The Book Embedding Problem from a SAT-Solving Perspective

[GD 2015]

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Universität Tübingen, Germany

The Book Embedding problem

- Vertices V $\rightarrow$ ordered, on a spine
- Edges E $\rightarrow$ assigned to p pages
- Pages $\rightarrow$ half planes, delimited by spine
  planar

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goal: minimize p - **book thickness**

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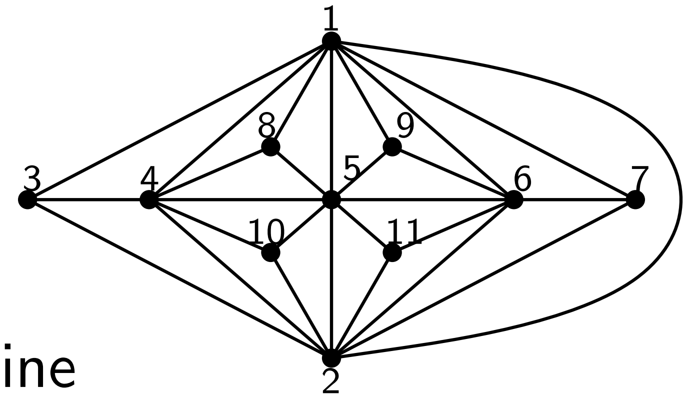

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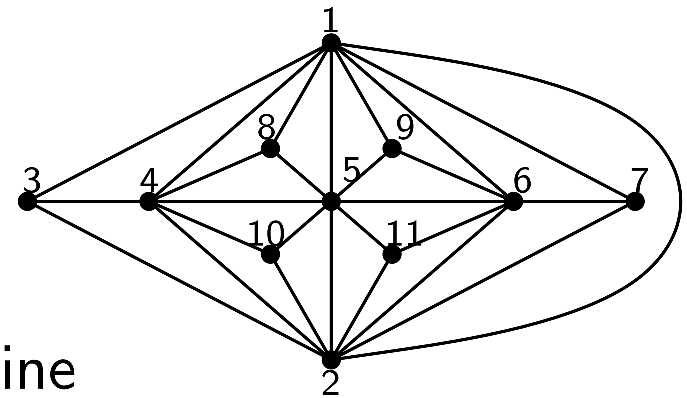

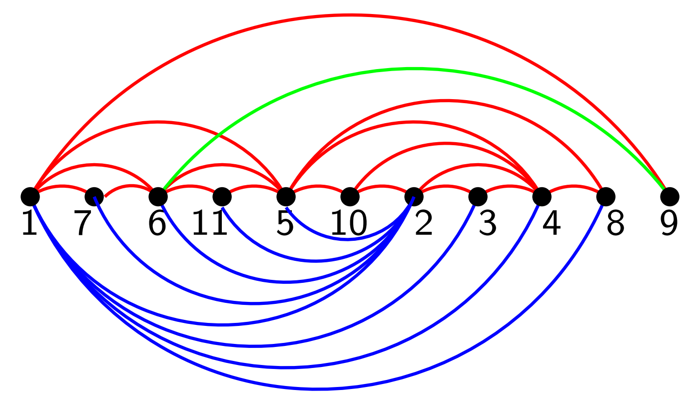

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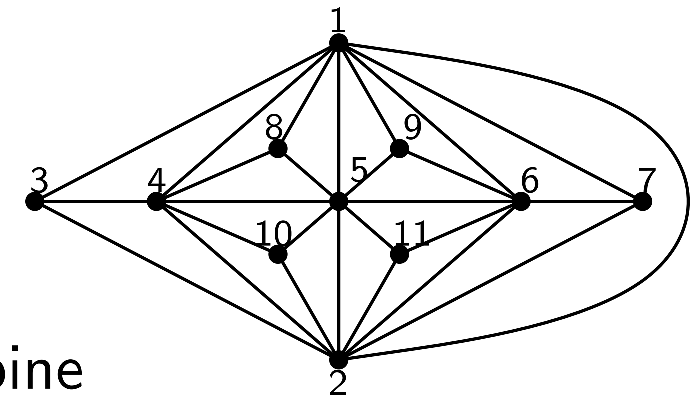

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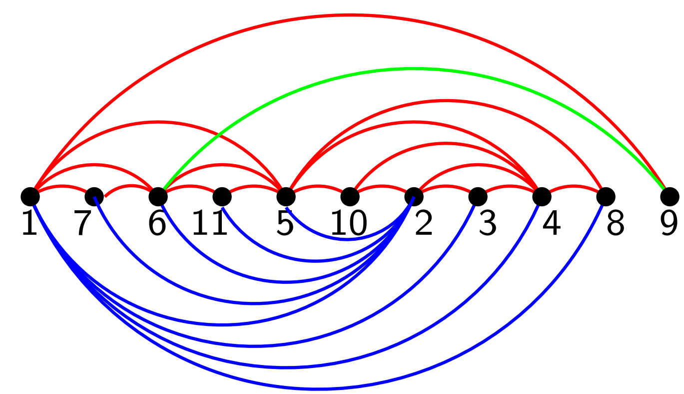

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Known results:

[Bernhard & Kainen, 1979]

$bt(G) = 1 \Leftrightarrow G$ is outerplanar, $bt(G) = 2 \Leftrightarrow G$ is subhamiltonian

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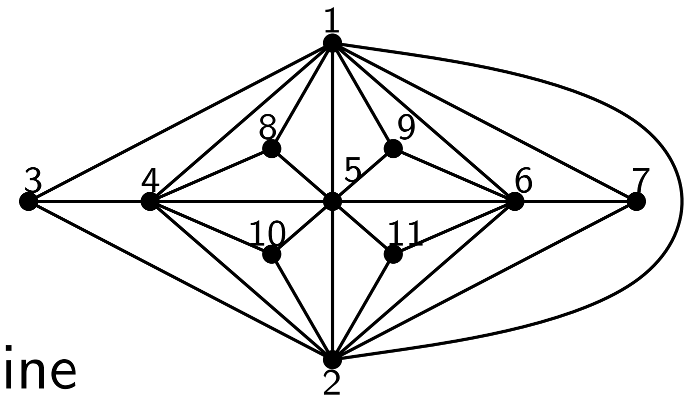

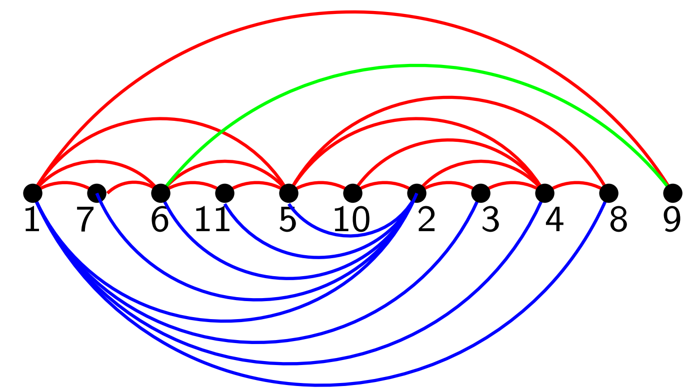

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all planar graphs can be embedded on 4 pages (no graph needing 4 pages is known)

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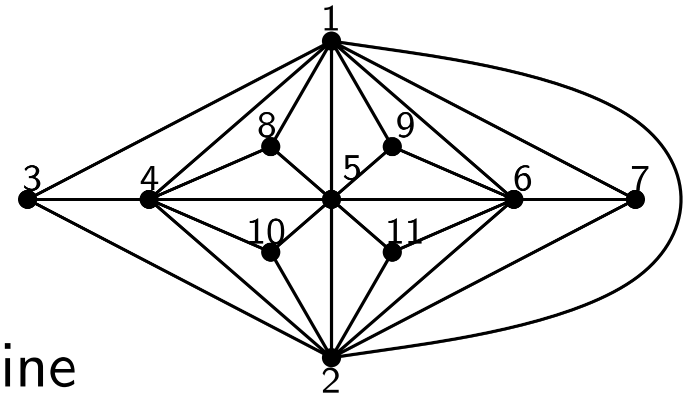

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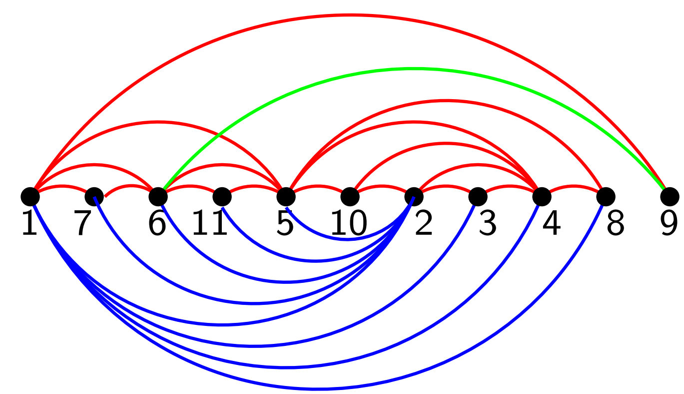

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[Bekos et al., 2015]

1-planar graphs have constant book thickness (39)

SAT solving

- Boolean variables "sun is shining", "road is wet", α, β, ...

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    $\rightarrow$ implications: "if ... then ..."

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use SAT solver as black box / oracle

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Applications of SAT:

[Zeranski & Chimani, 2012]

Upward planarity via SAT

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Applications of SAT:

[Zeranski & Chimani, 2012]
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[Biedl et al, 2013]
SAT for grid-based graph problems (pathwidth, vis. representation, ...)

