Maximizing the Degree of (Geometric) Thickness-t Regular Graphs

Christian A. Duncan

Department of Mathematics and Computer Science
Quinnipiac University

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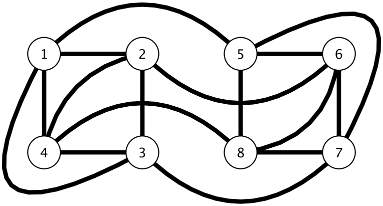
Thickness

Definition

The *thickness* $\Theta(G)$ of a graph G is the minimum number of planar subgraphs whose union forms G. The edges of these subgraphs form a partitioning of $E(G)$. For convenience, we identify each partition with a unique color.



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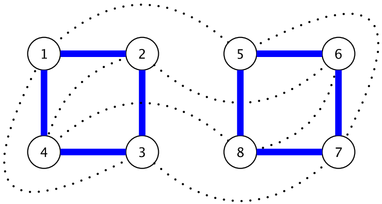

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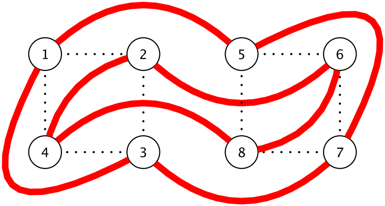
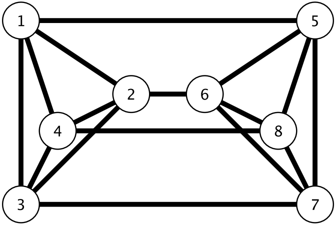


Geometric Thickness

Definition

The *geometric thickness* of G, $\bar{\Theta}(G)$, is the smallest integer t such that there is a *straight-line drawing* $\Gamma(G)$ whose edges can be colored with t colors such that no two edges with the same color intersect, except at the endpoints. That is, each coloring (layer) is a planar drawing.



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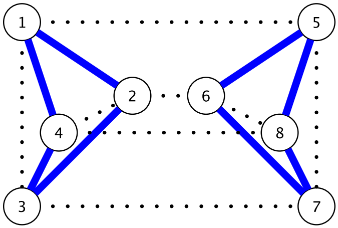

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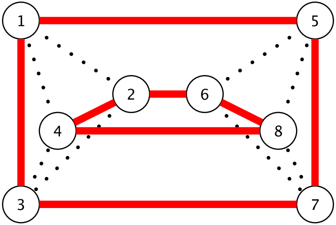

Motivation

- Durocher et al. [2013*] explored the relationship between colorability and thickness.
- Coloring: Fewest number of colors needed to color vertices of a graph so that no two adj. vertices have same color.
- Trivial to color a k-degenerate graph with $k+1$ colors
  1. Delete a degree-k vertex v.
  2. Color the remaining graph (with $k+1$ colors).
  3. Insert v back using one of the available colors
- k-regular graphs are k-degenerate graph.

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For (geometric) thickness-t graphs, what is the maximum k-regular graph possible?

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Previous Bounds

- $k = 5$ for planar graphs
  - There exist 5-regular planar graphs
- $k \leq 6t - 1$ for (geometric) thickness-t graphs
  - Based on edge counting
  - $|E| \leq (3n - 6)t$
  - Average degree $= \frac{2|E|}{n} \leq \frac{(6n-12)t}{n} = 6t - \frac{12t}{n}$
  - Must be at least one node with degree $< 6t$
- $k = 11$ for thickness-2 graphs [Durocher et al., 2013*]

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For $t > 2$, is $k < 6t - 1$?

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Our Results

Theorem

There exist $(6t-1)$-regular thickness-t graphs.
Thus, we show that $k = 6t - 1$ for thickness-t graphs.

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There exist $5t$-regular graphs with geom. thickness at most t.
For $t < 7$, the geometric thickness is exactly t.

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For $t < 7$, the geometric thickness is exactly t.

