Intersection-Link Representations of Graphs

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Motivations

Node-Link Diagrams vs Intersection Representations

- good for sparse graphs
- good for dense graphs

- We want to represent graphs that are **globally sparse** but contain **dense subgraphs**, e.g.:
  - *social networks*: dense subgraphs are **communities**
  - *computer networks*: dense subgraphs are **backbone subnetworks**
Intersection-link Representations

\[ G = (V, E_1 \cup E_2) \]

- Each vertex \( v \) is a geometric object \( R(v) \)
- \( R(u) \) and \( R(v) \) intersect iff \( (u, v) \) is an intersection-edge
- \( R(u) \) and \( R(v) \) are connected by a curve iff \( (u, v) \) is a link-edge
Intersection-link Representations

\[ G = (V, E_1 \cup E_2) \]

MODELS:
- intersecting subgraphs \((K_n, K_{n,m}, \triangle\text{-free}, \text{etc.})\)
- geometric objects (rectangles, circles, convex polygons, etc.)
- link-edges style (topological or geometric)

Hybrid Model

dense parts

sparse parts

intersection-edges

link-edges
Clique Planarity

Problem definition

- **input**: pair \((G, S = \{s_1, \ldots, s_k\})\) where:
  - \(G = (V, E)\) is a graph
  - \(S\) is a partition of \(V\) into cliques \(s_i\)
- **question**: is there a *clique planar drawing* of \((G, S)\)?

<table>
<thead>
<tr>
<th>(s_1 = K_{10})</th>
<th>(s_2 = K_9)</th>
<th>(s_3 = K_7)</th>
</tr>
</thead>
</table>

*clique planar drawing

- intersection-link representation of 
  \((G, S)\) in which:
  - no two link-edges intersect
  - no link-edge intersects the interior of a rectangle

\(s_3\) is a cool clique!!
Our Results

- NP-completeness of Clique Planarity

- Related Problems
  - clustered planarity
  - book embedding
  - level planarity

- P-time cases
Canonical Representations

Lemma (Canonical)

\((G, S)\) is clique-planar \iff \((G, S)\) has a **canonical representation**

- vertices are **axis-aligned unit squares**

- \(\forall s \in S\), all the intersecting squares representing vertices in \(s\) have their upper-left corner along a **common line with slope 1**
Canonical Representations: **proof idea**

**Lemma (Canonical)**

\((G, S)\) is clique-planar \(\iff\) \((G, S)\) has a **canonical representation**

**Key Property for rerouting the link-edges without introducing crossings:**

The circular sequence of link-edges crossing the **boundary of** \(\Gamma_s\) contains edges incident to a **subsequence of**:

\[ R(u_1), R(u_2), \ldots, R(u_{|S|}), R(u_{|S|-1}), \ldots, R(u_2) \]

for some permutation \(u_1, \ldots, u_{|S|}\) of the vertices of \(s\)
Clique Planarity and Clustered Planarity

Problem definition

- **input**: a $c$-graph $(G, T)$, i.e. a graph $G$ together with a clustering $T$ of its vertex set
- **question**: is there a $c$-planar drawing of $(G, T)$?

Our focus:

- $c$-graphs whose clusters induce independent sets of vertices

**c-planar drawing**

- **planar drawing** of $G$
- clusters are simple closed regions enclosing their vertices
  - no region-region overlaps
  - no edge-region crossings

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### Clustered Planarity with Linear Saturator (CPLS)

<table>
<thead>
<tr>
<th>Theorem [Feng, Cohen, and Eades, ESA ’95]</th>
</tr>
</thead>
<tbody>
<tr>
<td>A c-graph ((G, \mathcal{T})) is c-planar iff ((G, \mathcal{T})) can be augmented with a set of edges (saturator) to a <strong>c-planar</strong> and <strong>c-connected</strong> c-graph</td>
</tr>
</tbody>
</table>