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General idea: build formula $\mathcal{F}(G, p)$ via

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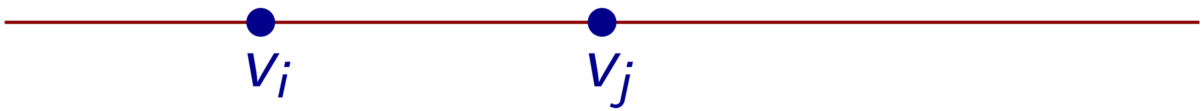

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$\sigma(v_i, v_j)$: v_i is left of v_j on spine

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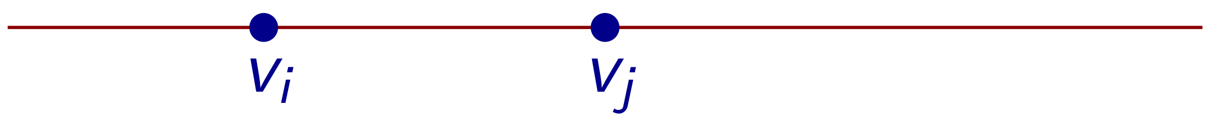

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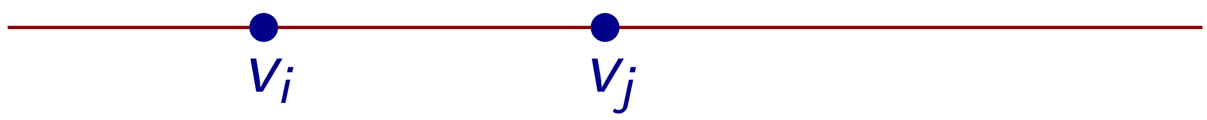

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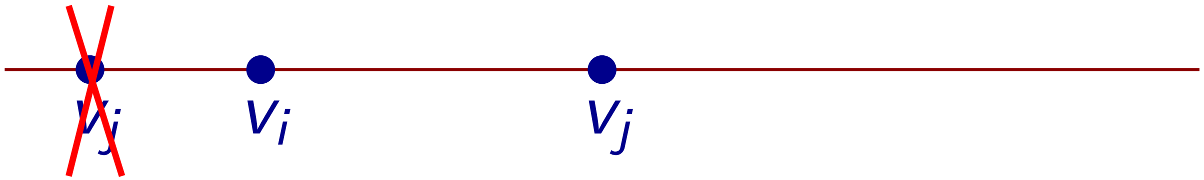

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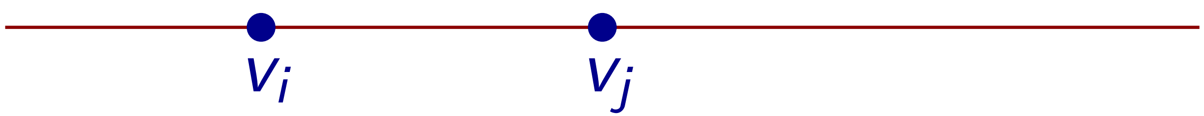

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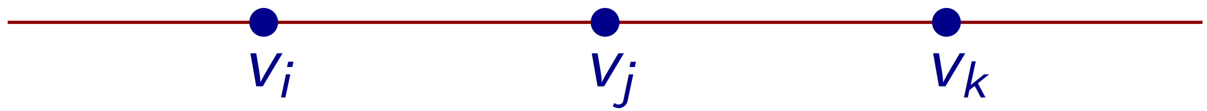

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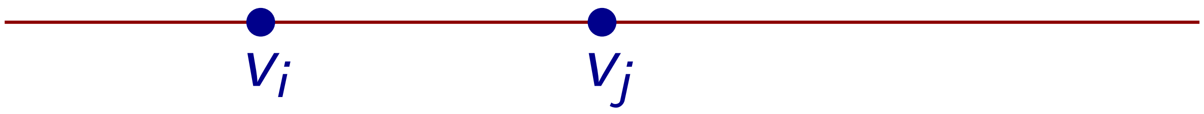

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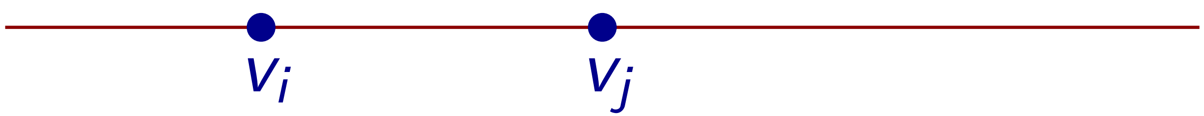

$\sigma(v_i, v_j)$

2. assure that every edge is assigned to one of p pages

Variables:

$\phi_p(e_i)$: edge e_i is assigned to page p

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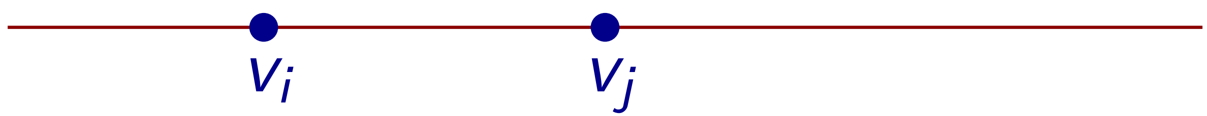

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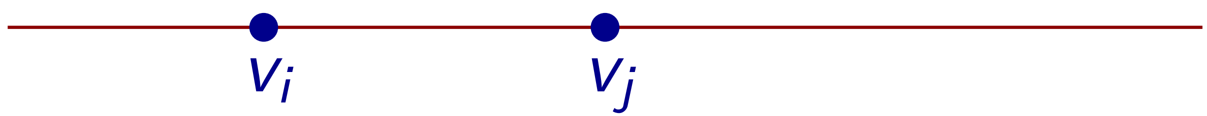

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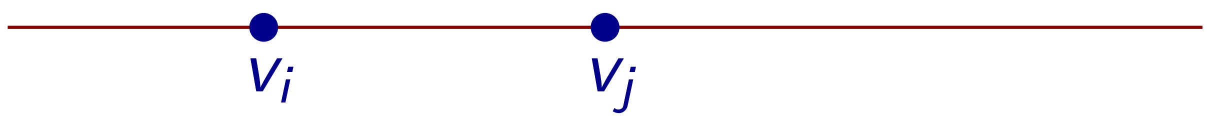

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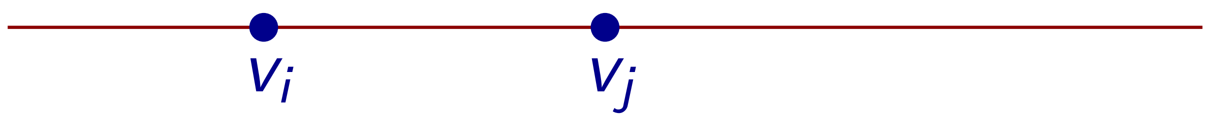

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same page: $(\phi_k(e_i) \wedge \phi_k(e_j)) \to \chi(e_i, e_j)$

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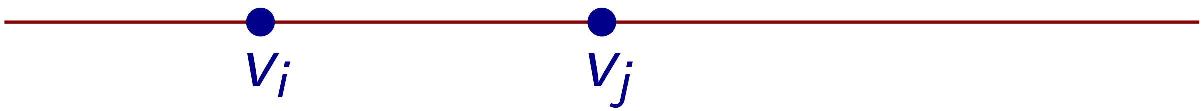

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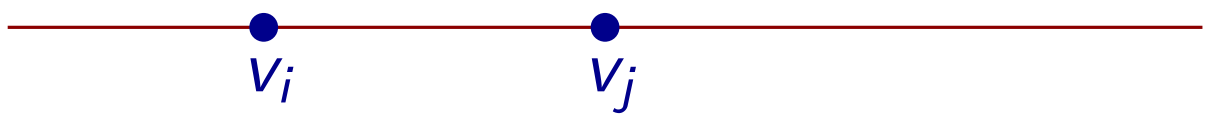

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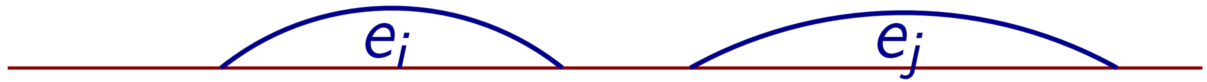

$\chi(e_i, e_j)$

3. forbid crossings on each page

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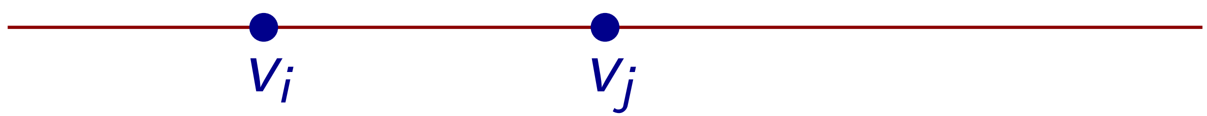

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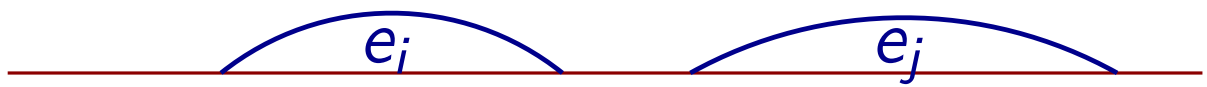

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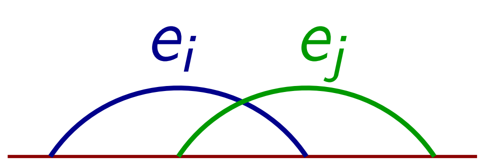

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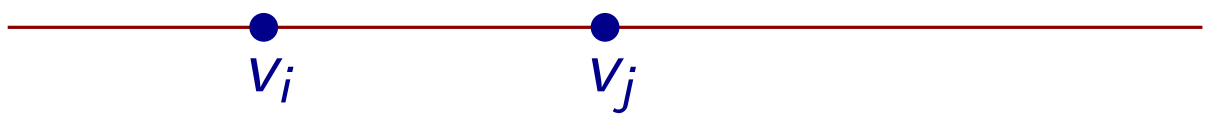

