Three Ways to Draw a Graph

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joint work with

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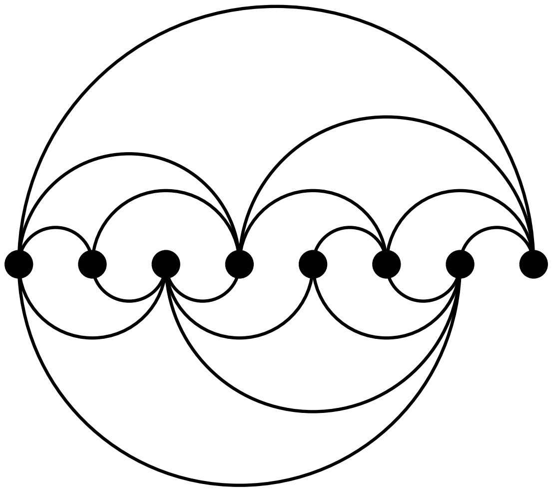
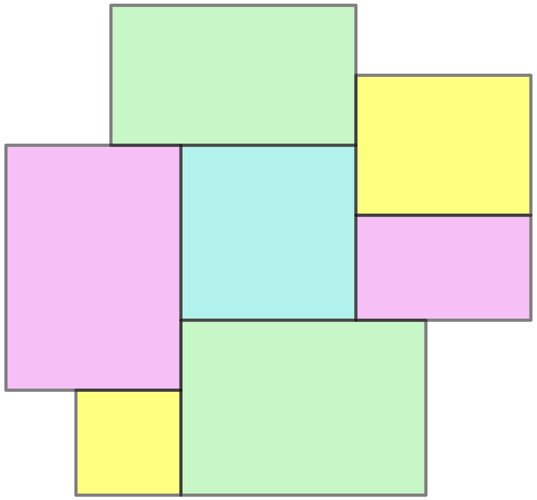
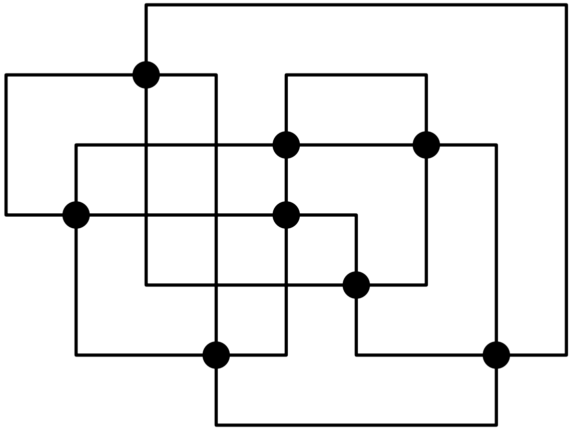
