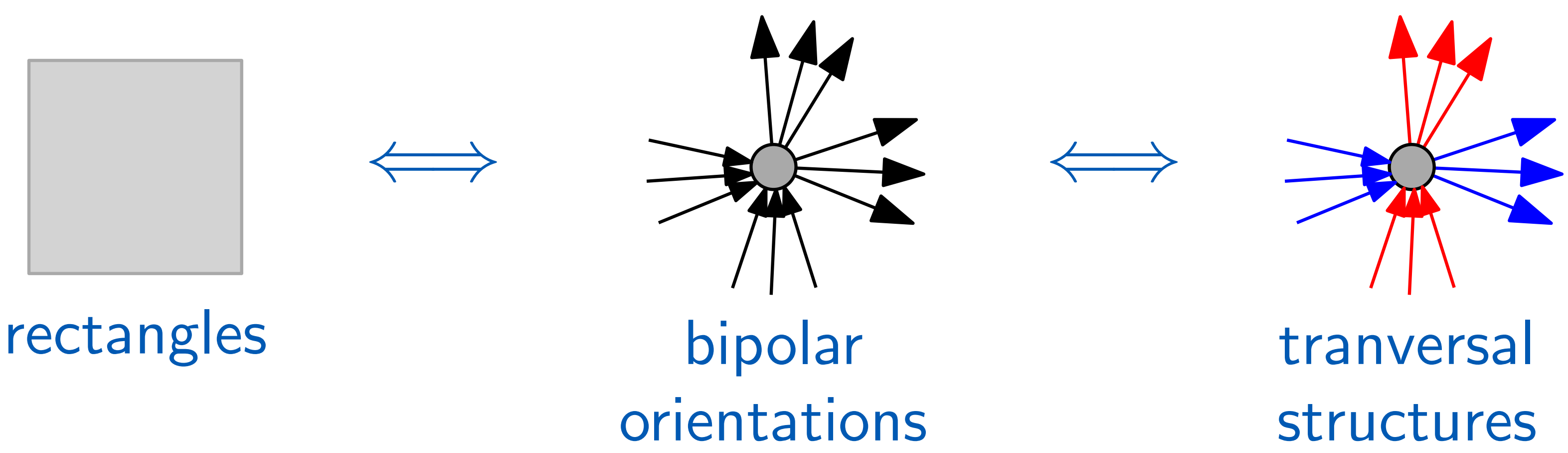
1. axis-aligned L-shape representations of G
2. 3-orientations of the vertex-face augmentation G'
3. angular edge labelings of G



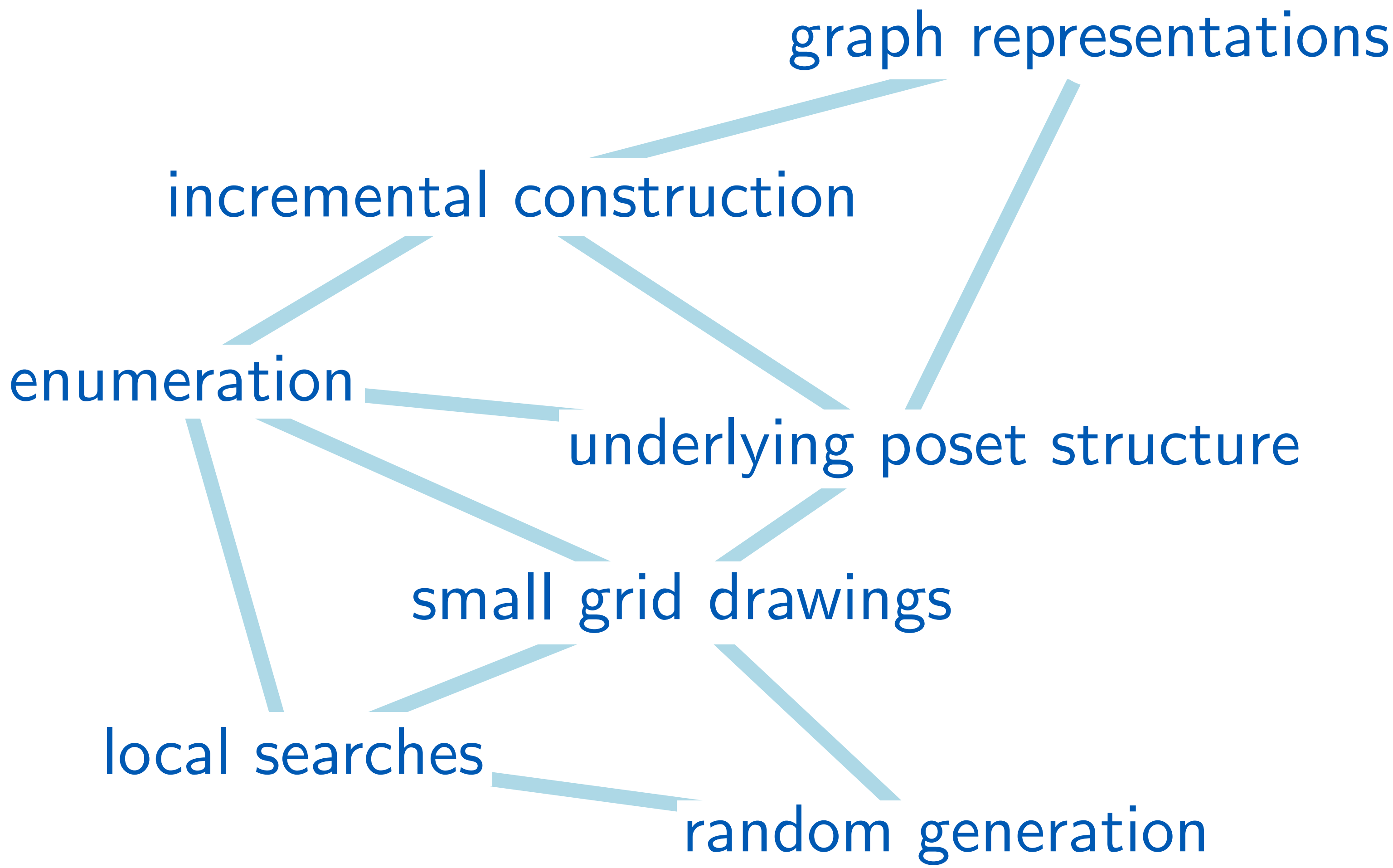
L-shapes ⟺ 3-orientations of G' ⟺ angular edge labelings

Thm. For any rectangular dual G each of the following are in bijection. [Kant-He 1997]

1. rectangle contact representations of G
2. bipolar orientations of G
3. transversal structures of G



Applications



corners = outgoing edges

- ▷ axis-aligned segments ⇄ separating decompositions
- ▷ bottom-aligned triangles ⇄ Schnyder realizer
- ▷ axis-aligned L-shapes ⇄ angular edge labelings

sides = several outgoing edges

- ▷ axis-aligned rectangles ⇄ transversal structures

corners = outgoing edges

- axis-aligned segments ⇄ separating decompositions
- bottom-aligned triangles ⇄ Schnyder realizer
- axis-aligned L-shapes ⇄ angular edge labelings

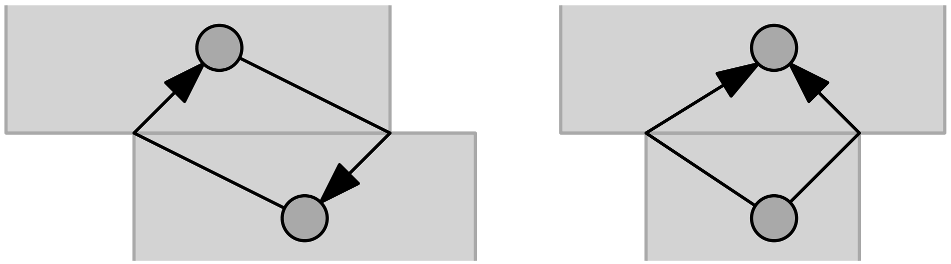
Our Contribution:

- axis-aligned rectangles ⇄ corner edge labelings

sides = several outgoing edges

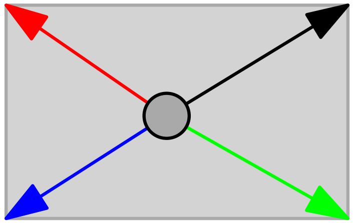
- axis-aligned rectangles ⇄ transversal structures

1) Orientation



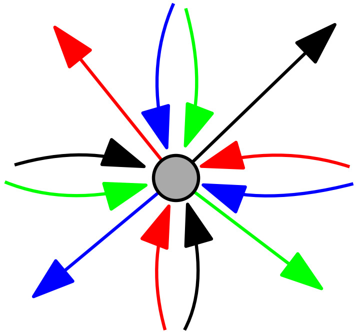
"who pokes who?"

