Augmenting Planar Straight Line Graphs to 2-Edge-Connectivity

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Definition. A graph is k-edge-connected if the removal of any $k-1$ edges does not disconnect the graph.

We consider edge-connectivity augmentation for planar straight line graphs (PSLGs) with n vertices in general position (no three collinear vertices).

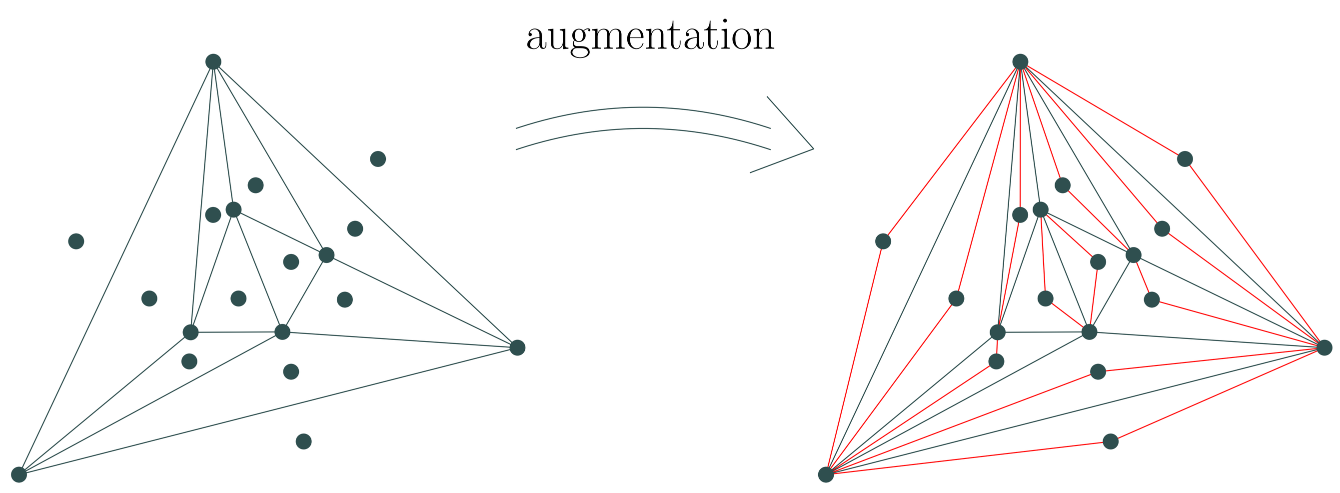
Question. What is the minimum number of new edges that can increase the edge connectivity of any PSLG with n vertices to two?

Theorem. *Every PSLG with $n \geq 3$ vertices can be augmented to a 2-edge-connected PSLG with the addition of at most $\lfloor (4n-4)/3 \rfloor$ new edges. This bound is the best possible.*

Table 1: The minimum number of new edges sufficient for raising the edge-connectivity of any PSLG on n vertices in general position to a target k = 1, 2, 3. Tight bounds and lower bounds. Adapted from [3].

Target edge-connectivity	1	2	3
Arbitrary PSLG	$n-1$	$\lfloor (4n-4)/3 \rfloor$	$2n-2$
Connected PSLG	—	$\lfloor (2n-2)/3 \rfloor$	$\geq \lfloor (4n-4)/3 \rfloor$
2-edge-connected PSLG	—	—	$n-2$

Lower Bound Construction



Graph on n=19 vertices. Each of $\lfloor (2n-2)/3 \rfloor = 12$ singletons requires two edges to increase the edge connectivity of the graph to two.

Upper Bound Proof

Let G be a planar straight line graph on $n \geq 3$ vertices in general position.
Let c be the number of connected components of G.

Case I: $c \leq \lfloor (2n+1)/3 \rfloor$

1. Add $(c-1)$ new edges to make the graph connected
2. Add $\lfloor (2n-2)/3 \rfloor$ edges to make the graph 2-edge-connected ([4], cf. Table 1)

$$(c-1) + \left\lfloor \frac{2n-2}{3} \right\rfloor \leq \left\lfloor \frac{2n+1}{3} \right\rfloor - 1 + \left\lfloor \frac{2n-2}{3} \right\rfloor \leq \left\lfloor \frac{4n-4}{3} \right\rfloor .$$

Case II: $c \geq \lfloor (2n+4)/3 \rfloor$

1. Construct a convex subdivision H of G (Fig. 1b)
2. Create a simple polygon within each cell of H (Fig. 1c and 2)
   [$c+h-1$ new edges]
3. Replace each bridge of the graph by a double edge (Fig. 1d)
   [b new edges; we obtain a 2-edge-connected multigraph]
4. Transform the multigraph into a simple graph as in [1]
   [the number of edges does not increase]

Total number of new edges: $c+h-1+b$.