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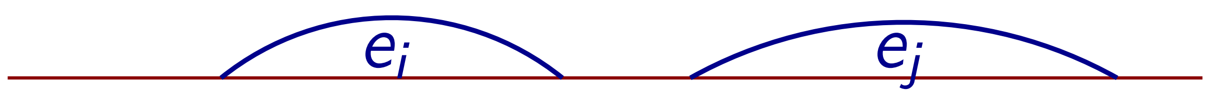

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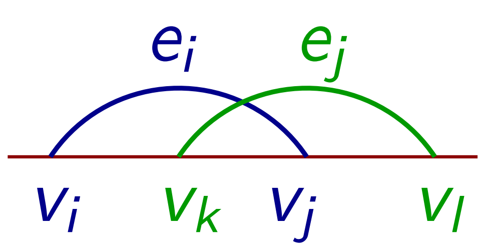

forbid $\chi(e_i, e_j)$ together with $\sigma(v_i, v_k), \sigma(v_k, v_j), \sigma(v_j, v_l)$

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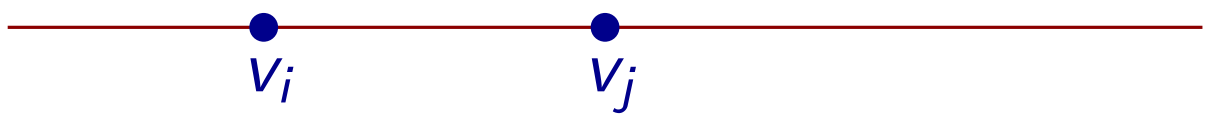

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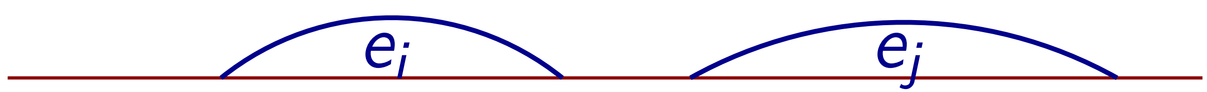

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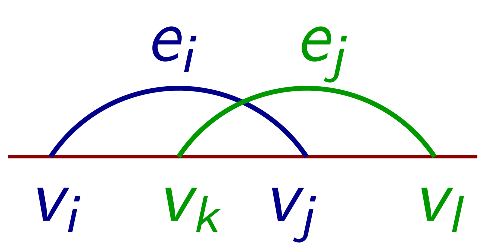

forbid $\chi(e_i, e_j)$ together with $\sigma(v_i, v_k), \sigma(v_k, v_j), \sigma(v_j, v_l)$

(for every pair of edges 8 forbidden configurations)

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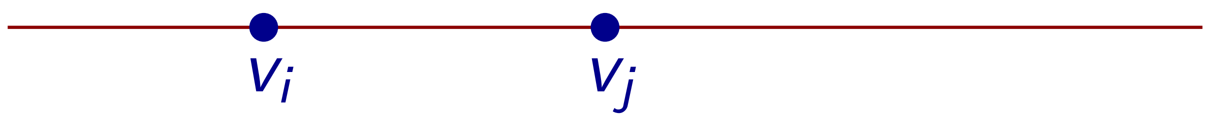

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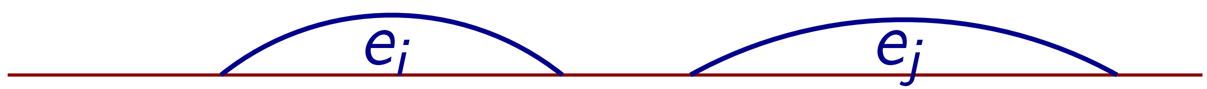

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solve optimization problem: $\mathcal{F}(G, k-1)$ **is UNSAT**, $\mathcal{F}(G, k)$ **is SAT**

Experiments

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    (some graphs have high book thickness)

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- Hypothesis:
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Stellation — Octahedron Creation — Graph Insertion

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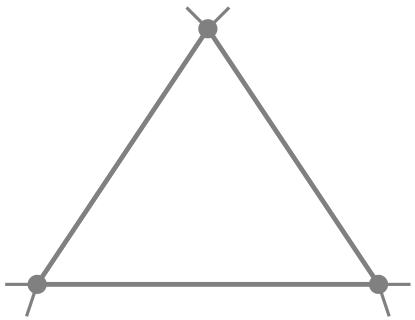

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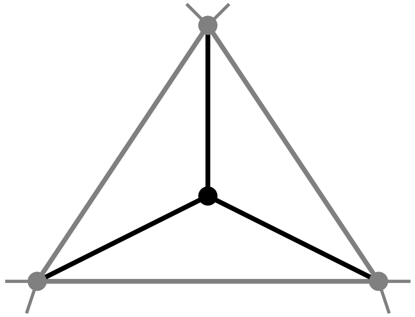

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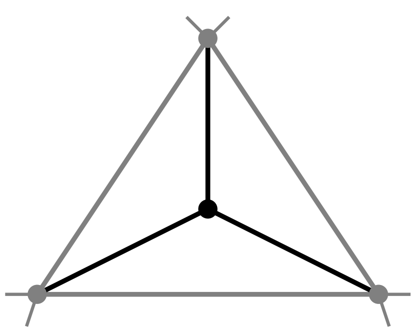
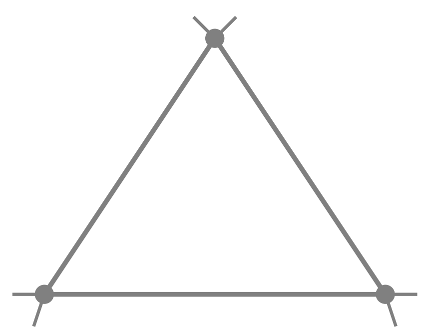


Octahedron Creation



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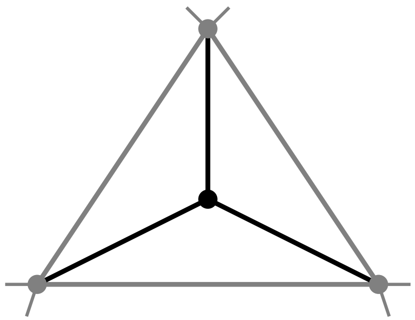

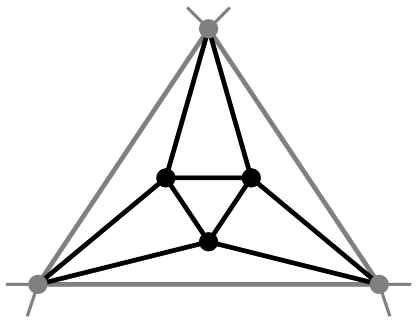

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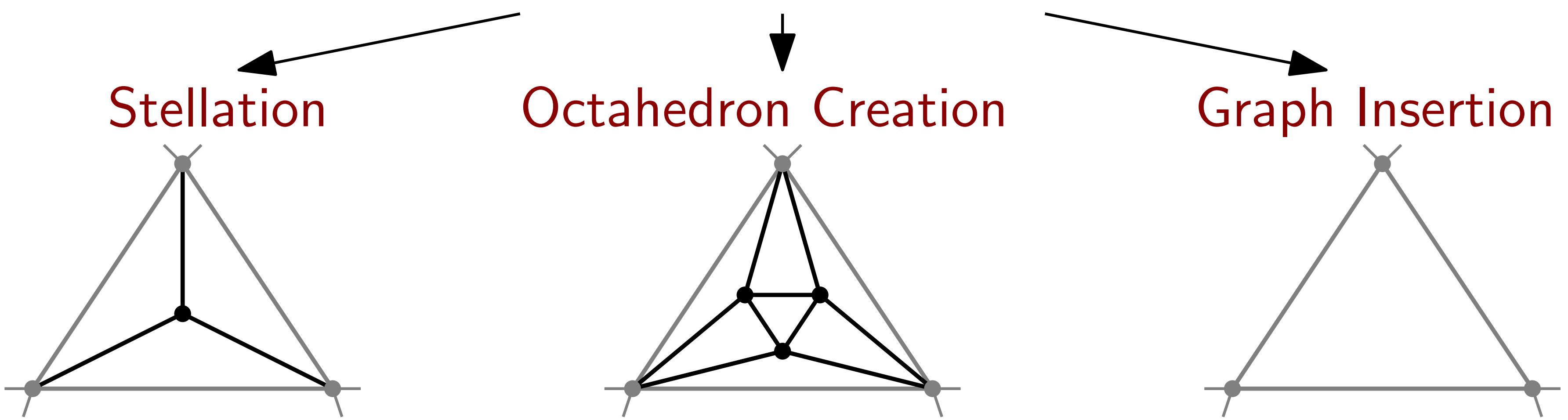