$(6t-1)$-regular Thickness-t Graphs (Overview)

- Construct a planar graph $\mathcal{G}$ having $48(t-1)$ degree-6 vertices and 48 degree-5 vertices.
- $\mathcal{G}_C \rightarrow \; C = 48t$ disjoint copies of $\mathcal{G}$
- Create t layers of $\mathcal{G}_C$ on same vertex set permuting the vertices to ensure every vertex has degree 5 in exactly one layer and no edge is repeated in different layers
  - $G \leftarrow$ Union of t layers of $\mathcal{G}_C$
  - Every vertex in G has degree $6t-1$
  - $\Theta(G) \leq t$ because every layer is planar — t layers
  - $\Theta(G) \geq t$ because $2|E| = (6t-1)n$ — too many edges

Constructing $\mathcal{G}_C$

- $16(t-1)$ nested triangles
  - 6 degree-4 vertices
  - Rest are degree-6 vertices
- Add vertices to get desired degrees.
  Observe symmetry
- deg. 4 verts now have deg. 6
- All new vertices have deg. 5.
- Repeat process for the inner triangle.
- Total $3(16(t-1)) + 2(24) = 48t$ vertices
  $48(t-1)$ are degree-6
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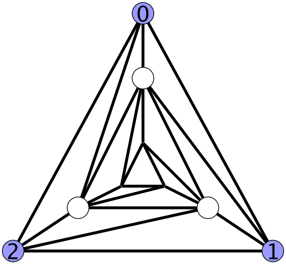

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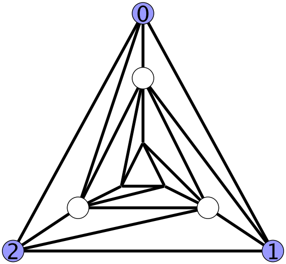

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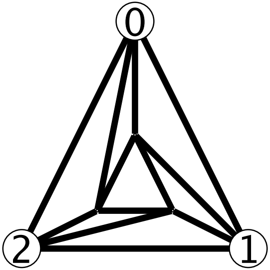

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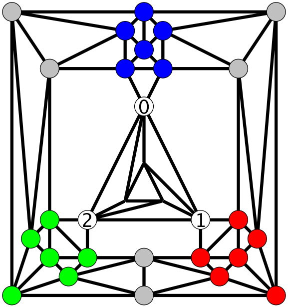

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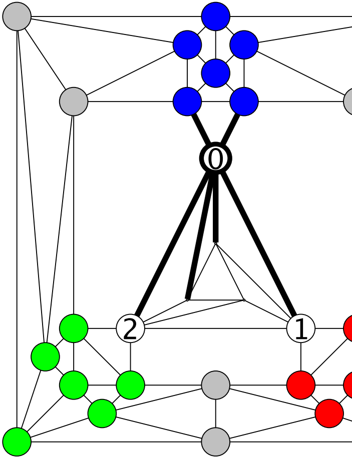

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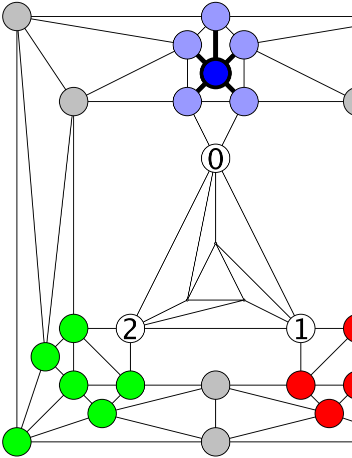

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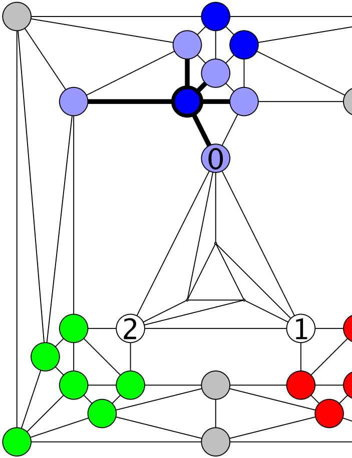