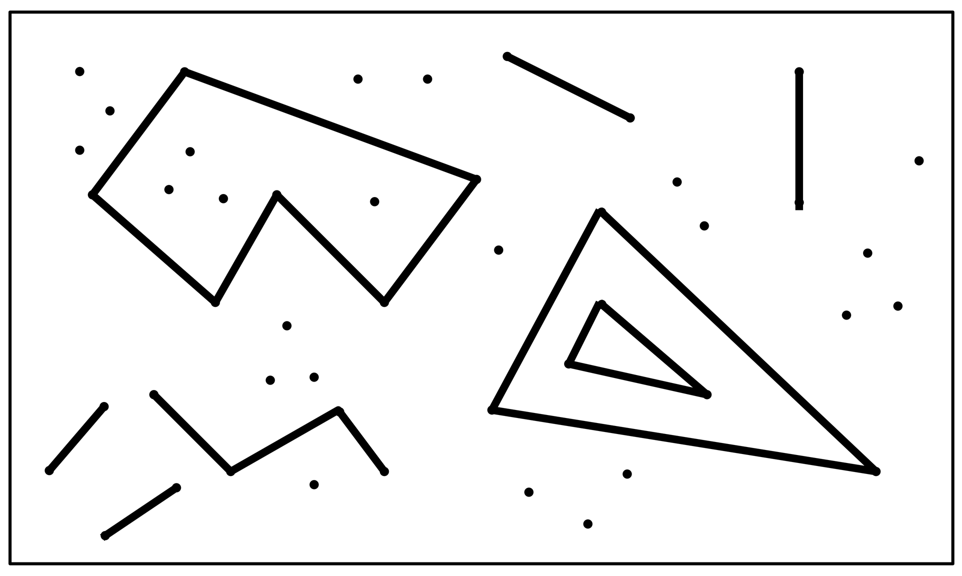
Lemma. *Let G be a PSLG with n vertices, b bridges, and c connected components. Then every convex subdivision of G has at most $h \leq 2n-2c-b+1$ cells.*

By Lemma , since $c \geq \lfloor (2n+4)/3 \rfloor$, the number of new edges is:

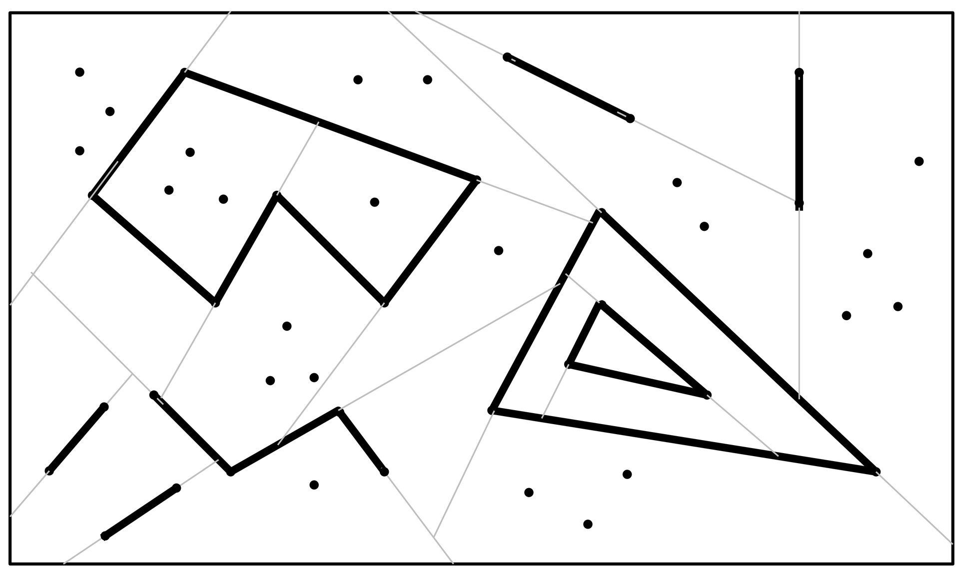
$$c+h-1+b \leq 2n-c \leq 2n - \left\lfloor \frac{2n+4}{3} \right\rfloor \leq \left\lfloor \frac{4n-4}{3} \right\rfloor .$$