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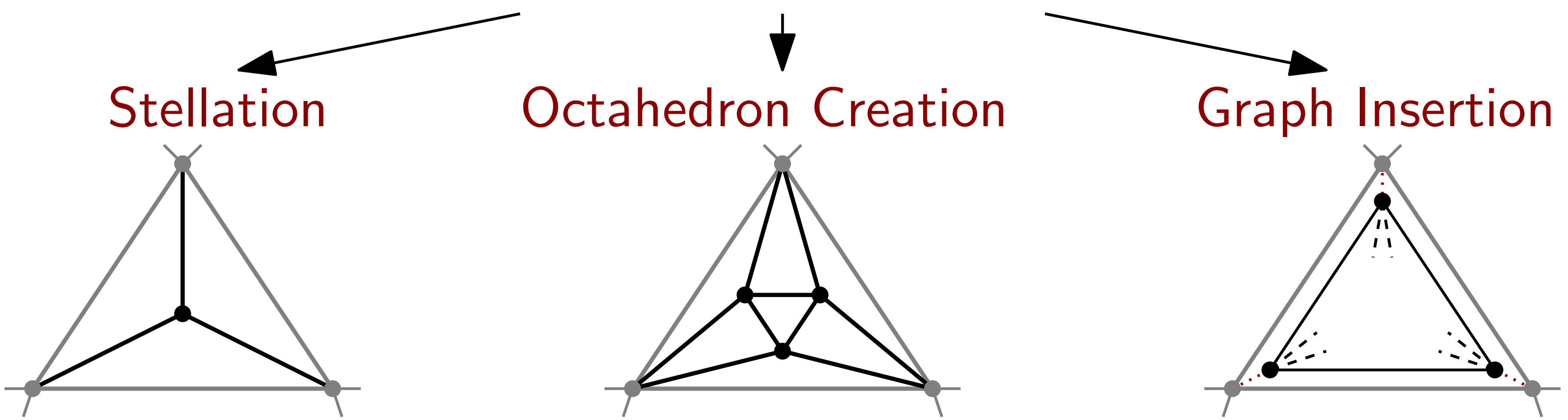

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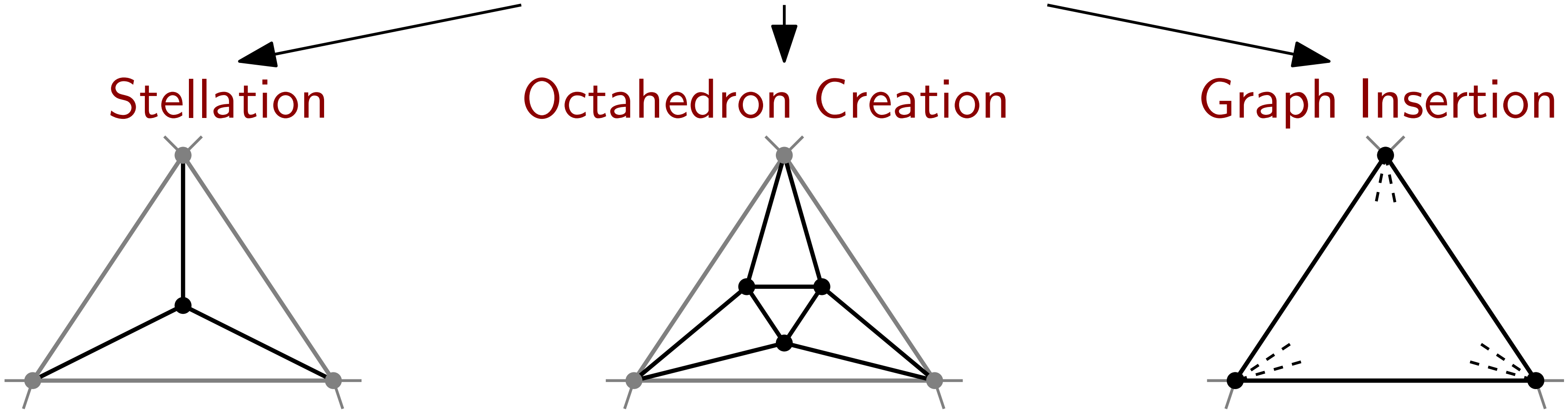

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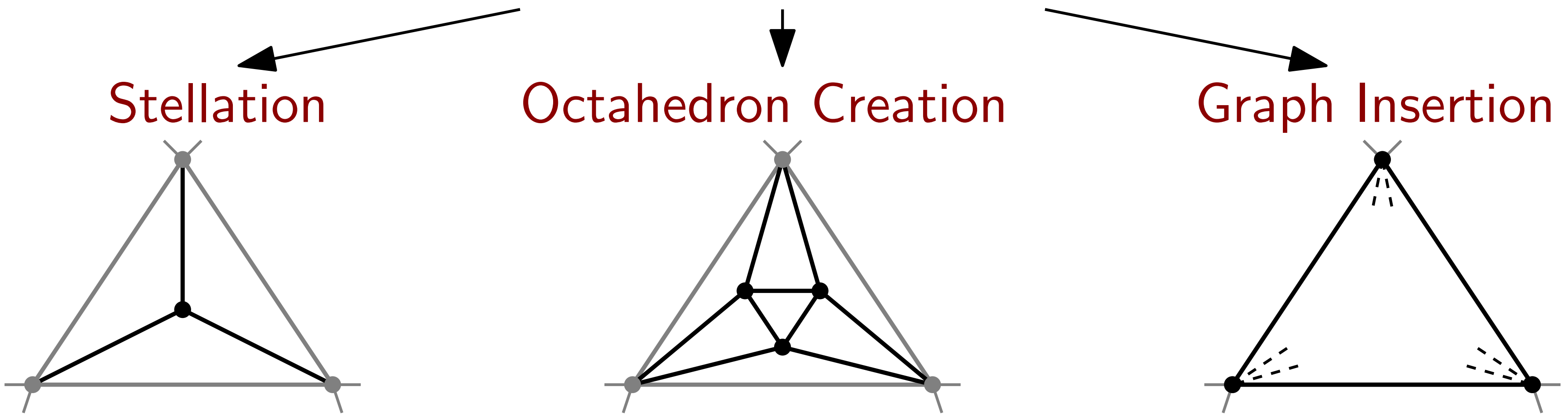

Experiments

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- **Idea:** possibly overcome the limit of $\approx$ 700 vertices

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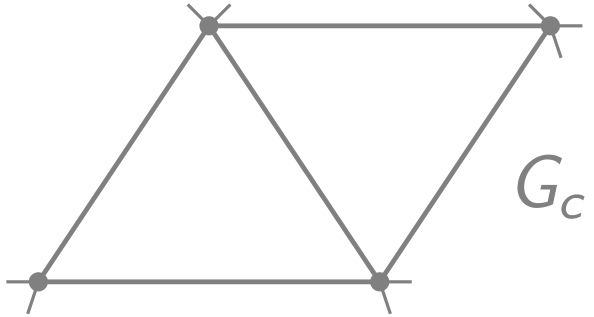

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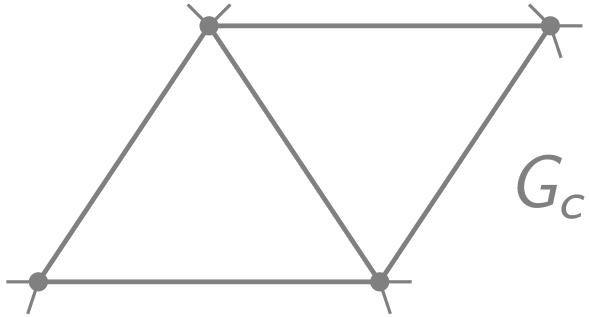

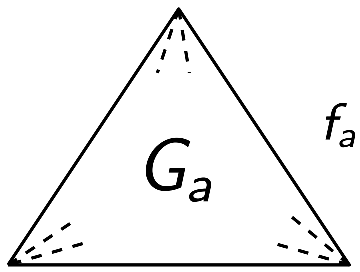

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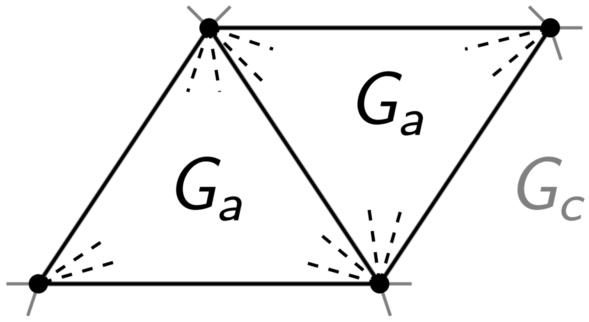

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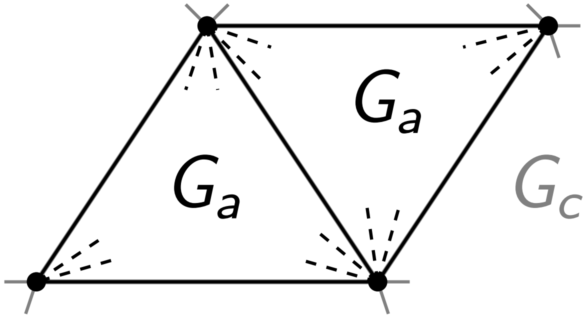

tested $\approx$ 284,000 graphs
with 60 to 125 vertices
$\rightarrow$ 0 confirmed Hypotheses

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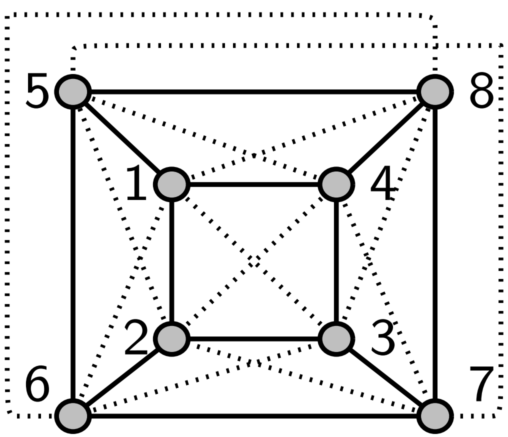

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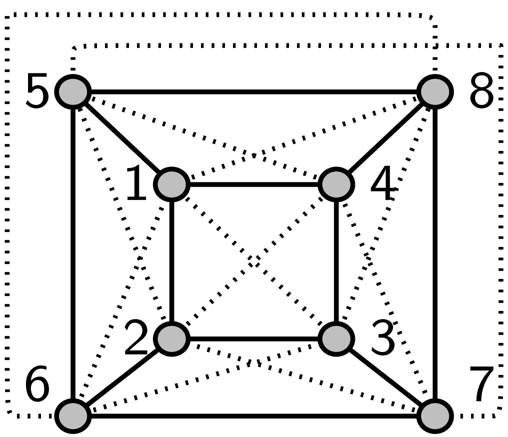