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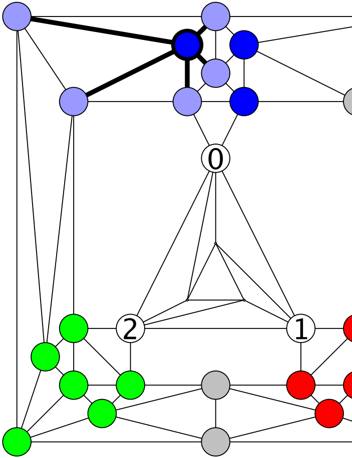

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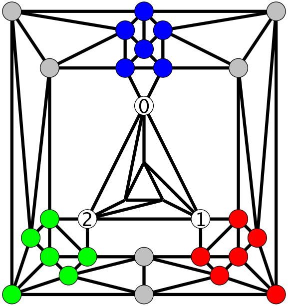

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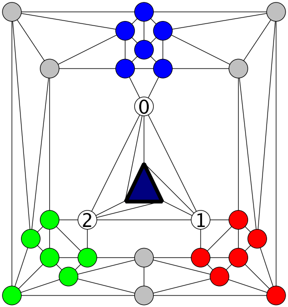

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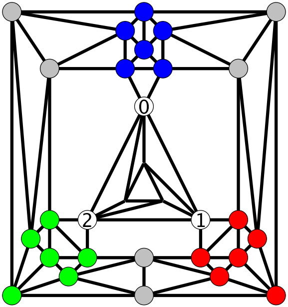

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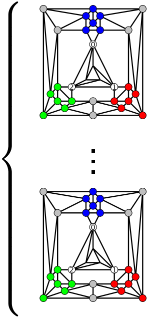
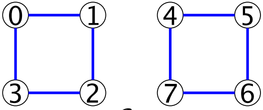


Now the fun part.
Merging multiple layers of this graph...

Merging Multiple Layers

- Suppose this is $\mathcal{G}_C$.
- Create multiple layers with each layer having a different permutation of the same vertices.
- $\pi_i(v) =$ permuted vertex in $\mathcal{G}_C$ of layer i, $0 \leq i < t$
- Example:



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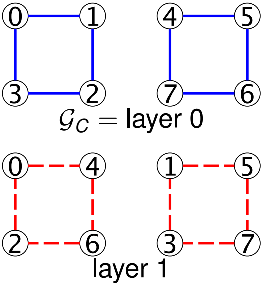

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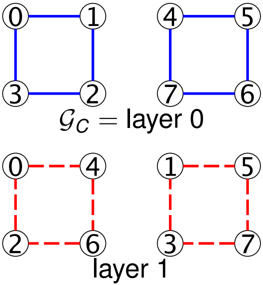

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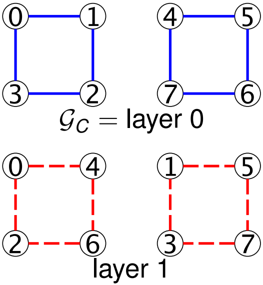

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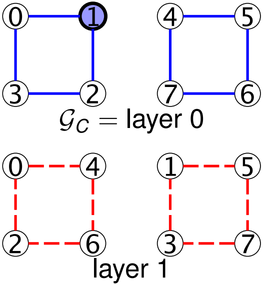

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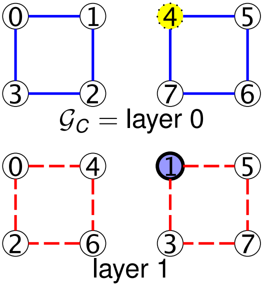

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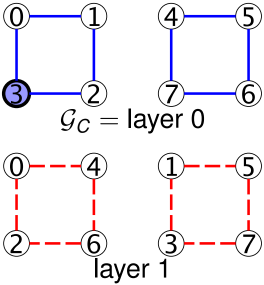