2) Coloring

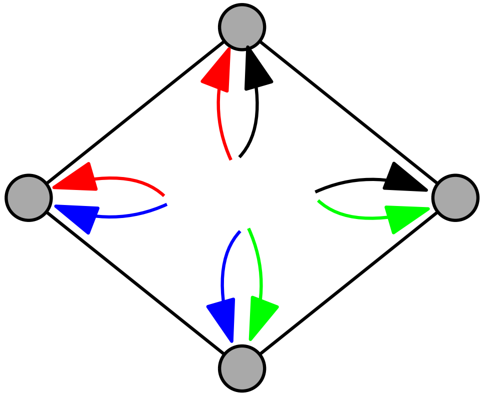


"with which feature?"

3) Local Rules



inner vertex



outer vertices

4) Graph Class

- planar
- maximal triangle-free

1) Orientation

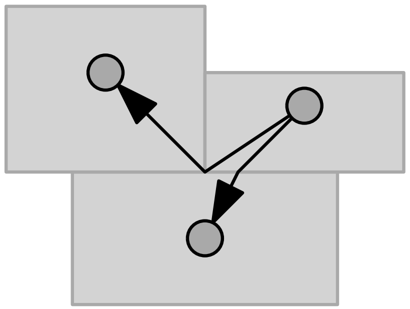
- “corners = outgoing edges”

4) Graph Class

- planar

1) Orientation

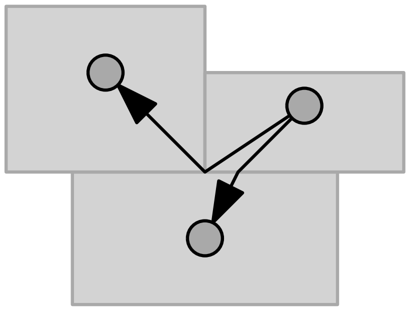
- “corners = outgoing edges”



triangle $\rightsquigarrow$ 2 edges for 1 corner

4) Graph Class

- planar



triangle $\rightsquigarrow$ 2 edges for 1 corner

1) Orientation

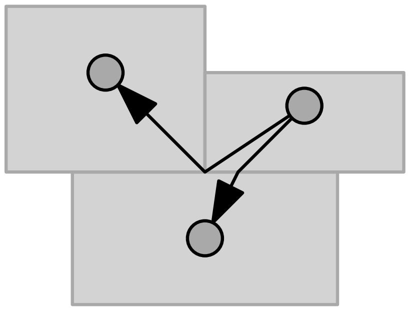
- "corners $=$ outgoing edges"

4) Graph Class

- planar
- maximal triangle-free
  (only 4-faces and 5-faces)

1) Orientation

- ▷ “corners $=$ outgoing edges”



triangle $\rightsquigarrow$ 2 edges for 1 corner

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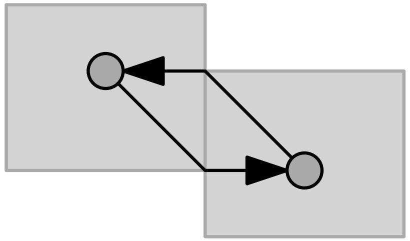
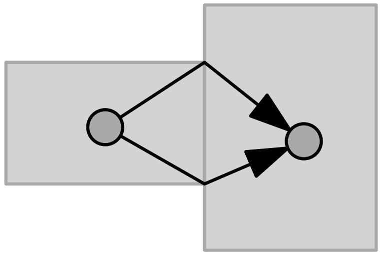
- ▷ maximal triangle-free (only 4-faces and 5-faces)

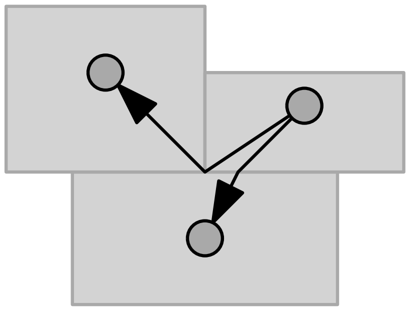




every contact involves 2 corners

1) Orientation

- “corners = outgoing edges”



triangle $\rightsquigarrow$ 2 edges for 1 corner

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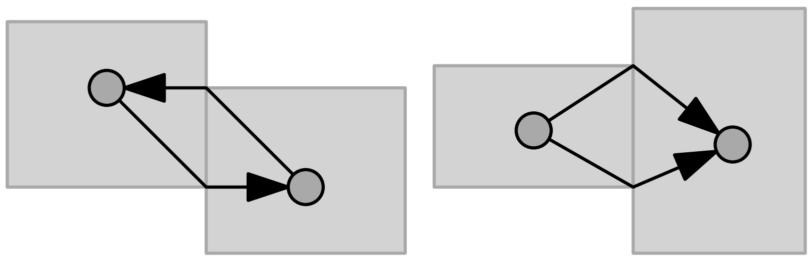
- maximal triangle-free
  (only 4-faces and 5-faces)



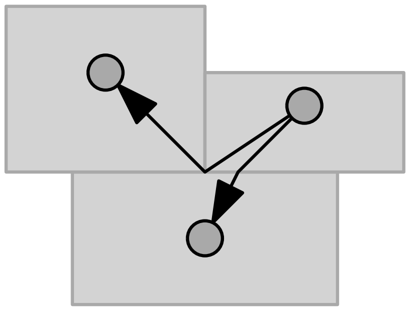
every contact involves 2 corners

5) Augment Input Graph

- double each edge

1) Orientation

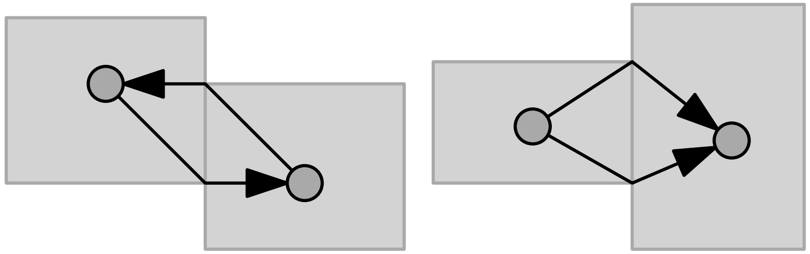
- ▷ "corners = outgoing edges"



triangle ⇝ 2 edges for 1 corner

4) Graph Class

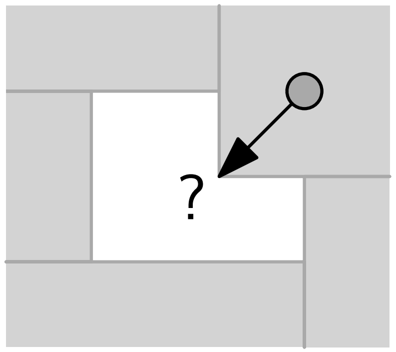
- ▷ planar
- ▷ maximal triangle-free
  (only 4-faces and 5-faces)



every contact involves 2 corners

5) Augment Input Graph

- ▷ double each edge



1 unused corner in each 5-face

1) Orientation

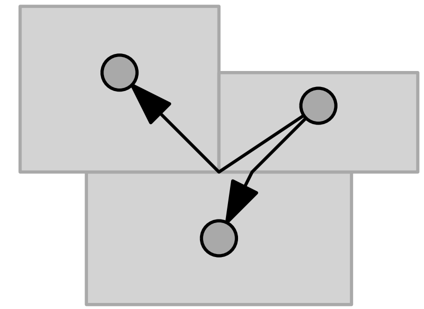
- ▷ "corners = outgoing edges"