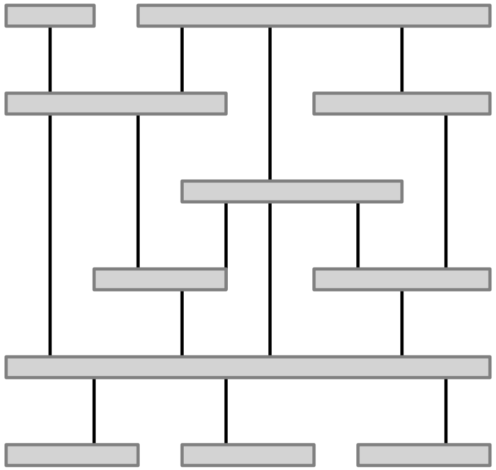
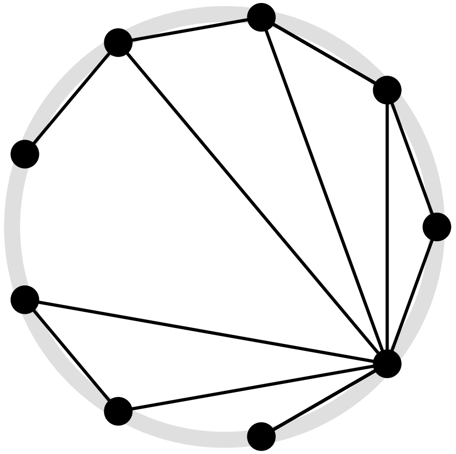
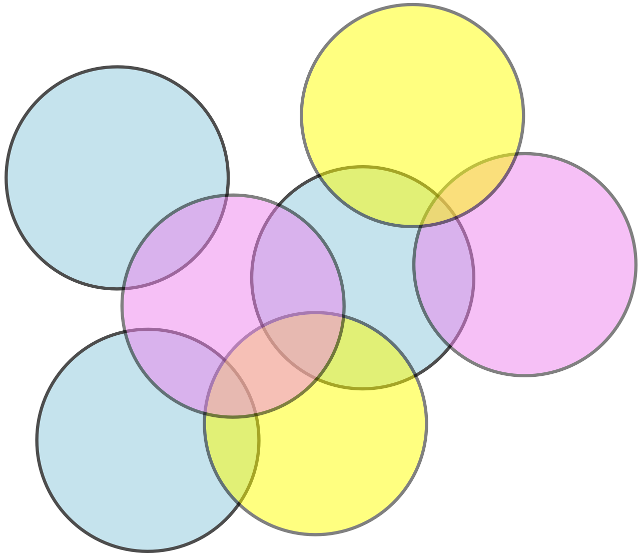
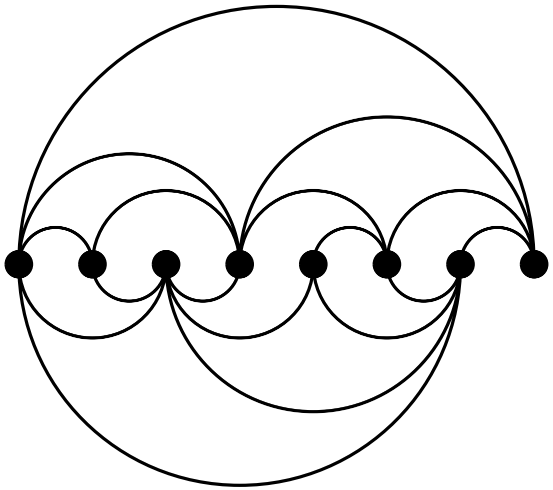
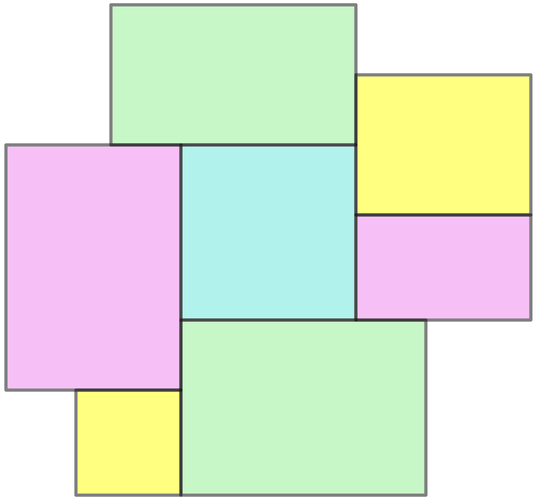
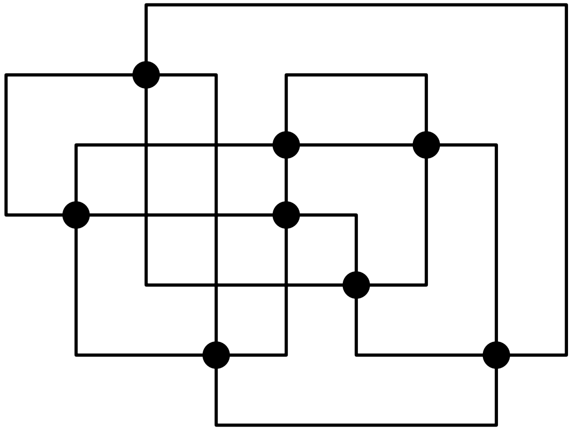
Peter Stumpf
Thomas Bläsius