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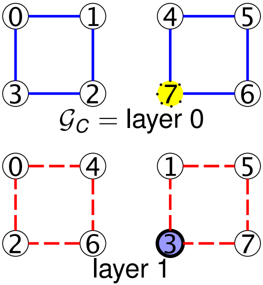

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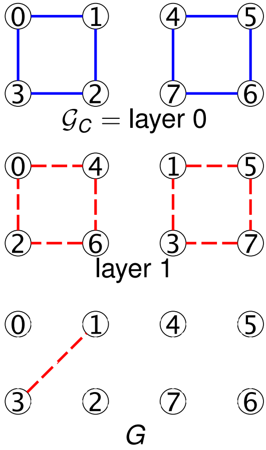

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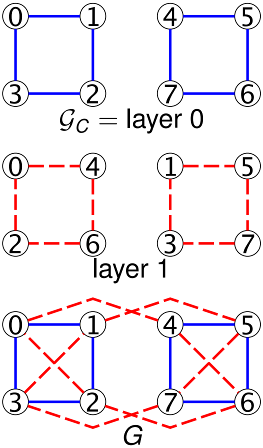

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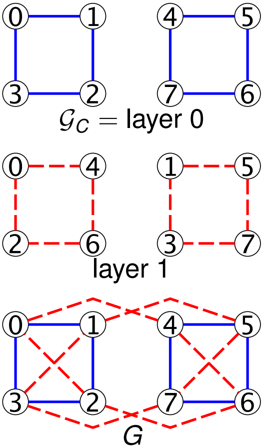
- $\pi_i(v) =$ permuted vertex in $\mathcal{G}_C$ of layer i, $0 \leq i < t$
- Example:
  $\pi_0(1) = 1$, $\pi_1(1) = 4$
  $\pi_0(3) = 3$, $\pi_1(3) = 7$
  $(1,3) \in E(G)$ because
  $(\pi_1(1), \pi_1(3)) = (4,7) \in E(\mathcal{G}_C)$
- Strategy: Do the same for t layers of $\mathcal{G}_C$. With certain conditions...



Merging Multiple Layers of $\mathcal{G}_C$ to form G

Conditions

Want to create permutations $\pi_i(v)$ such that:

1. Every vertex gets mapped to a degree 5 vertex exactly once.
2. No duplicate edges: no edge is in more than one layer.

Conditions 1 and 2 (and our construction of $\mathcal{G}_C$) guarantee that G is $(6t-1)$-regular.

Vertex Mapping

To complete our mapping, it helps to group portions of the triangles from $\mathcal{G}$ into t levels, ℓ.

- $\ell = 0$ is set of outer/inner (degree 5) vertices
- Other $t - 1$ levels are groups of 16 triangles in $\mathcal{G}$
- Each level has 48 vertices $\rightarrow$ t levels.

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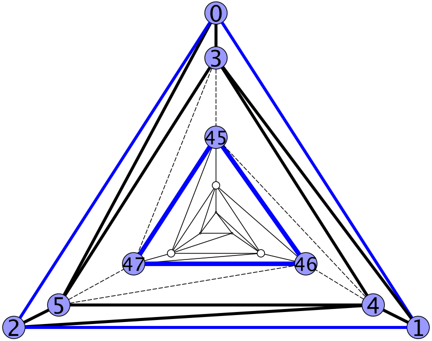

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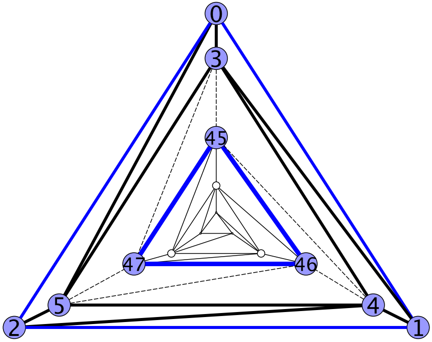

Vertex Mapping

- Label vertices of $\mathcal{G}_C$ as $\rho_{a,\ell,c}$
- a is an ordering of vertices within one level of $\mathcal{G}$
- ℓ is the level $(0 \leq \ell < t)$
- c is the cluster $(0 \leq c < C)$
- Label vertices of G such that $\pi_0(v_{a,\ell,c}) = \rho_{a,\ell,c}$.