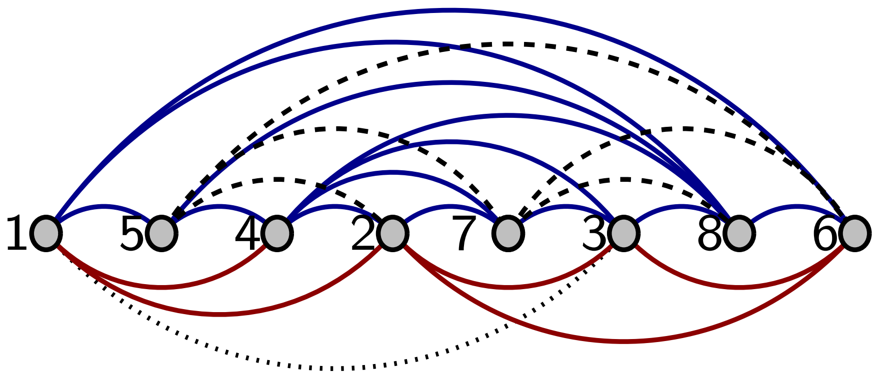

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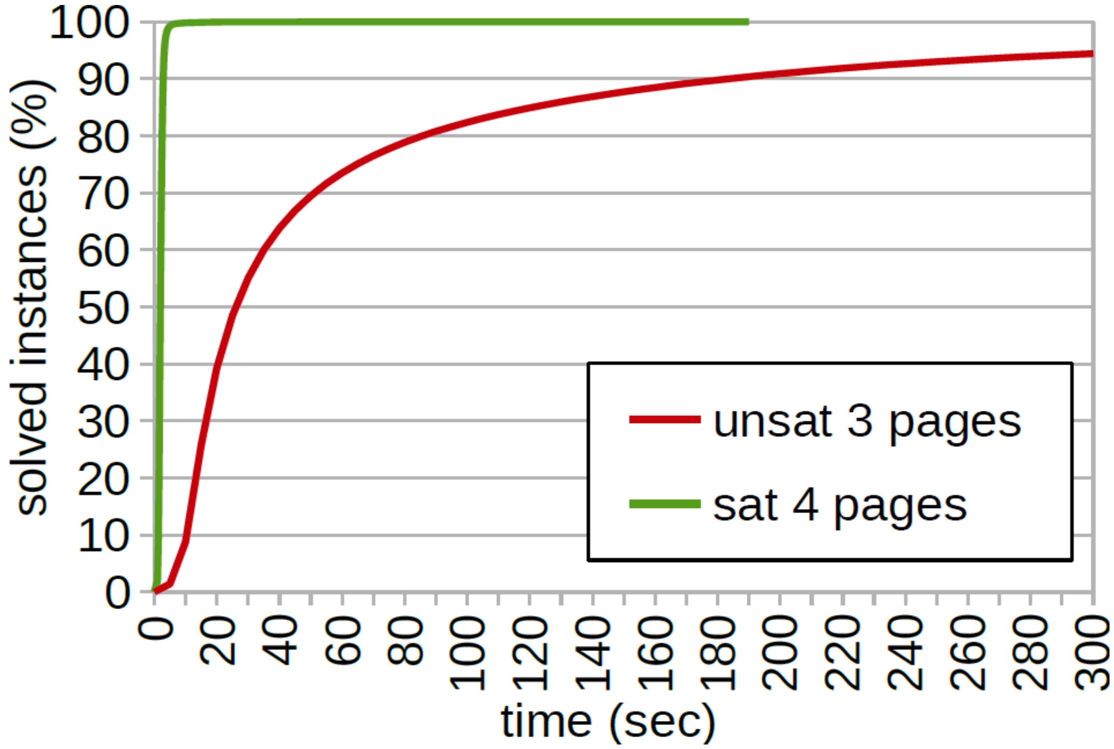

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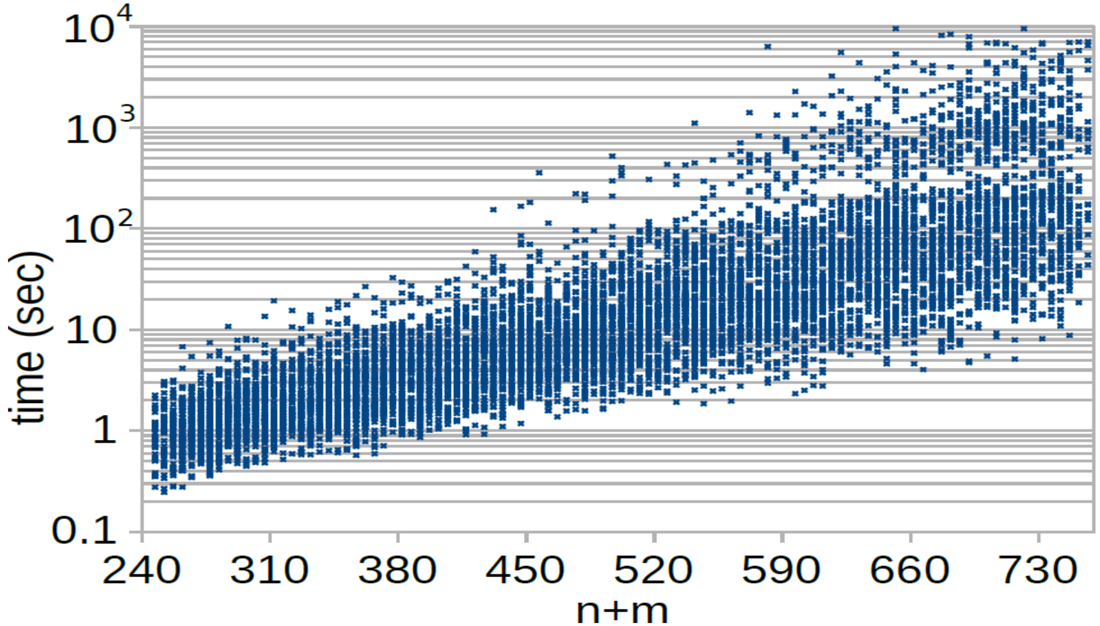

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$\rightarrow$ additional constraints to force acyclic subgraphs increase runtime

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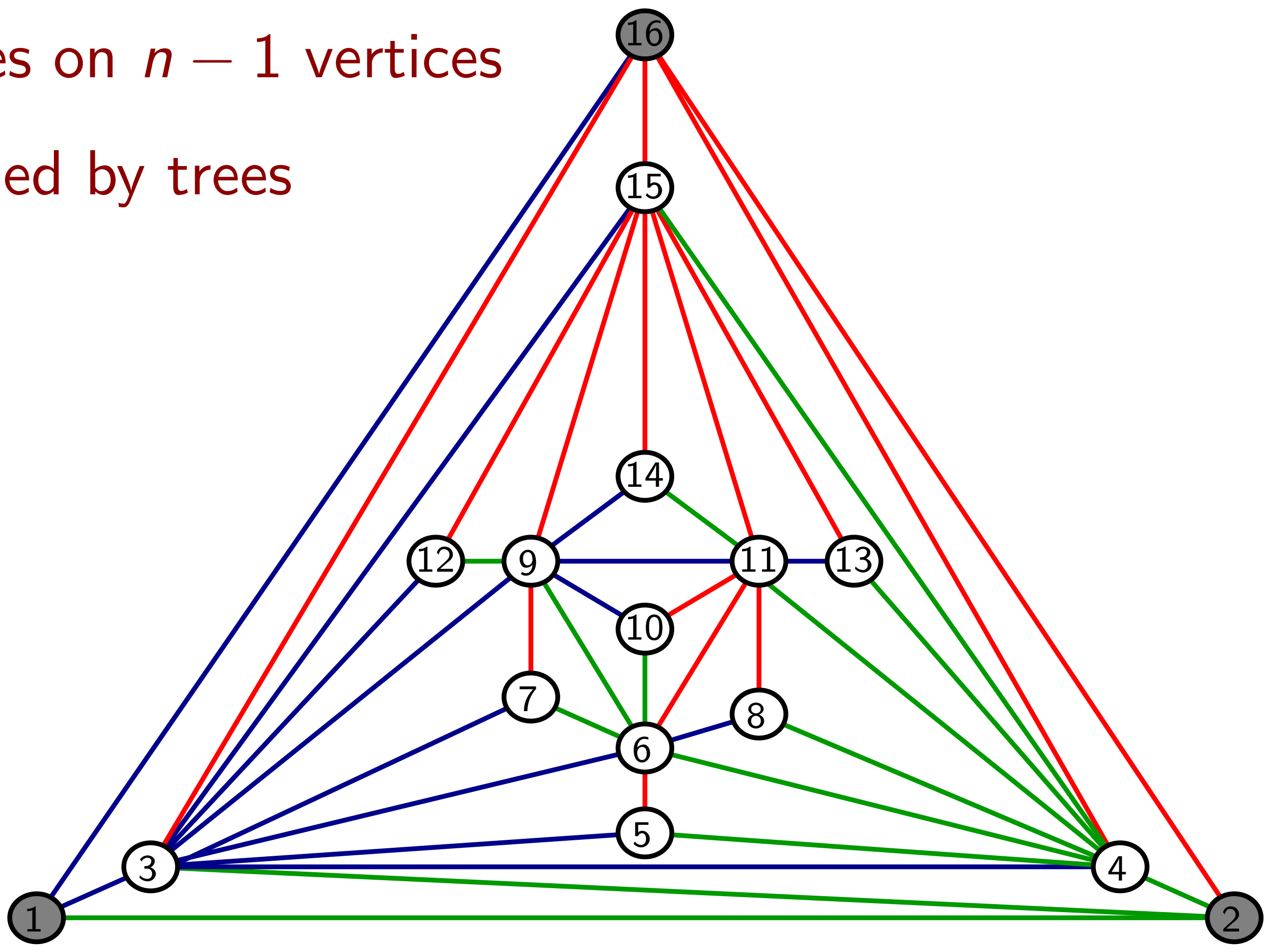