#### C-Planarity
- The saturator induces a **tree** in each cluster

#### C-Planarity with Linear Saturators
- The saturator induces a **path** in each cluster
Relationship between CPLS and Clique Planarity

Theorem

\[ \text{CPLS } \propto \text{ Clique Planarity} \]

\[ \mu \in \mathcal{T}(G, T) \]
\[ s \in \mathcal{S}(G', S) \]

\[ \text{cluster } \mu \implies \text{clique } s_\mu \]

\[ e \in E(G) \implies e \text{ is a link-edge} \]

\[ u, v \in \mu \implies (u, v) \text{ is an intersection-edge of } s_\mu \in \mathcal{S} \]
Relationship between CPLS and Clique Planarity

**Theorem**

CPLS $\propto$ Clique Planarity

$(G, T)$

$\Gamma$: C-Planar Drawing with Linear Saturators

$\mu$

$(G', S)$

$\Gamma^*$: Canonical Clique Planar Drawing

$\Gamma^*$

$S_\mu$

Key property:

the endpoints of edges incident to $\mu$ in $\Gamma$ (to $s_\mu$ in $\Gamma^*$) form a subsequence of $u_1, u_2, \ldots, u_{k-1}, u_k, u_{k-1}, \ldots, u_2$
Computational Complexity

**Theorem**

HamPath in 2-connected planar graphs $\propto$ CPLS

**Corollary**

Clique Planarity is $\mathcal{NP}$-complete

\[
(G', \mathcal{T}) \text{ has a linear saturator iff } G \text{ is Hamiltonian}
\]
**Clique Planarity with Given Vertex Representations**

**Input**: a pair $(G, S)$ with a geometric representation $\Gamma$ of all the cliques of $S$

**Goal**: test whether the link-edges of $(G, S)$ can be drawn in $\Gamma$ to obtain a clique-planar drawing

**Partial Embedding Planarity problem**: test whether a planar drawing of a graph $H$ exists extending a given drawing $H'$ of a subgraph $H'$ of $H$

Angelini et al. · Trans. Alg. ’15

$O(n)$-time testing algorithm
Clique Planarity with Given Vertex Representations

- All embedded cacti representing \( \Gamma \) lie in the **same face**
- Planar drawing \( \Gamma^* \) extending \( \Gamma \) with the **link-edges**

\[ \Gamma_s \]

**occurrences of rectangles** along the outer boundary of \( \Gamma_s \)

**outerplane binary cactus** representing \( \Gamma_s \)
**Cliques Planarity with 2 Cliques**

**Complexity status w.r.t. |S|:**

- |S| = 1: there are no link-edges, always YES
- |S| = 2: unknown!
- |S| ∈ O(n): $\mathcal{NP}$-complete even if $S = \{K_q, K_1, \ldots, K_1\}$, with $q > 1$

canonical clique planar drawing of a pair $(G, \{s_1, s_2\})$
Bipartite 2-Page Book Emb. with Spine Crossings

Problem definition (without spine crossings)

- **input**: bipartite planar graph $G = (V_1 \cup V_2, E)$
- **question**: is there a 2-page book embedding of $G$ in which vertices in $V_1$ ($V_2$) are **consecutive along the spine**?

B2PBE with spine crossings = Clique Planarity with $|S| = 2$
- edges may cross the spine once between the two portions of the spine delimited by a vertex of $V_1$ and a vertex of $V_2$
Partitioning the Link-Edges

2-Partitioned Clique Planarity

each link-edge has to be incident to a **prescribed side of each clique** (top or bottom)

Partitioned B2PBE (with spine crossings)

each of the two end-parts of an edge has to be incident to a **prescribed side of the spine**
Simultaneous Embedding with Fixed Edges

Problem definition

- **input**: $G_1(V, E_1)$, $G_2(V, E_2)$
- **question**: existence of planar drawings $\Gamma_1$ and $\Gamma_2$

such that

1. **vertices**:  
   - $\forall v \in V$, $\Gamma_i(v) = \Gamma_j(v)$

2. **shared edges**:  
   - $\forall e \in E_i \cap E_j$, $\Gamma_i(e) = \Gamma_j(e)$
Algorithm for 2-Partitioned Clique Planarity

2-Partitioned Bipartite Clique Planarity

Partitioned Bipartite 2-Page Book Emb.