General Mapping

$$\pi_i(v_{a,\ell,c}) = \rho_{a,(\ell+i) \bmod t,(c+i) \bmod C}$$

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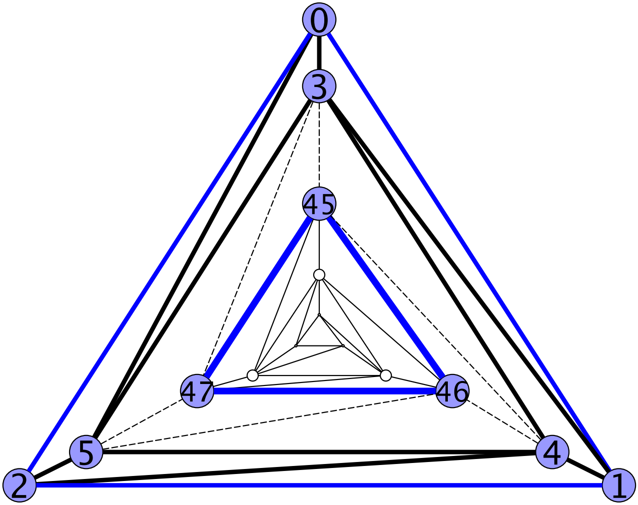

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$(6t-1)$-regular Thickness-t Graphs

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G is a $(6t-1)$-regular thickness-t graph

Proof.

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Proof.

Condition 1: Exactly one degree-5 assignment:

- $\rho_{\cdot,0,\cdot}$ are the only degree-5 vertices
- $\pi_i(v_{a,\ell,c}) = \rho_{\cdot,0,\cdot}$ only when $(\ell+i) \bmod t \equiv 0$.

$(6t-1)$-regular Thickness-t Graphs

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$$\pi_i(v_{a,\ell,c}) = \rho_{a,(\ell+i) \bmod t,(c+i) \bmod C}$$

Proof.

Condition 2: No duplicate edges:

- Suppose $v_{a,\ell,c}$ and $v_{a',\ell',c'}$ share edge in layers i, j with $i < j$.
- $\nexists$ edges in $\mathcal{G}_C$ between two nodes with same a
- $\nexists$ edges in $\mathcal{G}_C$ between two nodes with different c
- So, their "assignment" in i-th layer must have same c value.
- That is, $c + ai \equiv c' + a'i \bmod C$ (and similarly for j)
- Therefore, $a(j-i) \equiv a'(j-i) \bmod C$.
- But $0 \leq a, a' < 48$, $j - i < t$ and $C = 48t$
- So, only holds when $j = i$ $\Rightarrow\Leftarrow$

□

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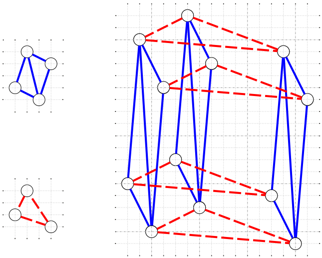
What about geometric thickness-t graphs?