4) Graph Class

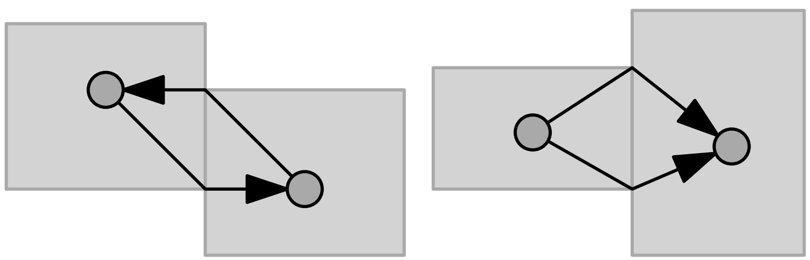
- ▷ planar
- ▷ maximal triangle-free
  (only 4-faces and 5-faces)

5) Augment Input Graph

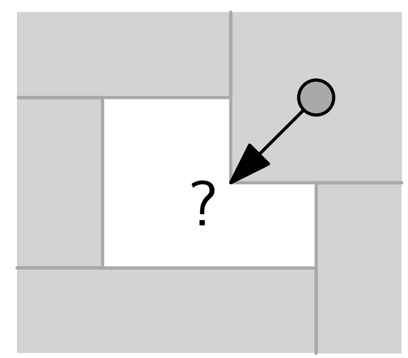
- ▷ double each edge
- ▷ add vertices for inner faces



triangle $\rightsquigarrow$ 2 edges for 1 corner



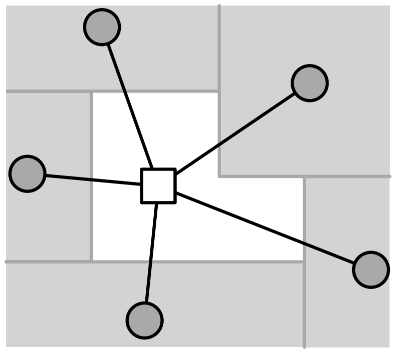
every contact involves 2 corners

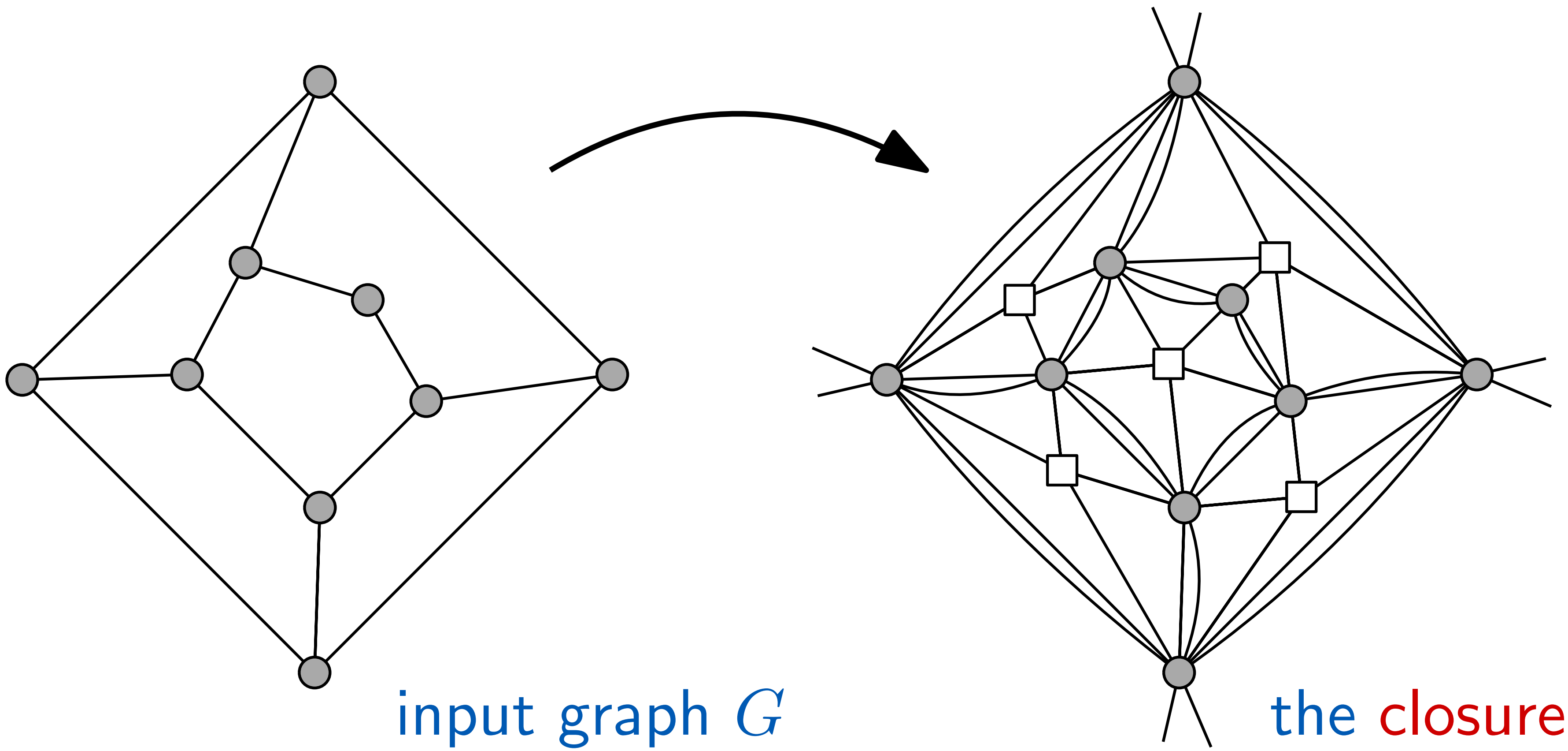


1 unused corner in each 5-face

5) Augment Input Graph

- ▷ double each edge
- ▷ add vertices for inner faces
- ▷ add 2 half-edges to each outer vertex