Partial Instance

SEFE instances \( \langle G_1, G_2 \rangle \) with

\( G_1 \) and \( G_2 \) biconnected

\( G = G_1 \cap G_2 \) connected

Bläsius and Rutter · SODA’13

\( O(n^2) \)-time testing algorithm

\( V_1 \)

\( V_2 \)

Page 1

Page 2
Algorithm for 2-Partitioned Clique Planarity

2-Partitioned Clique Planarity \rightarrow Partitioned Bipartite 2-Page Book Emb. \rightarrow SEFE instances \langle G_1, G_2 \rangle with $G_1$ and $G_2$ biconnected $G = G_1 \cap G_2$ connected

Complete Instance

Bläsius and Rutter · SODA'13

$O(n^2)$-time testing algorithm
Level Clique Planarity

What if cliques are given together with a hierarchical relationship among them?

- attempt to extend the notion of Level Planarity to Clique Planar Graphs

Input: pair \((G, S)\) together with leveling function \(\psi: S \rightarrow \{1, \ldots, k\}\)

Question: is there a canonical clique planar drawing of \((G, S)\) s.t.:

- the top side of the BBOX of clique \(s\) lies along a line \(y = 2\psi(s)\)
- link-edge \((u, v)\) is a \(y\)-monotone curve from the top of \(R(u)\) to the bottom of \(R(v)\)

Properness:
- it is always possible to subdivide link-edges spanning non adjacent levels with a dummy clique containing a single vertex
- vertices are assigned to levels
- edges only connect vertices belonging to different levels
- cliques are assigned to levels
- link-edges only connect cliques belonging to different levels
What if cliques are given together with a hierarchical relationship among them?

- attempt to extend the notion of Level Planarity to Clique Planar Graphs

Properties:
- (P1) the top side of the BBOX of a clique $s$ lies on the corresponding level
- (P2) link-edge $(u, v)$ is a $y$-monotone curve from the top of $R(u)$ to the bottom of $R(v)$
Testing Algorithm for Level Clique Planarity

- **Level Clique Planarity**
- **Equivalent proper instance**
- **Proper Instance of \( T \)-Level Planarity**

**\( T \)-Level Planarity**

- the order of the vertices of Level \( i \) has to be **compatible** with a tree \( T_i \) whose leaves are the vertices in Level \( i \)

**Degrees of freedom:**
- order of the cliques along each level
- order of the squares in a canonical representation of their clique

\( O(n^2) \)-time testing algorithm

Angelini et al. · TCS’14
Open Problems

Intersection-link Representations

- how about considering other families of dense graphs?
  - e.g.: interval graphs, complete bipartite graphs, triangle-free graphs

- how about using different geometric objects?
  - unfortunately even triangles and circles might not have simple canonical representation

Clique Planarity

- What is the complexity of Bipartite 2-Page Book Embedding with and without spine crossings?
  - recall that B2PBE with Spine Crossings = CPLS with 2 clusters = Clique Planarity with $|S| = 2$
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Thanks for your attention!!

Open Problems

- Intersection-link representations
  - how about considering other families of dense graphs?
    - e.g.: interval graphs, complete bipartite graphs, triangle-free graphs
  - how about using different geometric objects?
    - observe that triangles and unit circles might not have simple canonical representation

- Clique Planarity
  - What is the complexity of the Bipartite 2-Page Book Embedding problem (with and without spine crossings)?
    - recall that B2PBE with Spine Crossings = CPLS with 2 clusters = Clique Planarity with \(|S| = 2\)