On Degree Properties of Crossing-critical Families of Graphs

Drago Bokal[1], Mojca Bračič[1], Marek Derňár[2], Petr Hliněný[2]

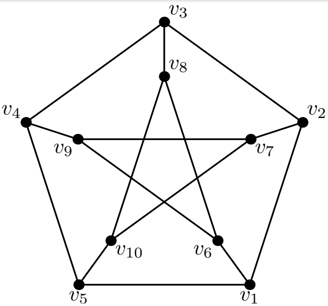
1 Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia,
2 Faculty of Informatics, Masaryk University, Brno, Czech Republic

September 24, 2015

Definition (drawing)

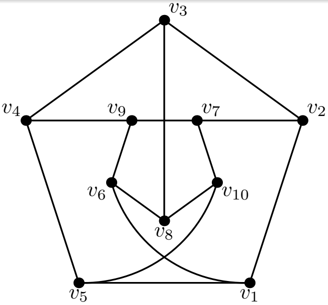
Drawing of a graph *G*:

- the vertices of *G* are distinct points, and every edge $e = uv \in E(G)$ is a simple curve joining *u* to *v*
- no edge passes through another vertex, and no three edges intersect in a common point



Definition (crossing number)

Crossing number cr(G) is the smallest number of edge crossings in a drawing of G.



Warning. There are slight variations of the definition of crossing number, some giving different numbers! (Like counting odd-crossing pairs of edges. [Pelsmajer, Schaeffer, Štefankovič, 2005]. . .)

Theorem (Kuratowski)

The graph G is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

Theorem (Kuratowski)

The graph G is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

Definition (crossing-critical graph)

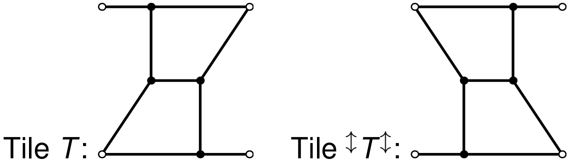
We say that a graph G is *k-crossing-critical*, if $\mathrm{cr}(G) \geq k$ but $\mathrm{cr}(G - e) < k$ for each edge $e \in E(G)$.

Definition (tile)

A tile is a triple $T = (G, \lambda, \rho)$ where $\lambda, \rho \subseteq V(G)$ are two disjoint sequences of distinct vertices of G, called the *left and right wall* of T, respectively.

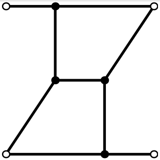
Definition (tile)

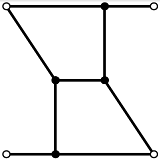
A tile is a triple $T = (G, \lambda, \rho)$ where $\lambda, \rho \subseteq V(G)$ are two disjoint sequences of distinct vertices of G, called the *left and right wall* of T, respectively.



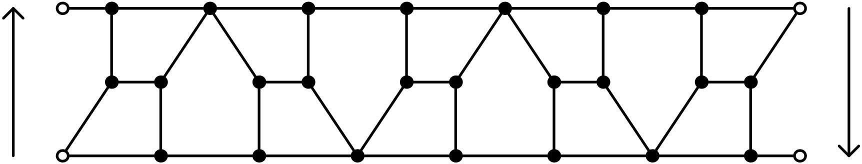
Definition (tile)

A tile is a triple $T = (G, \lambda, \rho)$ where $\lambda, \rho \subseteq V(G)$ are two disjoint sequences of distinct vertices of G, called the *left and right wall* of T, respectively.

Tile T: 

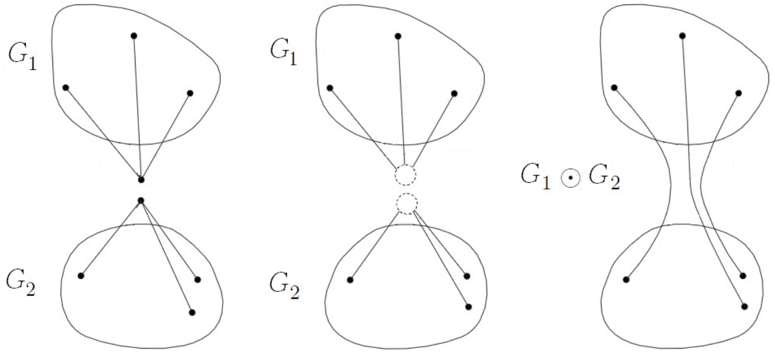
Tile ${}^{\updownarrow}T^{\updownarrow}$: 