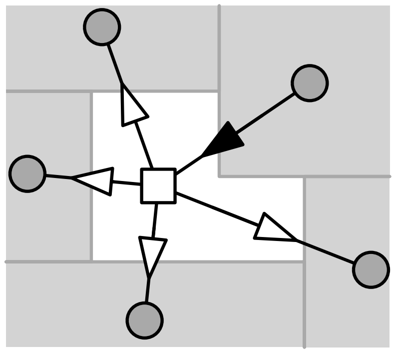


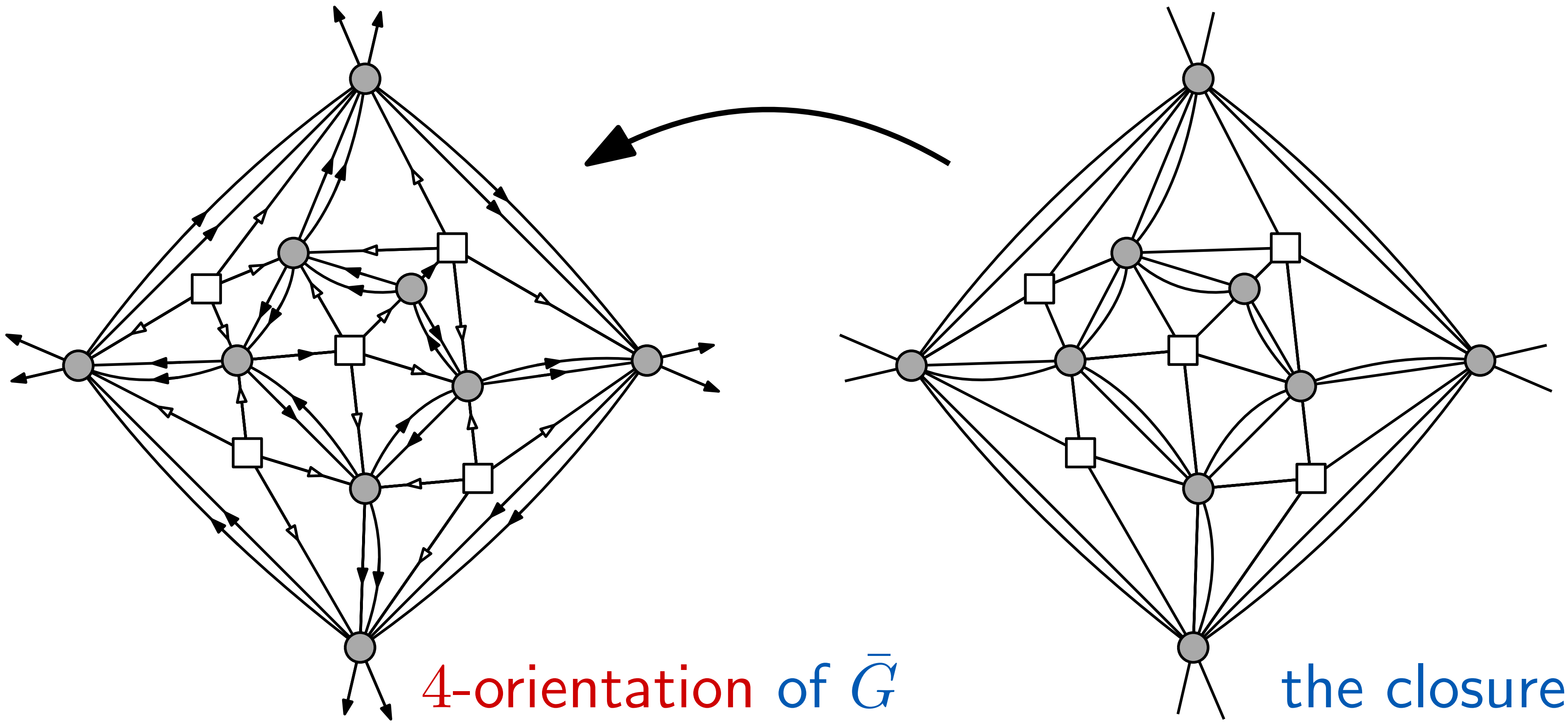
input graph G

the closure $\bar{G}$

1) Orientation

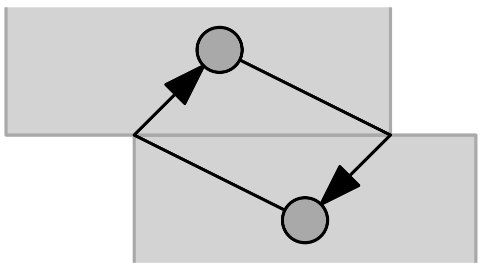
- ▷ original vertices: "outgoing edges = corners"
- ▷ face-vertices: "outgoing edges = extremal sides"
- ▷ 4-orientation: outdegree 4 at every vertex





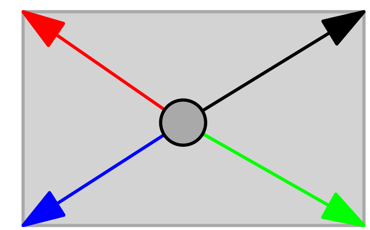
4-orientation of $\bar{G}$ the closure $\bar{G}$

1) Orientation



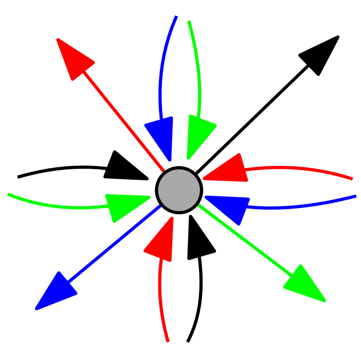
"who pokes who?"

2) Coloring

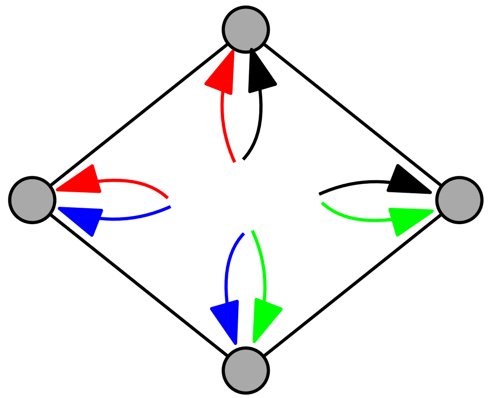


"with which feature?"

3) Local Rules



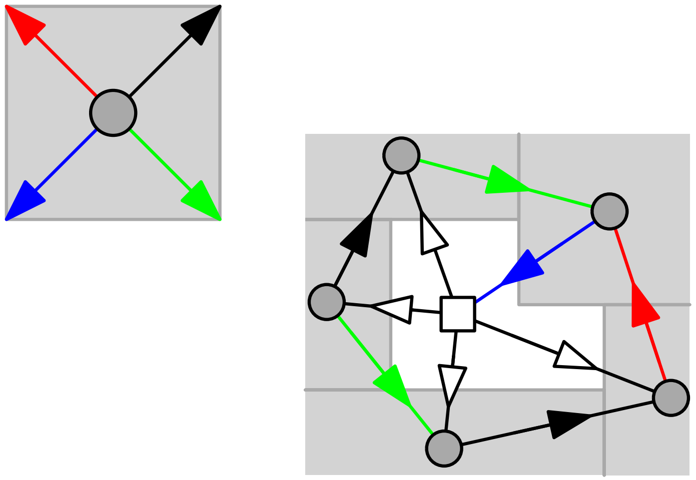
inner vertex



outer vertices

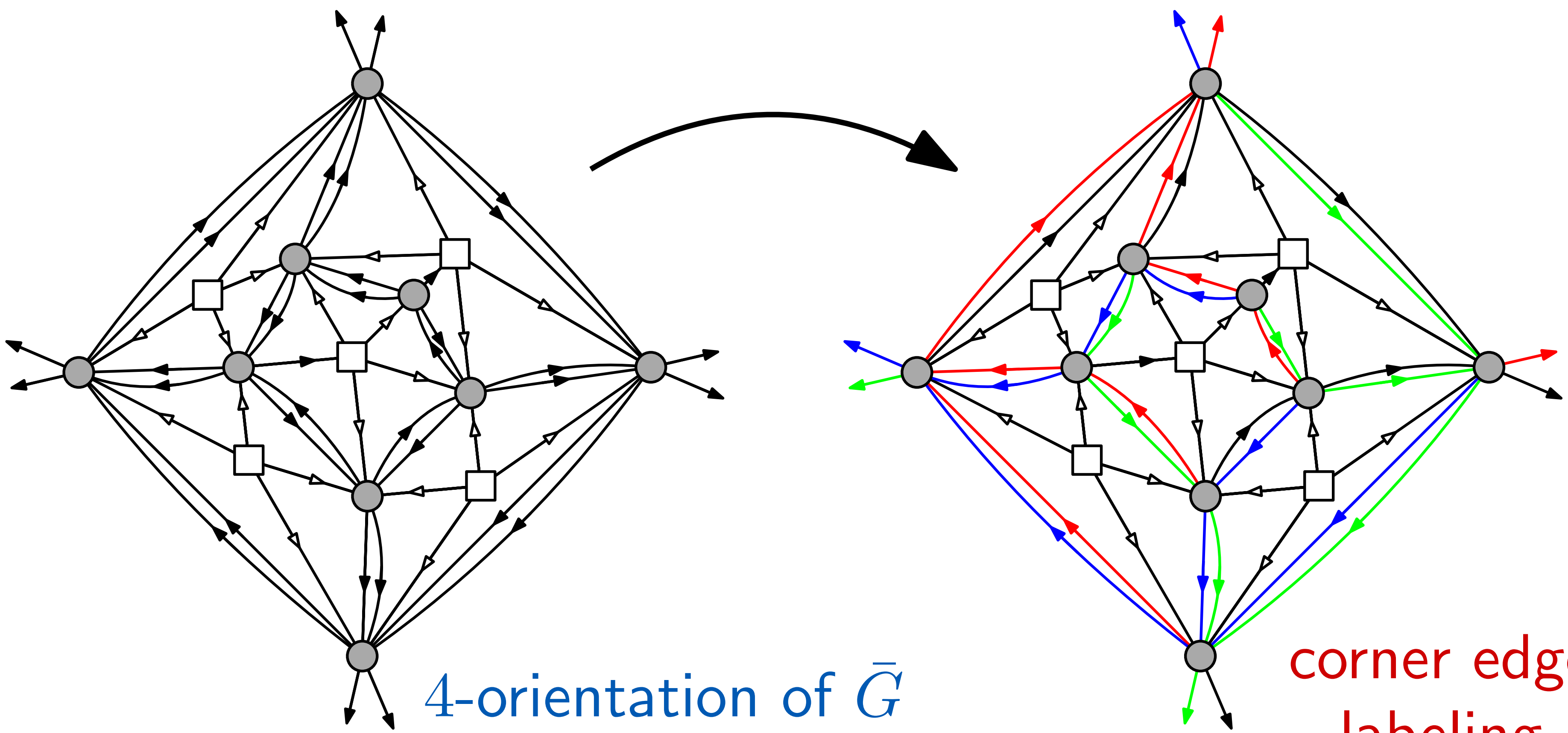
4) Graph Class

- planar
- maximal triangle-free



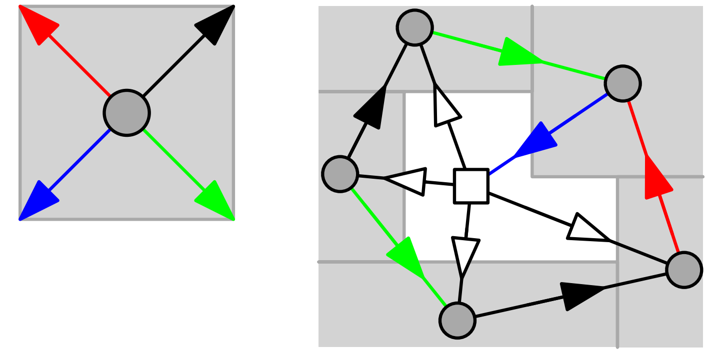
2) Coloring

- ▷ original vertices: one color per corner
- ▷ face-vertices: outgoing edges not colored