Augmentation Algorithm

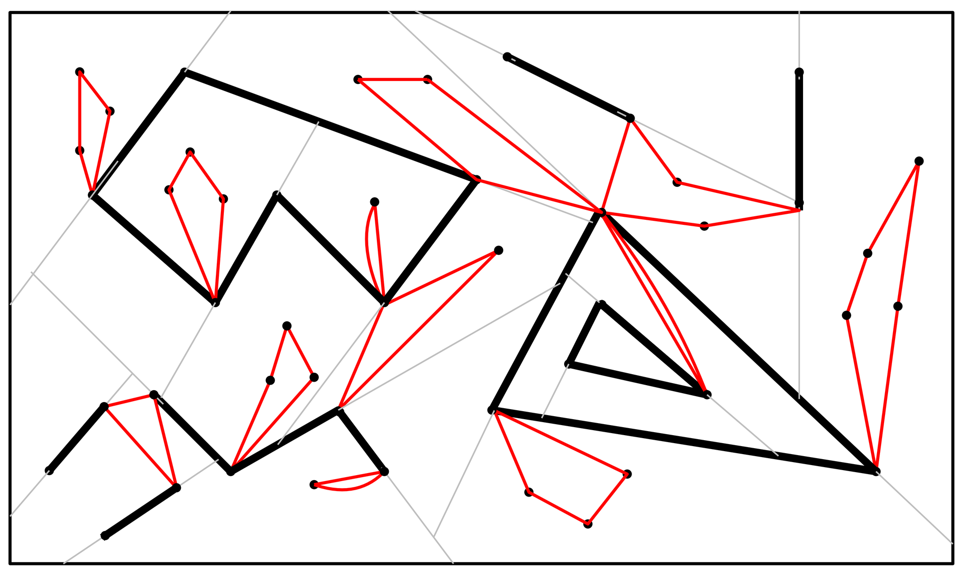
(a) We are given a graph G



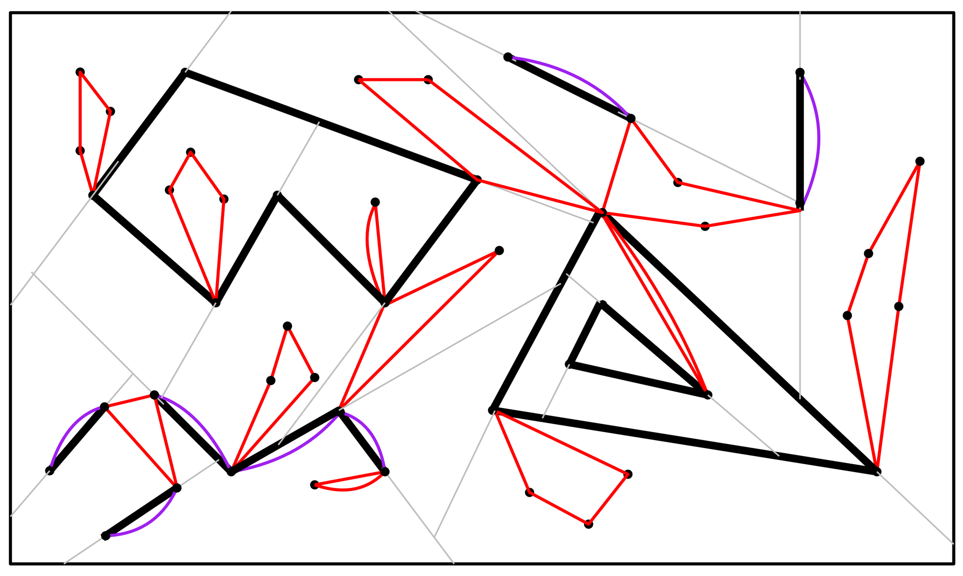
(b) Step 1. Create a convex subdivision H similar to [2]