Tile $\otimes\mathcal{T} = T \otimes {}^{\updownarrow}T^{\updownarrow} \otimes T \otimes {}^{\updownarrow}T^{\updownarrow} \otimes T$:

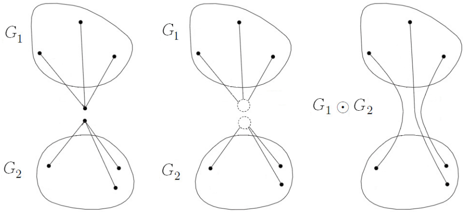


Kochol's construction: $\circ(\otimes\mathcal{T})^{\updownarrow}$

Zip product



Zip product



Theorem (Bokal)

Let G be a zip product of G_1 and G_2 according to degree-3 vertices. Then, $\text{cr}(G) = \text{cr}(G_1) + \text{cr}(G_2)$. Consequently, if G_i is k_i-crossing-critical for $i = 1, 2$, then G is $(k_1 + k_2)$-crossing-critical.

Motivation

- Average degree of infinite k-crossing-critical families in $(3, 6)$ (Salazar,...)

Open problem (Bokal, from 2007)

Is there any infinite k-crossing-critical families of graphs which contain (arbitrary many) vertices of any prescribed odd degrees, for sufficiently large k?

D-max-universal families

Definition (D-universal)

For a finite set $D \subseteq \mathbb{N}$, we say that a family of graphs $\mathcal{F}$ is *D-universal*, if and only if, for every integer m there exists a graph $G \in \mathcal{F}$, such that G has at least m vertices of degree d for each $d \in D$.

D-max-universal families

Definition (D-universal)

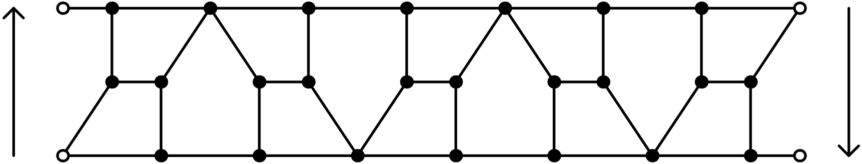
For a finite set $D \subseteq \mathbb{N}$, we say that a family of graphs $\mathcal{F}$ is *D-universal*, if and only if, for every integer m there exists a graph $G \in \mathcal{F}$, such that G has at least m vertices of degree d for each $d \in D$.

Definition (D-max-universal)

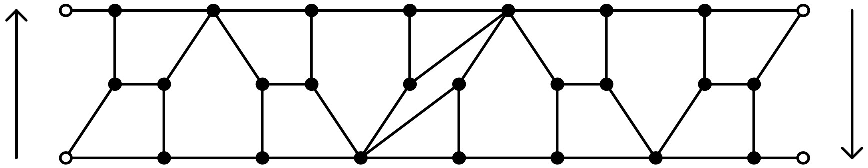
$\mathcal{F}$ is *D-max-universal*, if:

- it is D-universal
- there are only finitely many degrees appearing in graphs of $\mathcal{F}$ that are not in D
- there exists an integer M, such that any degree not in D appears at most M times in any graph of $\mathcal{F}$.