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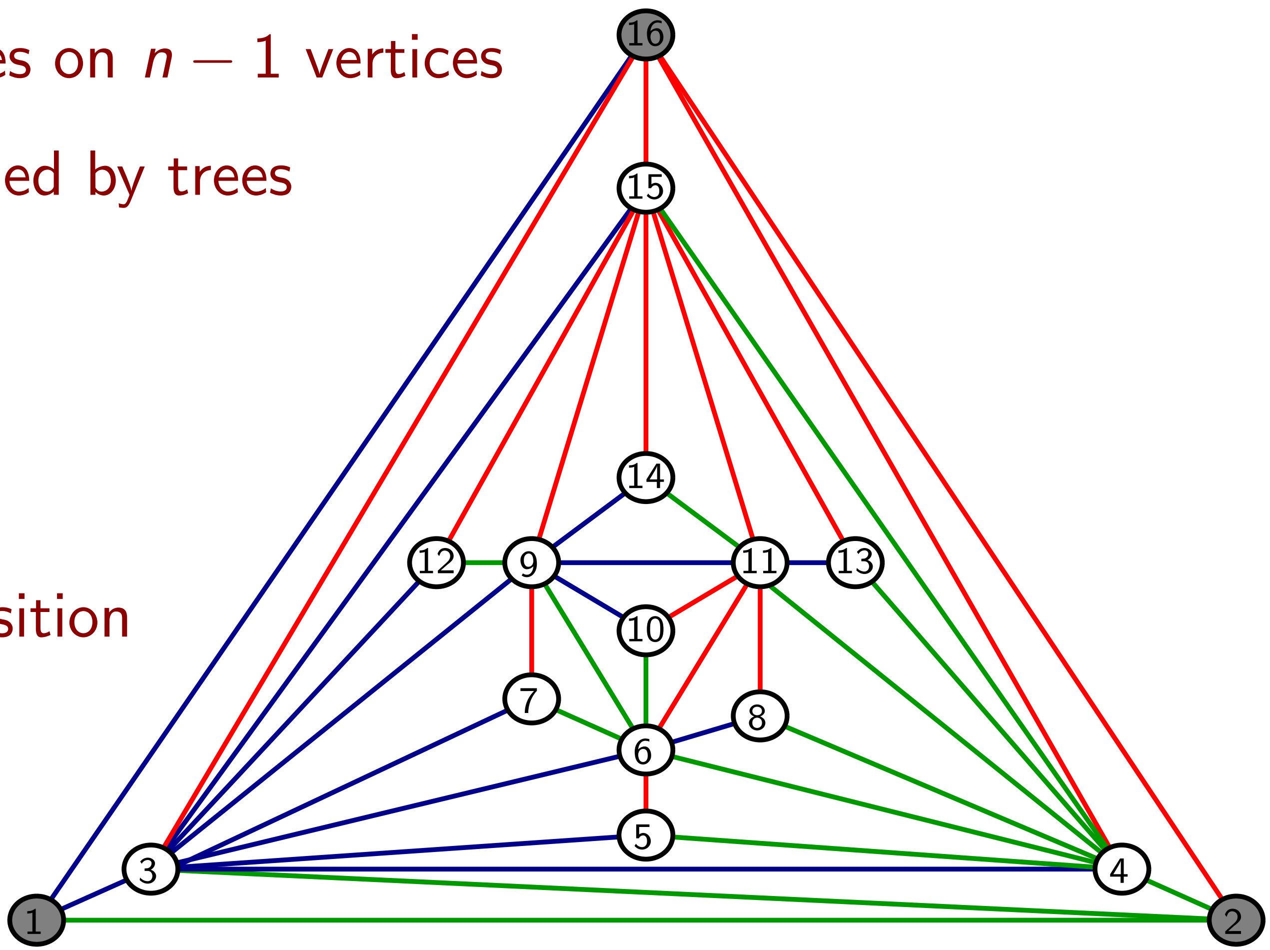
→ Schnyder decomposition not always possible



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Thanks!!!

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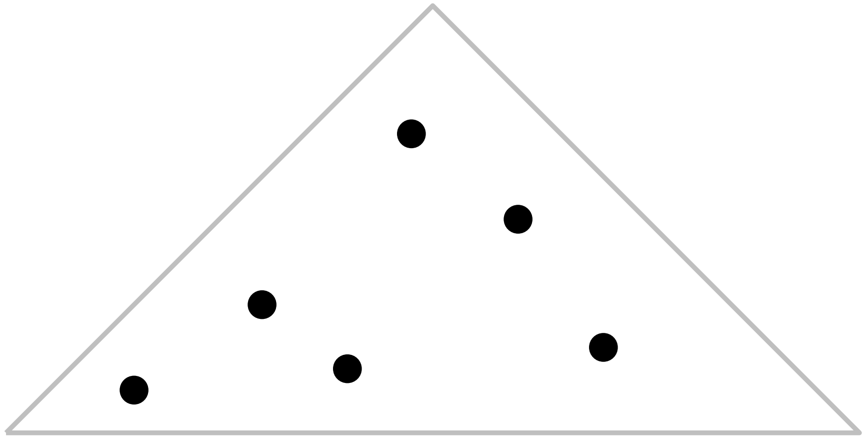

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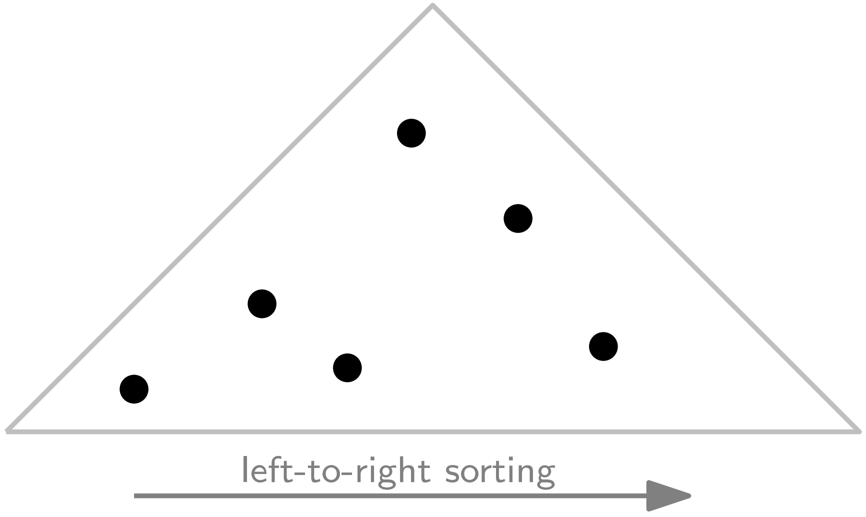

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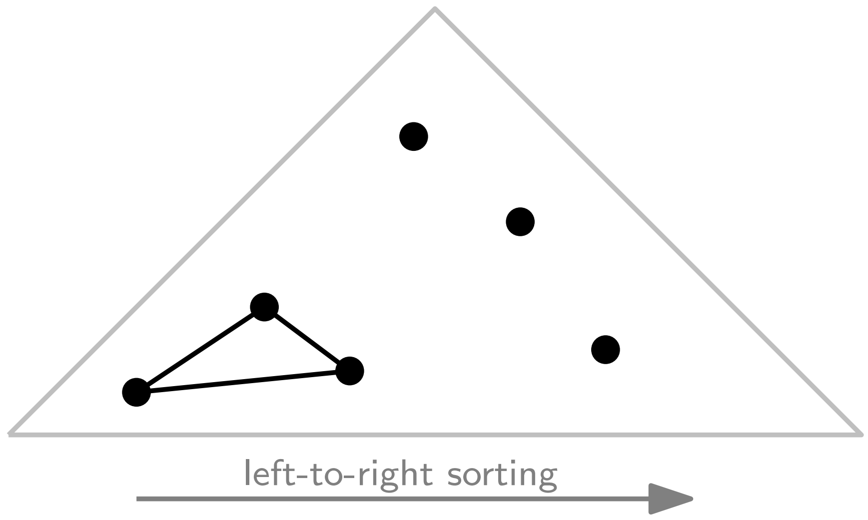

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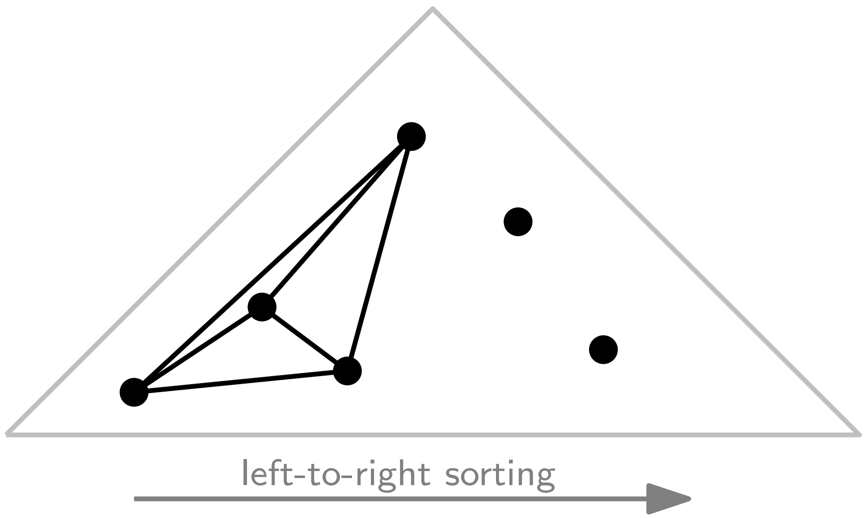