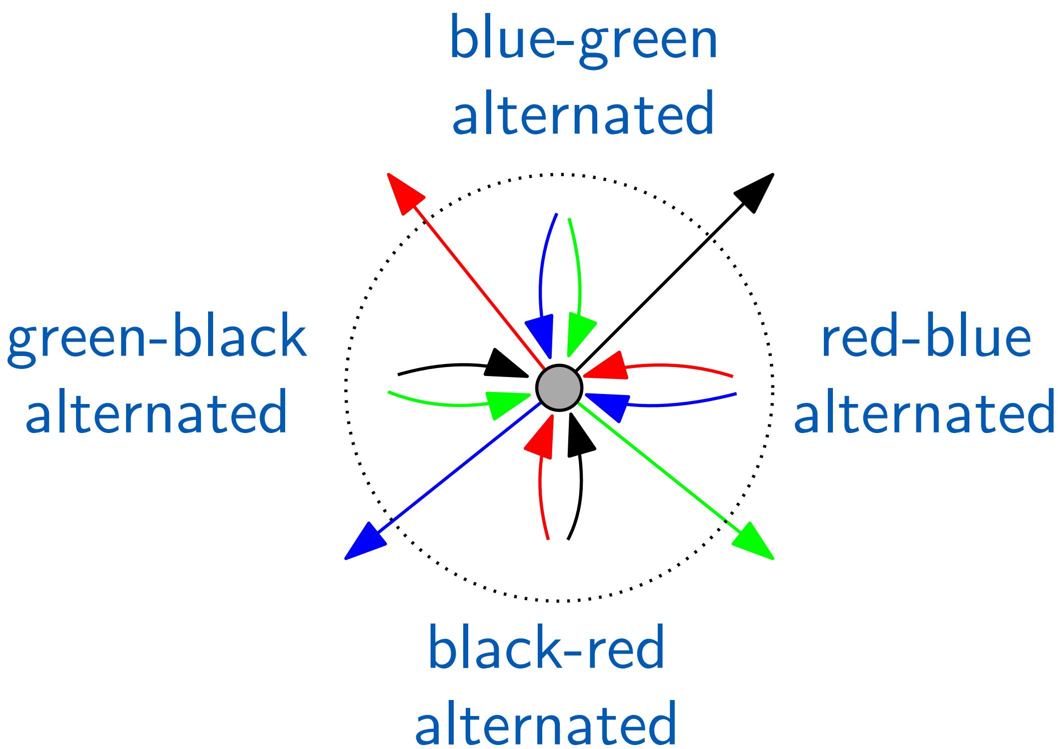


4-orientation of $\bar{G}$

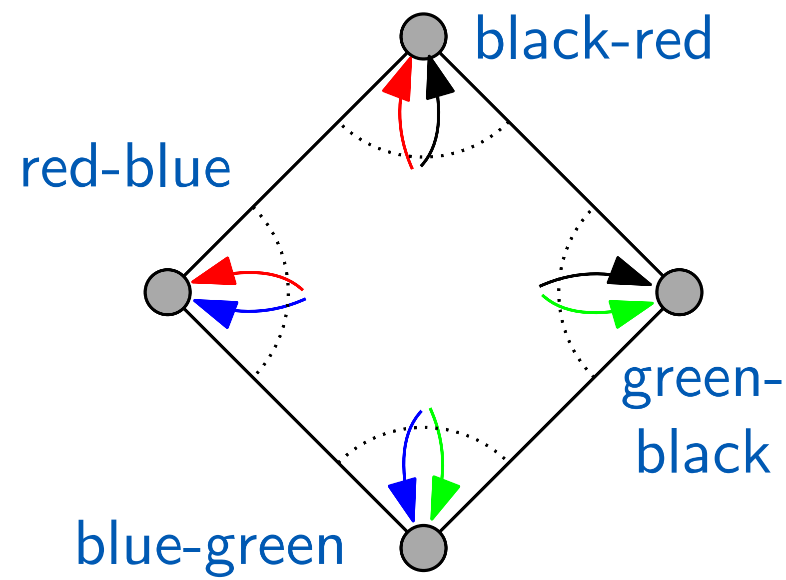
corner edge labeling



3) Local Rules



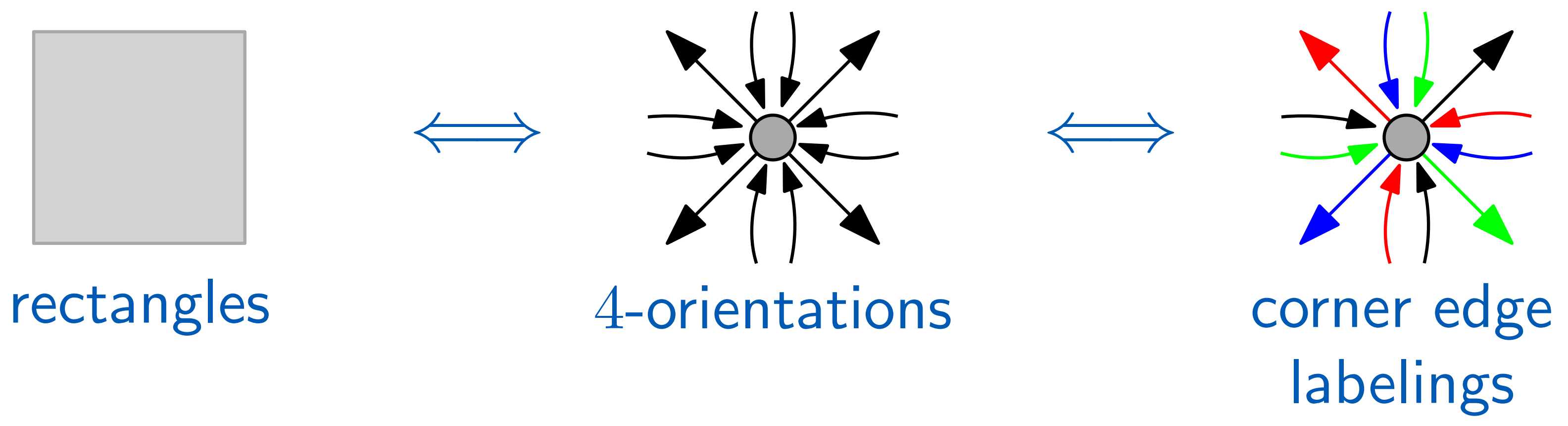
inner vertices



outer vertices

Thm. For any maximal triangle-free, plane graph G with quadrilateral outer face, each of the following are in bijection.