(c) Step 2. Create a simple polygon in each cell of H



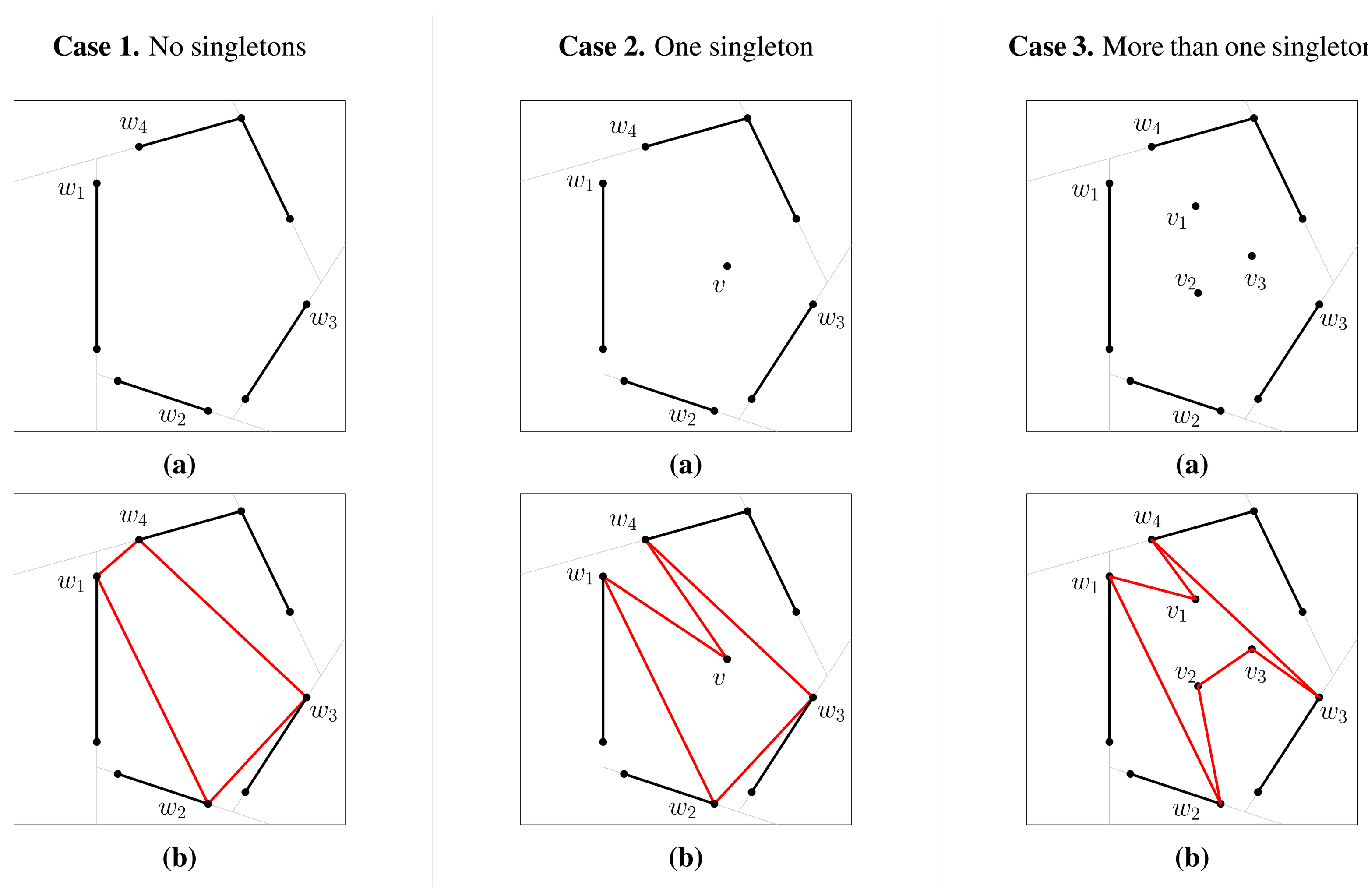
(d) Step 3. Create a double edge for each bridge



Steps 1–3 produce a 2-edge-connected *multigraph*. Step 4 transforms it into a 2-edge-connected *simple* PSLG by substituting or removing double edges (as in [1]).

Step 2. New edges in a single convex cell

Case 1. No singletons **Case 2.** One singleton **Case 3.** More than one singleton



We construct a simple polygon within each cell. No new bridge is created.

References

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