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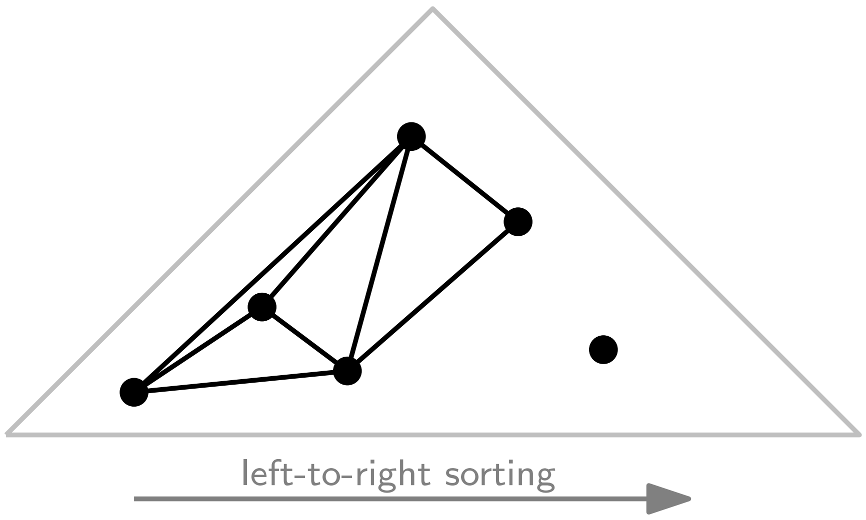

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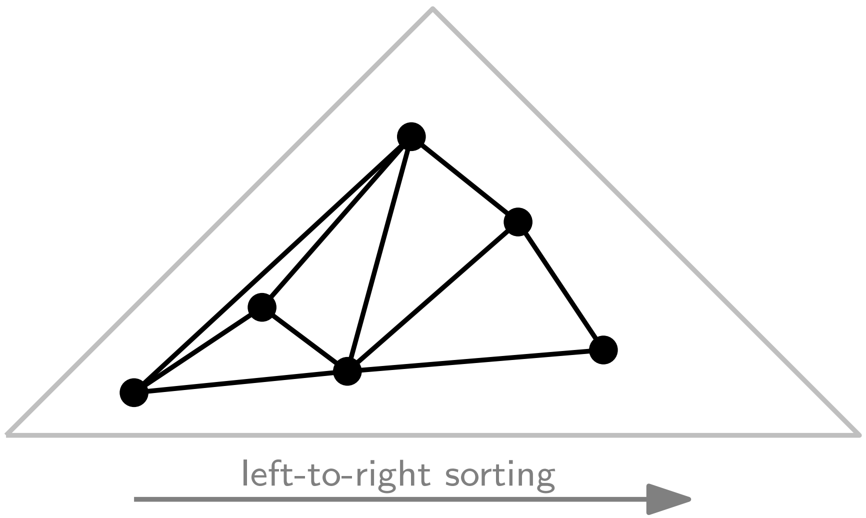

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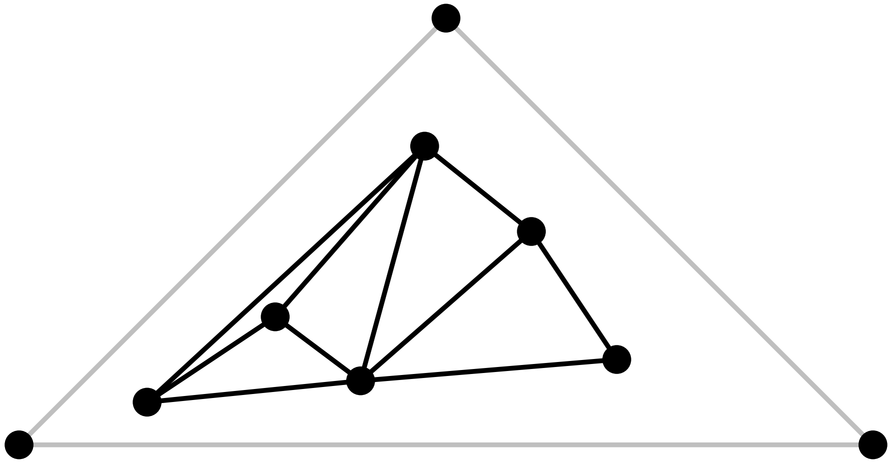

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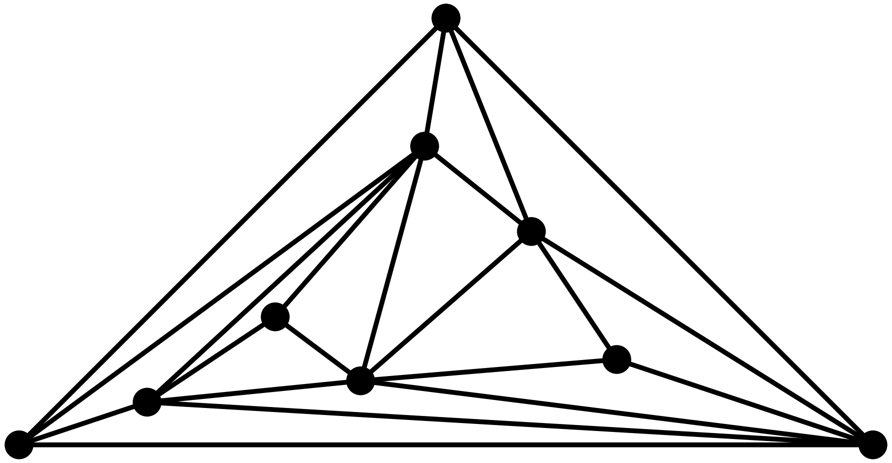

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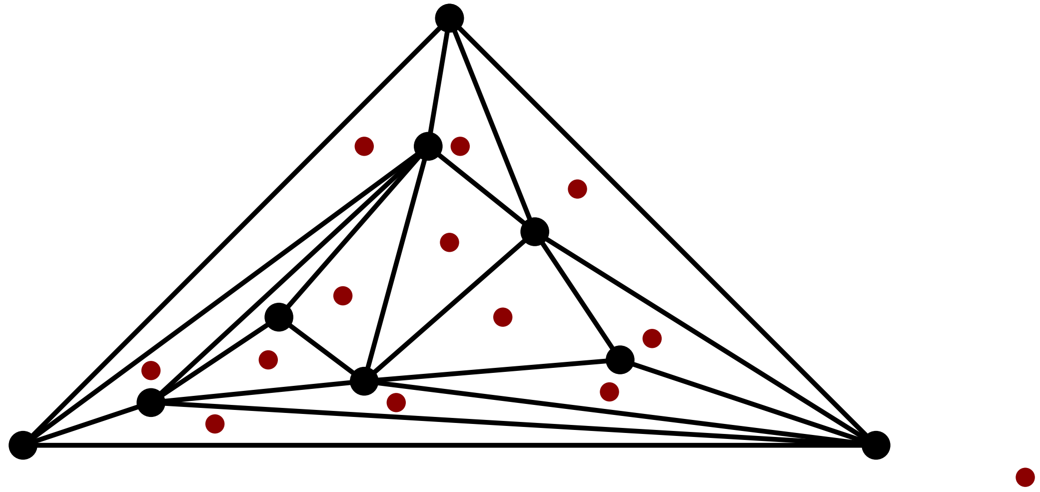

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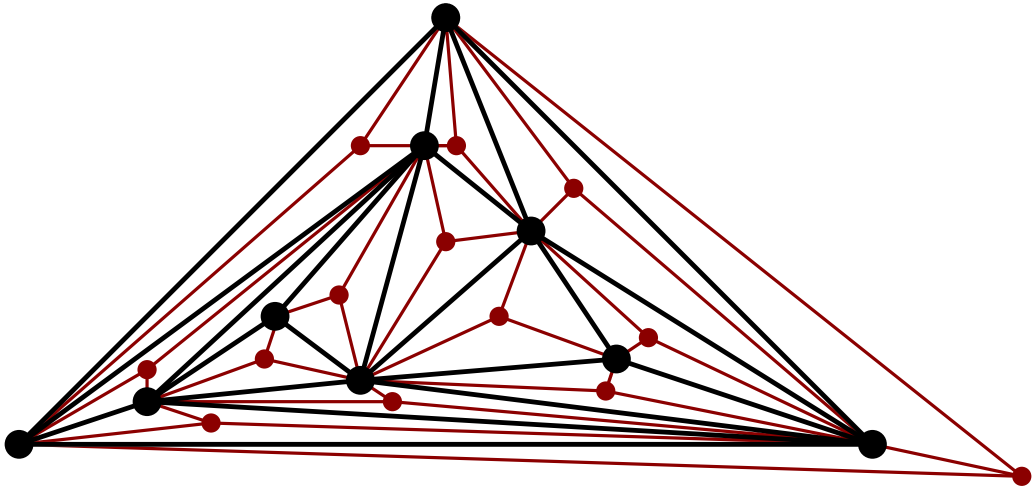

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  There is a 1-planar graph whose book thickness is (at least) 5.