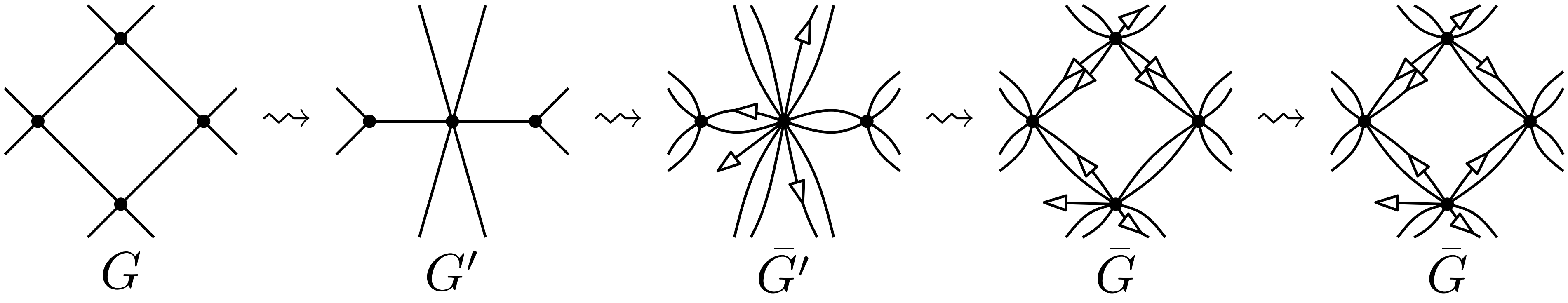
1. rectangle contact representations of G
2. 4-orientations of the closure $\bar{G}$
3. corner edge labelings of the closure $\bar{G}$



Lem. In every augmented corner edge labeling each of the following holds.

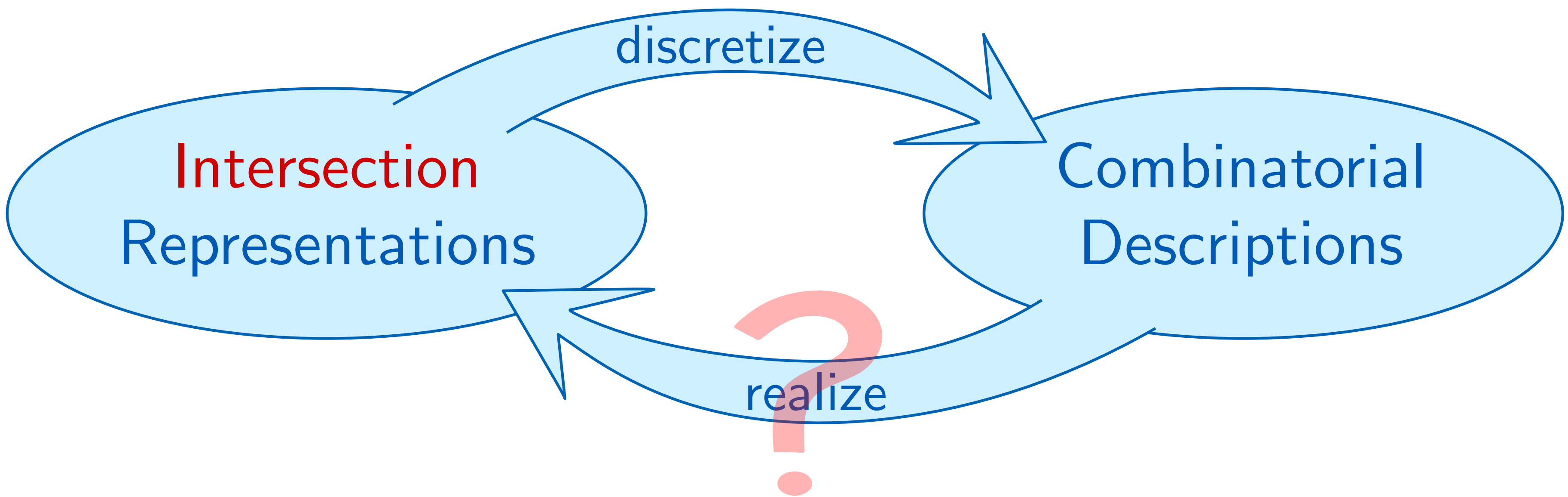
- ▷ Each color class is a tree.
- ▷ Every non-trivial directed cycle has all 4 colors.

Lem. For every maximal triangle-free plane graph G its closure $\bar{G}$ has a 4-orientation.
Hence, G has a rectangle contact representation.



Combinatorial Properties of Triangle-Free Rectangle Arrangements

The Squarability Problem


discretize
Intersection
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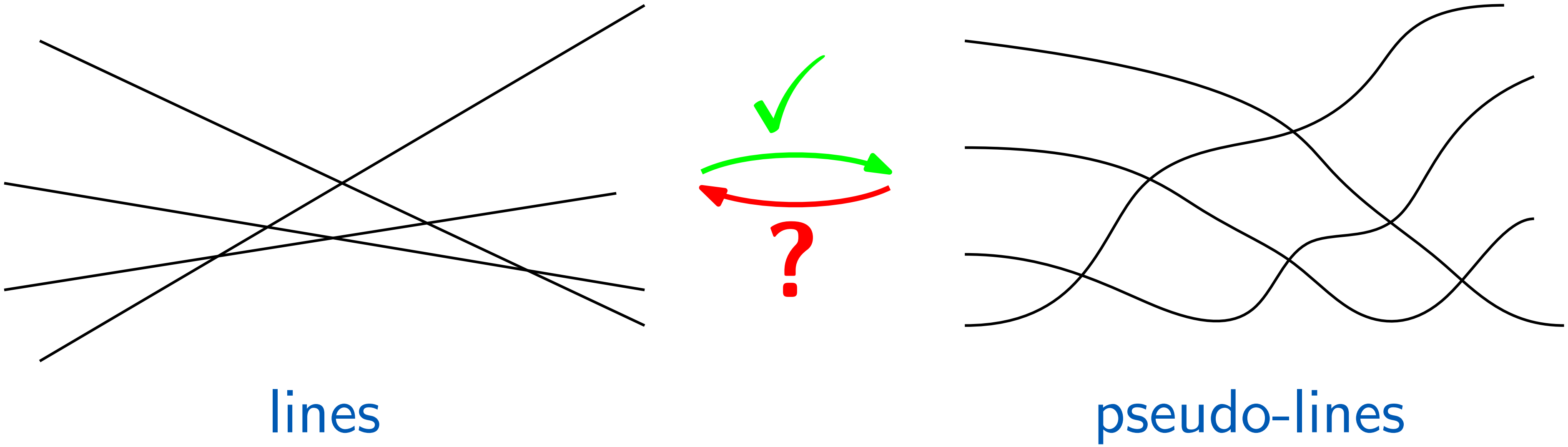
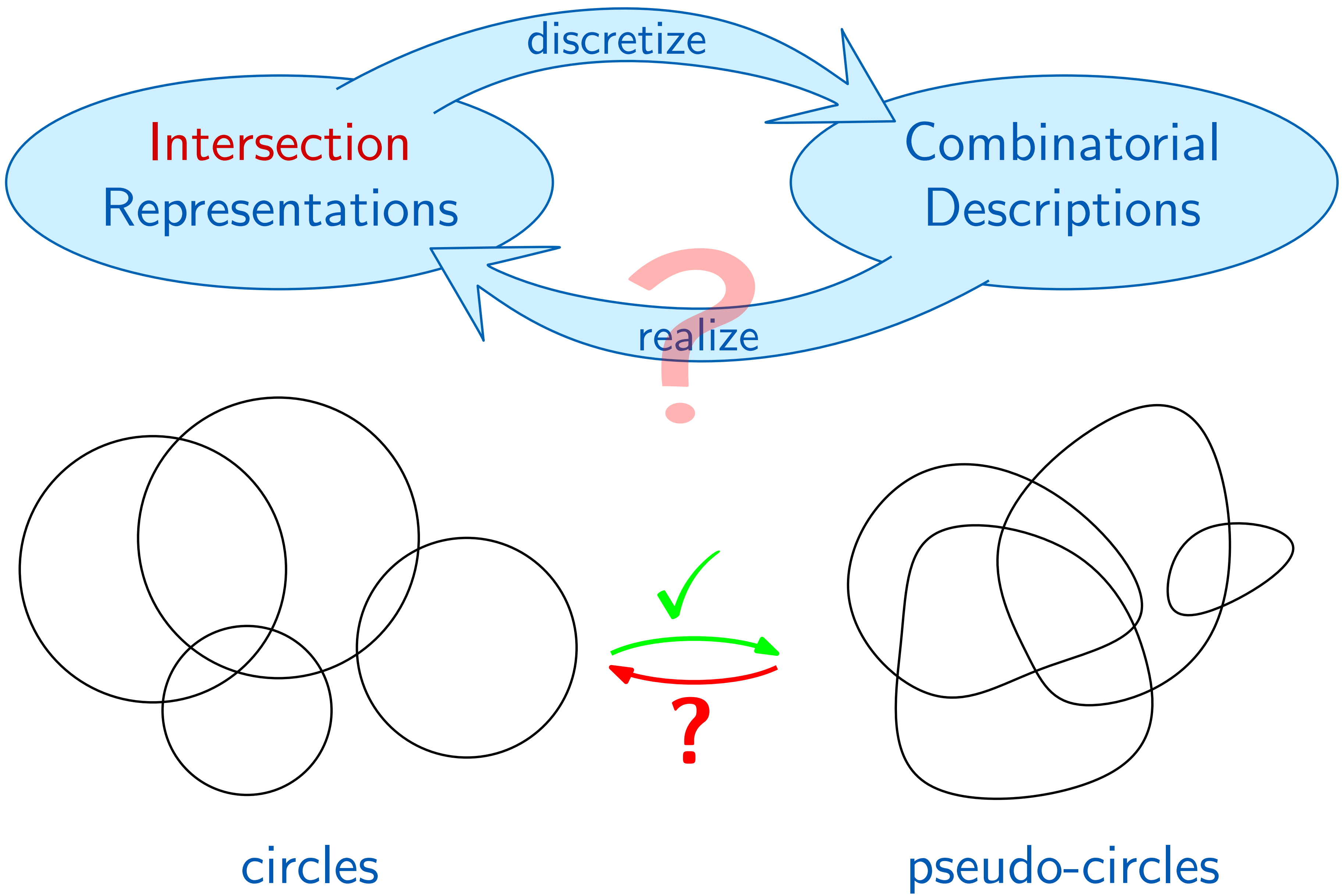