D-max-universal families

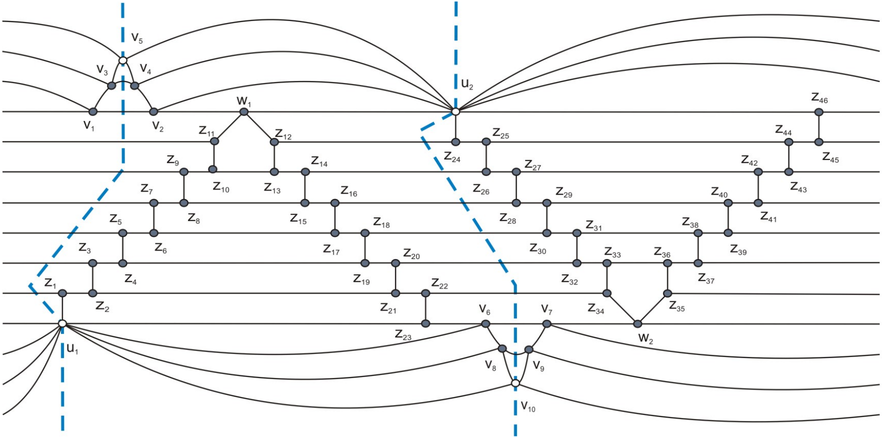


D-max-universal families

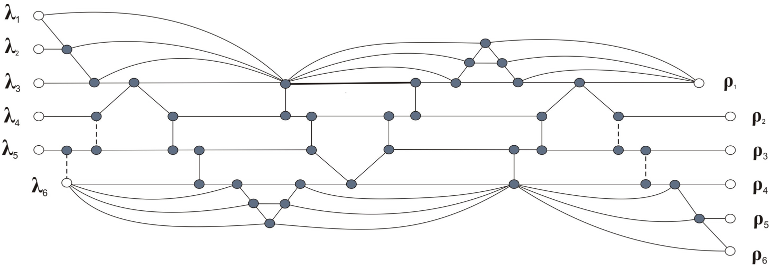


Main construction

Tile $H_{3,8}$:



Main construction



$$G_{\ell,n} = H_{\ell,n} \otimes {}^{\updownarrow}H_{\ell,n}{}^{\updownarrow} \otimes H_{\ell,n}$$
$$\mathcal{G}(\ell, n, m) = (G_{\ell,n}, {}^{\updownarrow}G_{\ell,n}{}^{\updownarrow}, G_{\ell,n} \ldots, {}^{\updownarrow}G_{\ell,n}{}^{\updownarrow}, G_{\ell,n})$$
$$G(\ell, n, m) = \circ\left(\mathcal{G}(\ell, n, m)^{\updownarrow}\right)$$

Main result

Theorem

Let $\ell \geq 1$, $n \geq 3$ be integers. Let $k = (\ell^2 + \binom{n}{2} - 1 + 2\ell(n-1))$ and $m \geq 4k - 1$ be odd. Then the graph $G(\ell, n, m)$ is k-crossing-critical.

Main result

Theorem

Let $\ell \geq 1$, $n \geq 3$ be integers. Let $k = (\ell^2 + \binom{n}{2} - 1 + 2\ell(n-1))$ and $m \geq 4k - 1$ be odd. Then the graph $G(\ell, n, m)$ is k-crossing-critical.

Theorem

Let D be any finite set of integers, $\min(D) \geq 3$. Then there is an integer $K = K(D)$, such that for every $k \geq K$, there exists a D-universal family of simple, 3-connected, k-crossing-critical graphs. Moreover, if either

- $3, 4 \in D$ or
- both $4 \in D$ and D contains only even numbers

then there exists a D-max-universal such family. All the vertex degrees are from $D \cup \{3, 4, 6\}$.

Theorem

Let D be any finite set of integers such that $\min(D) \geq 3$ and $A \subset \mathbb{R}$ an interval. Assume that at least one of the following assumptions holds:

a) $D \supseteq \{3, 4, 6\}$ and $A = (3, 6)$,

b) $D \supsetneq \{3, 4\}$ and $A = (3, 4]$, or $D = \{3, 4\}$ and $A = (3, 4)$,

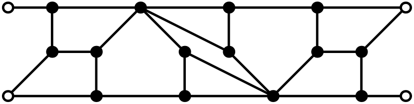
c) $D \supsetneq \{3, 4\}$ and $A = (3, 5 - \frac{8}{b+1})$, $b \geq 9$ is odd from D,

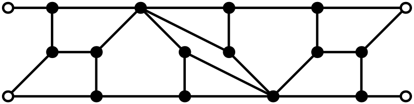
d) $D \supseteq \{4, 6\}$ has even n., $A = (4, 6)$, or $D = \{4\}$ and $A = \{4\}$.

Then, for every rational $r \in A \cap \mathbb{Q}$, there is an integer $K = K(D, r)$ such that for every $k \geq K$, there exists a D-max-universal family of simple, 3-connected, k-crossing-critical graphs of average degree precisely r.

Theorem

A simple, 3-connected 2-crossing-critical D-max-universal family exists if and only if $\{3\} \subsetneq D \subseteq \{3,4,5,6\}$. Without the simplicity requirement, such a family exists if and only if $D \subseteq \{3,4,5,6\}$, $|D| \geq 2$, and $D \cap \{3,4\} \neq \emptyset$.

$\{3,5\}$-max-universal family: 

$\{3,6\}$-max-universal family: 

Theorem

A simple, 3-connected, 2-crossing-critical infinite family of graphs with average degree $r \in \mathbb{Q}$ exists if and only if $r \in [3\frac{1}{5}, 4]$. Without the simplicity requirement, such a family exists if and only if $r \in [3\frac{1}{5}, 4\frac{2}{3}]$.

Finish

Thank you!