Cartesian Product

Definition

The Cartesian product of two graphs: $G = G_1 \square G_2$

- $V(G) = V(G_1) \times V(G_2)$
- $((v_1, v_2), (u_1, u_2)) \in E(G)$ if and only if
  - $v_1 = u_1$ and $(v_2, u_2) \in E(G_2)$, or vice versa



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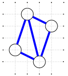

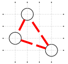

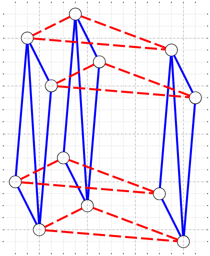

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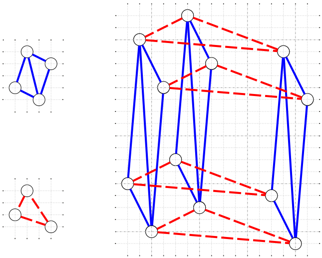

Cartesian Product

Lemma

$\bar{\Theta}(G_1 \Box G_2) \leq \bar{\Theta}(G_1) + \bar{\Theta}(G_2)$

The geometric thickness of the cartesian product is (at most) the sum of the geometric thicknesses of the two graphs.

Proof.

By picture... □

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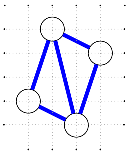

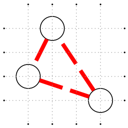

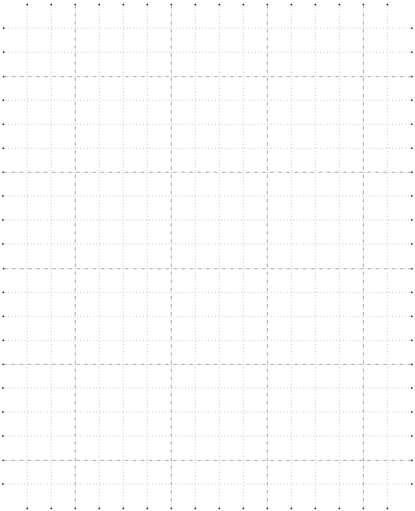

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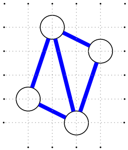

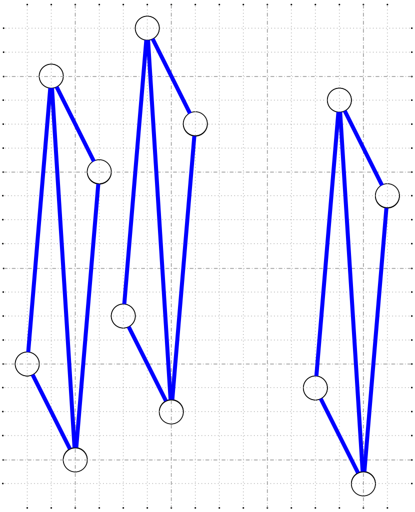

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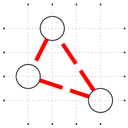

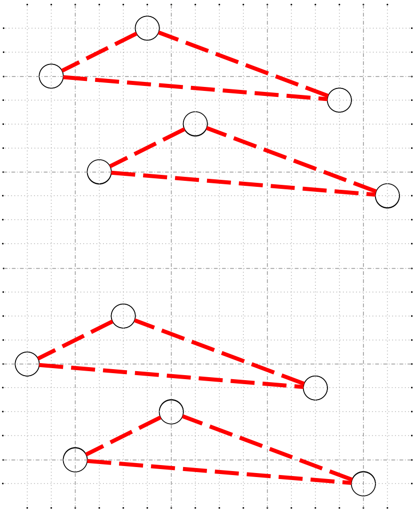

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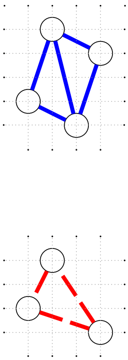

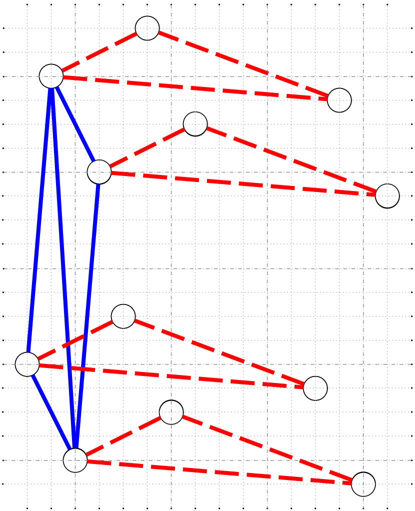

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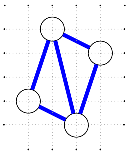