?
lines
pseudo-lines


discretize
Intersection
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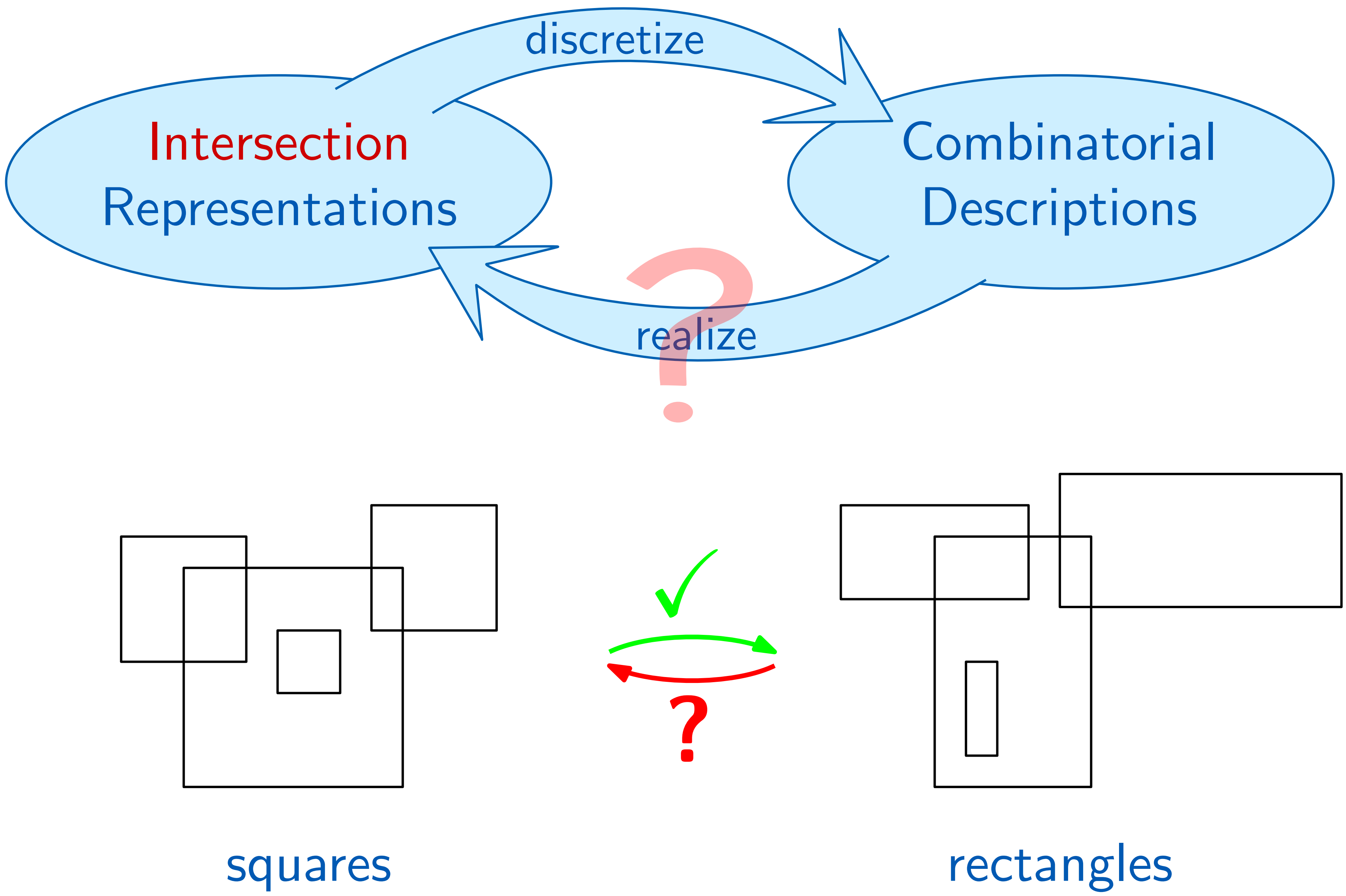
realize
?
?
circles
pseudo-circles


discretize
Intersection
Representations
Combinatorial
Descriptions
realize
?
squares
rectangles

Thm. It is NP-hard to decide whether a given pseudo-line arrangement is realizable with lines.
[Mnëv 1988, Shor 1991]

Thm. It is NP-hard to decide whether a given pseudo-circle arrangement is realizable with circles.
[Kang-Müller 2014]

Thm. It is NP-hard to decide whether a given pseudo-line arrangement is realizable with lines.
[Mnëv 1988, Shor 1991]

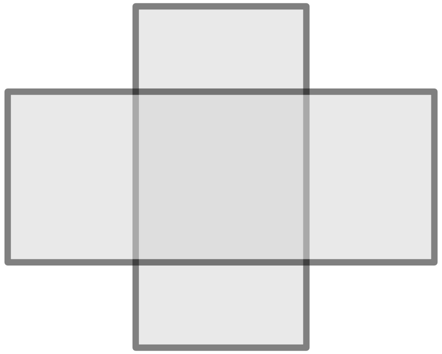
Thm. It is NP-hard to decide whether a given pseudo-circle arrangement is realizable with circles.
[Kang-Müller 2014]

Our Contribution:

Question Is it NP-hard to decide whether a given rectangle arrangement is realizable with squares?

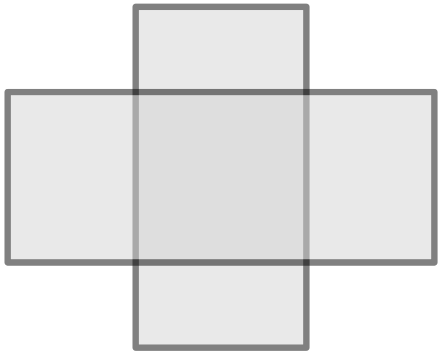
The Squarability Problem

some unsquarable rectangles

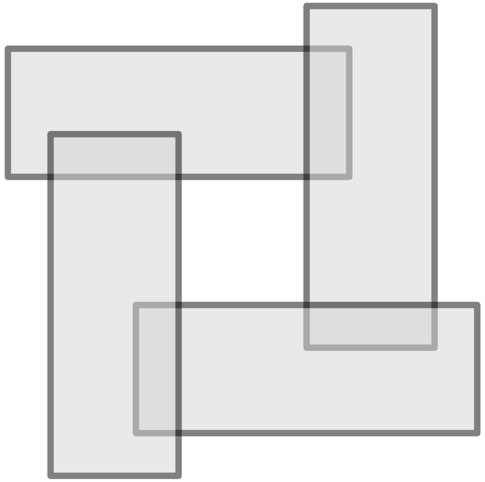


(a)

some unsquarable rectangles

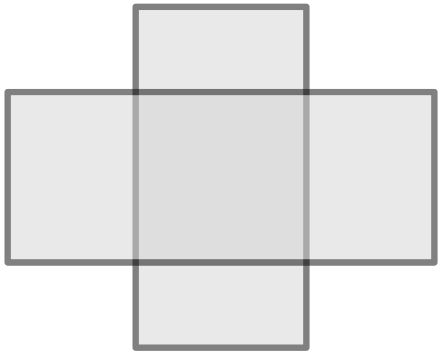


(a)