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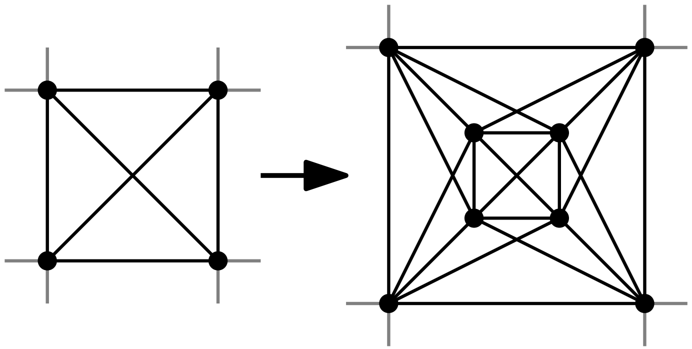

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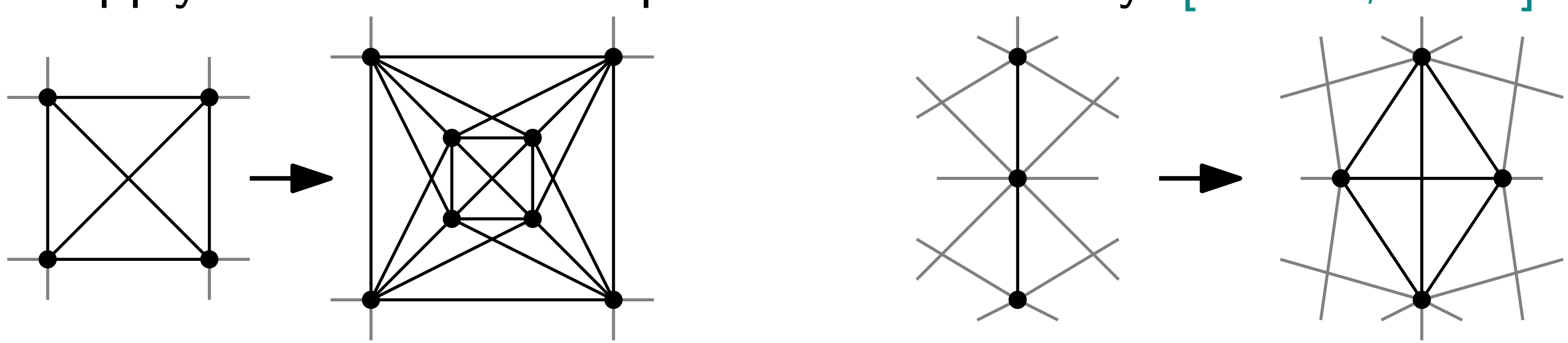

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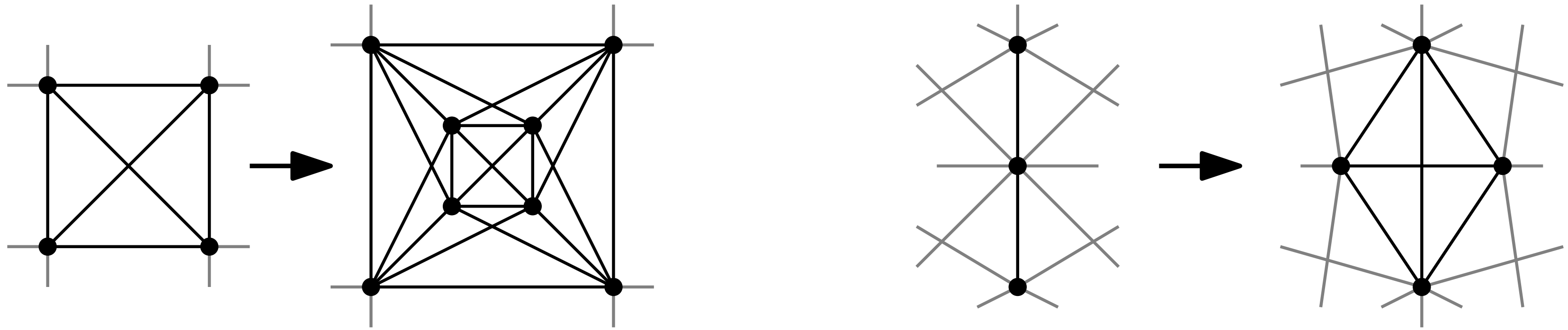

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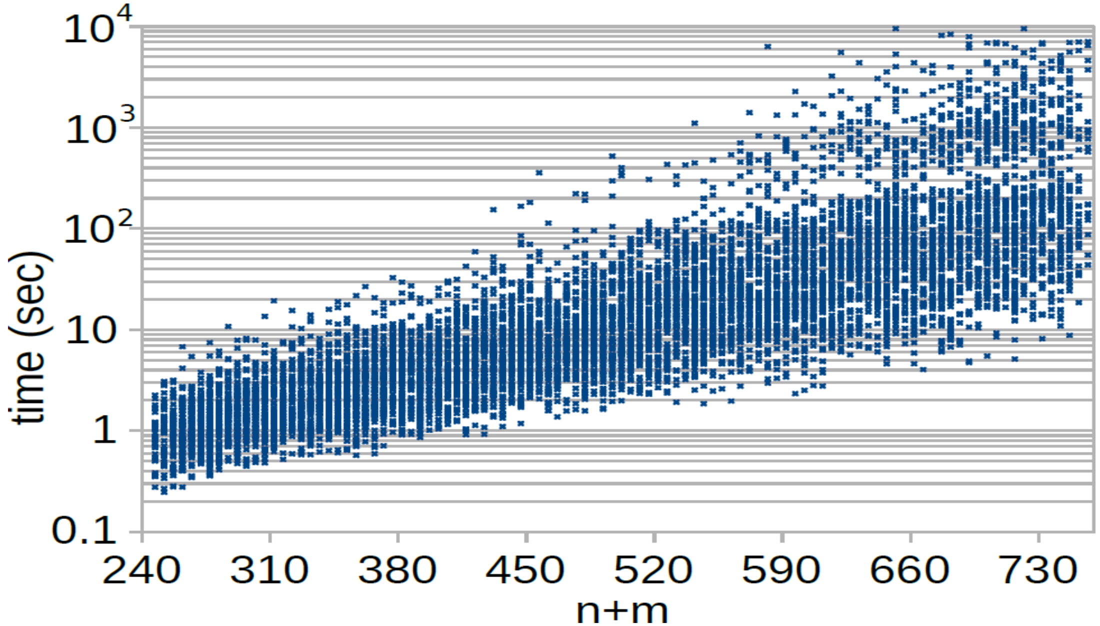
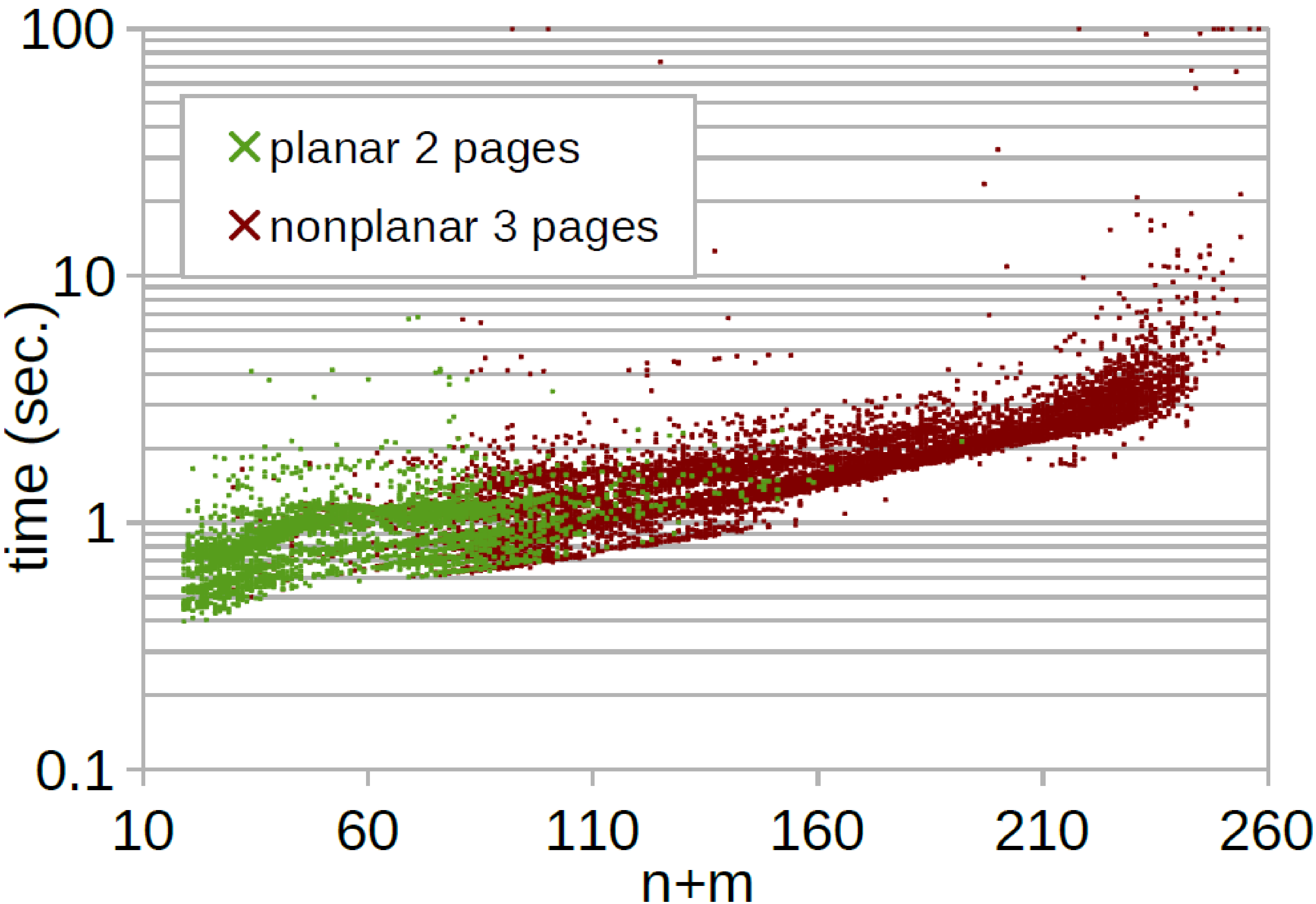


Runtime Rome



Runtime North