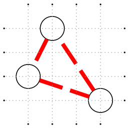

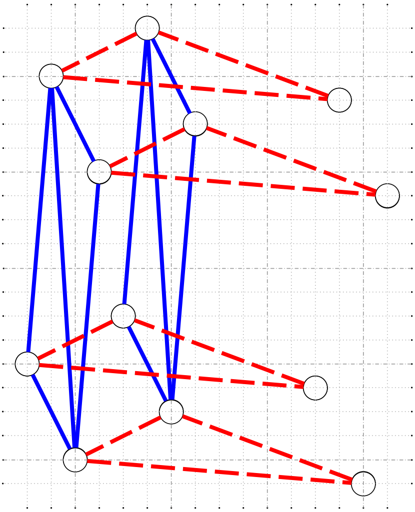

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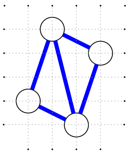
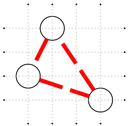
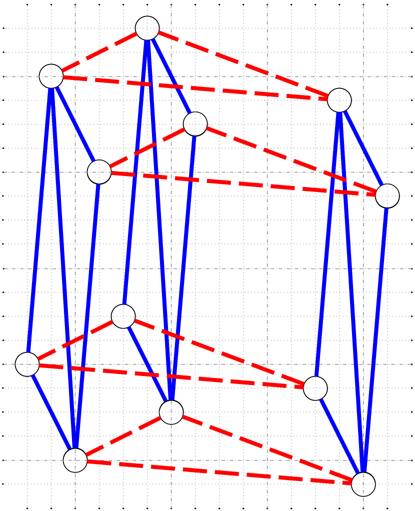
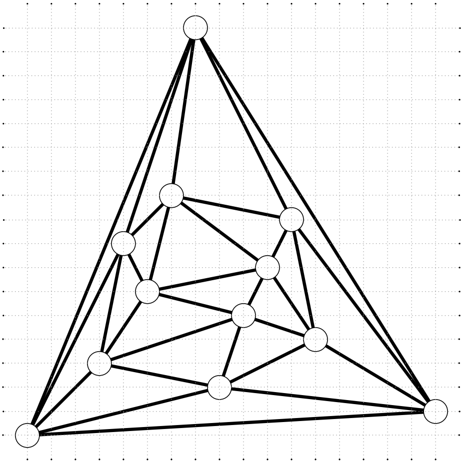






5-regular planar graph G_5



5t-regular graphs

Theorem

There exist 5t-regular graphs with geom. thickness at most t.

Proof.

- $\mathbb{G} = G_5 \square G_5 \square \cdots \square G_5$ ($t - 1$ times)
- $\bar{\Theta}(\mathbb{G}) \leq t$
- Every vertex has degree $5t$
- *Exactly* t for $t < 7$ — Edge counting
- If $\bar{\Theta}(\mathbb{G}) < t$, then $5tn < 6(t-1)n$ (or $t > 6$)

□

Conclusions and Open Questions

Theorem

There exist $(6t-1)$-regular thickness-t graphs.

Question #1

What is the smallest $(6t-1)$-regular graph of thickness t?
Our example had $(48t)^2$ vertices and we know that $|V| \geq 12t$.
Durocher et al. present a 32-vertex thickness-two graph.

Conclusions and Open Questions

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Question #2

What is the largest k such that there exists a k-regular graph of geom. thickness t? Is there an 11-regular graph with geom. thickness 2?

Question #3

Does the graph $\mathbb{G}$ have geom. thickness *exactly* t for all $t \in \mathbb{Z}^+$? We know it is true for $t < 7$.

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Thank You!