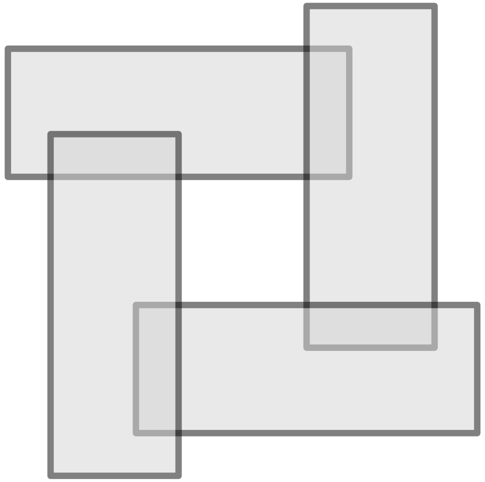


(b)

some unsquarable rectangles



(a)



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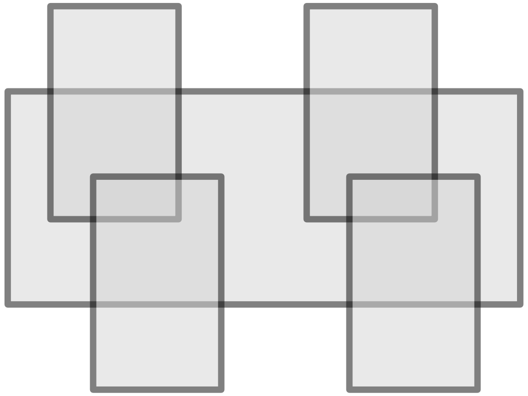
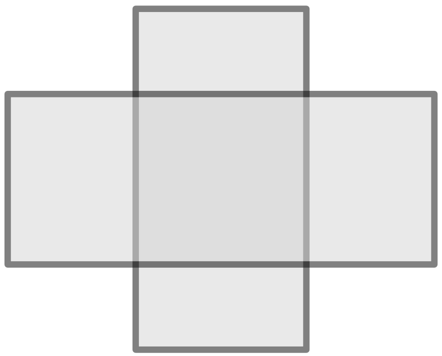
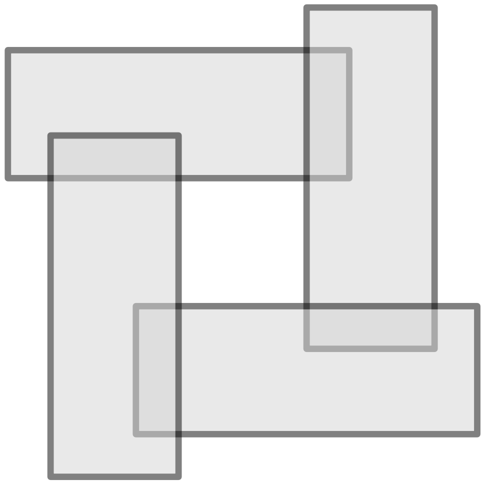


(c)

some unsquarable rectangles



(a)



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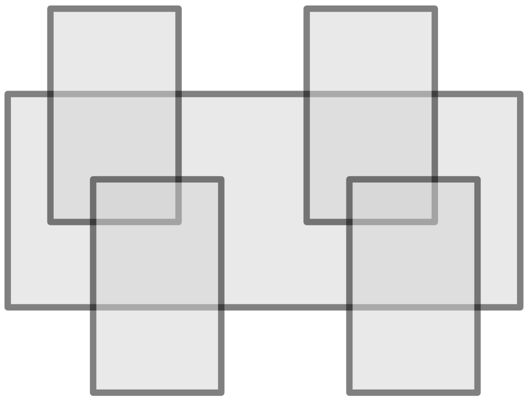
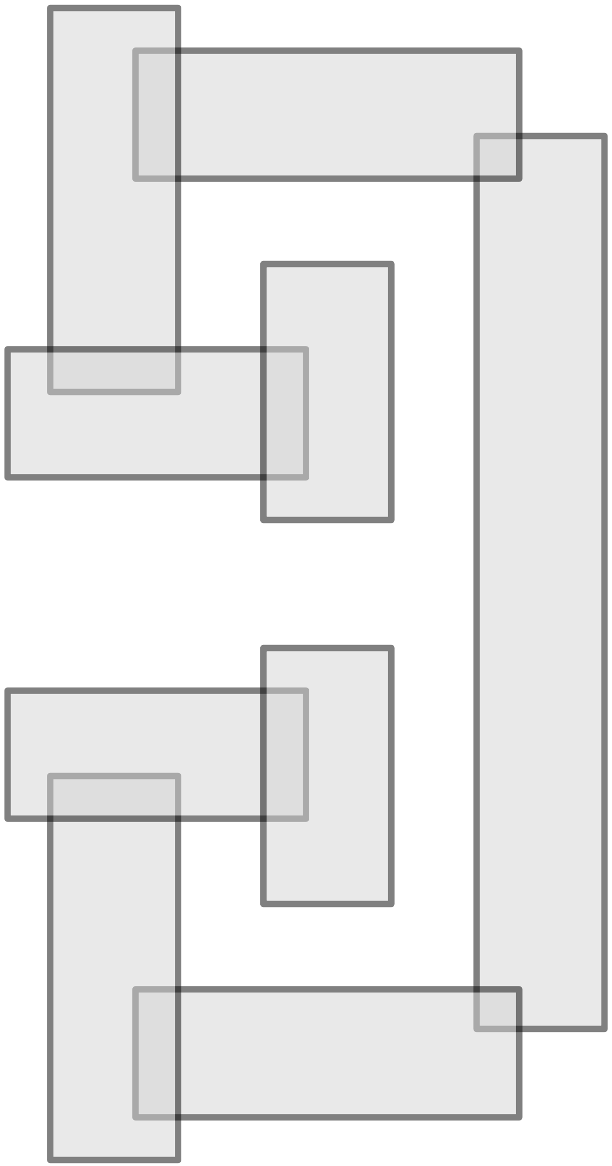
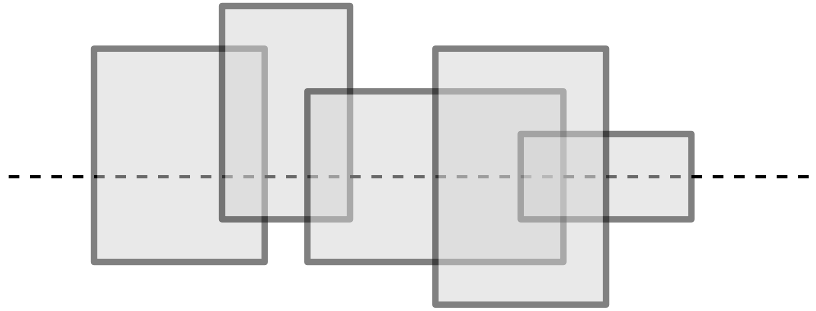


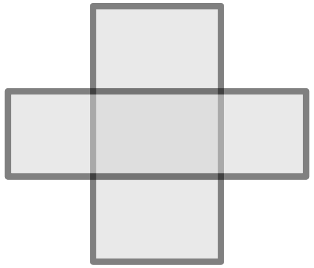
(c)



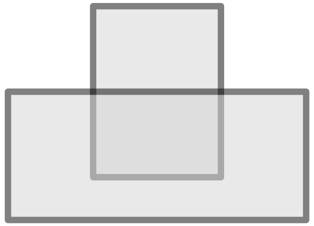
(d)



line-pierced arrangement



cross



side-intersection

Thm. Every line-pierced, triangle-free and cross-free rectangle arrangement is squarable.

Thm. Every line-pierced and cross-free rectangle arrangement without side-intersections is squarable.

Question Is every cross-free rectangle arrangement without side-intersections squarable?

Combinatorial Properties of Triangle-Free Rectangle Arrangements

The Squarability Problem

Thank you for your attention!

(joint work with Jonathan Klawitter and Martin Nöllenburg)