September 22, 2015
Recent Trends in Graph Drawing
Curves, Graphs, and Intersections
CSUN, Los Angeles

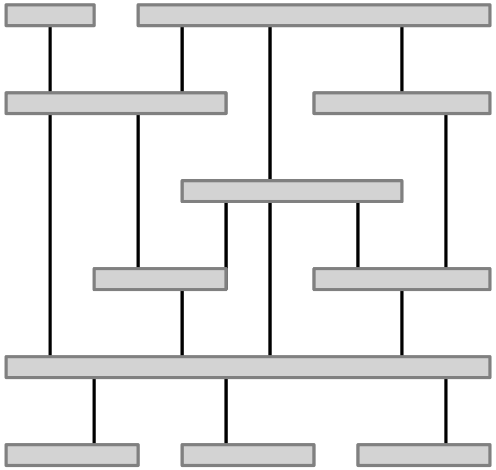
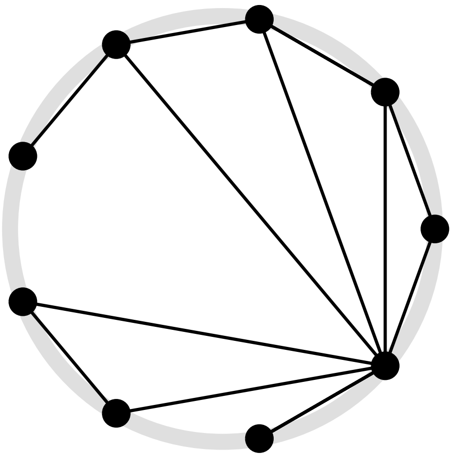
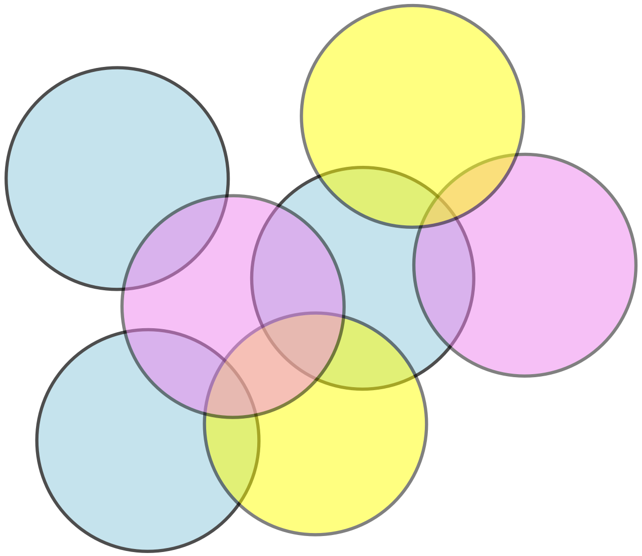
Your favorite way to draw a graph ...

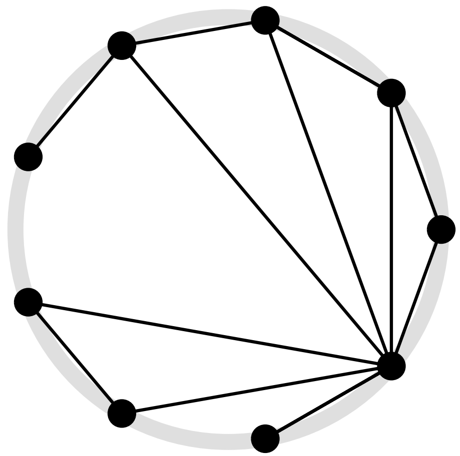




Your favorite way to draw a graph ...

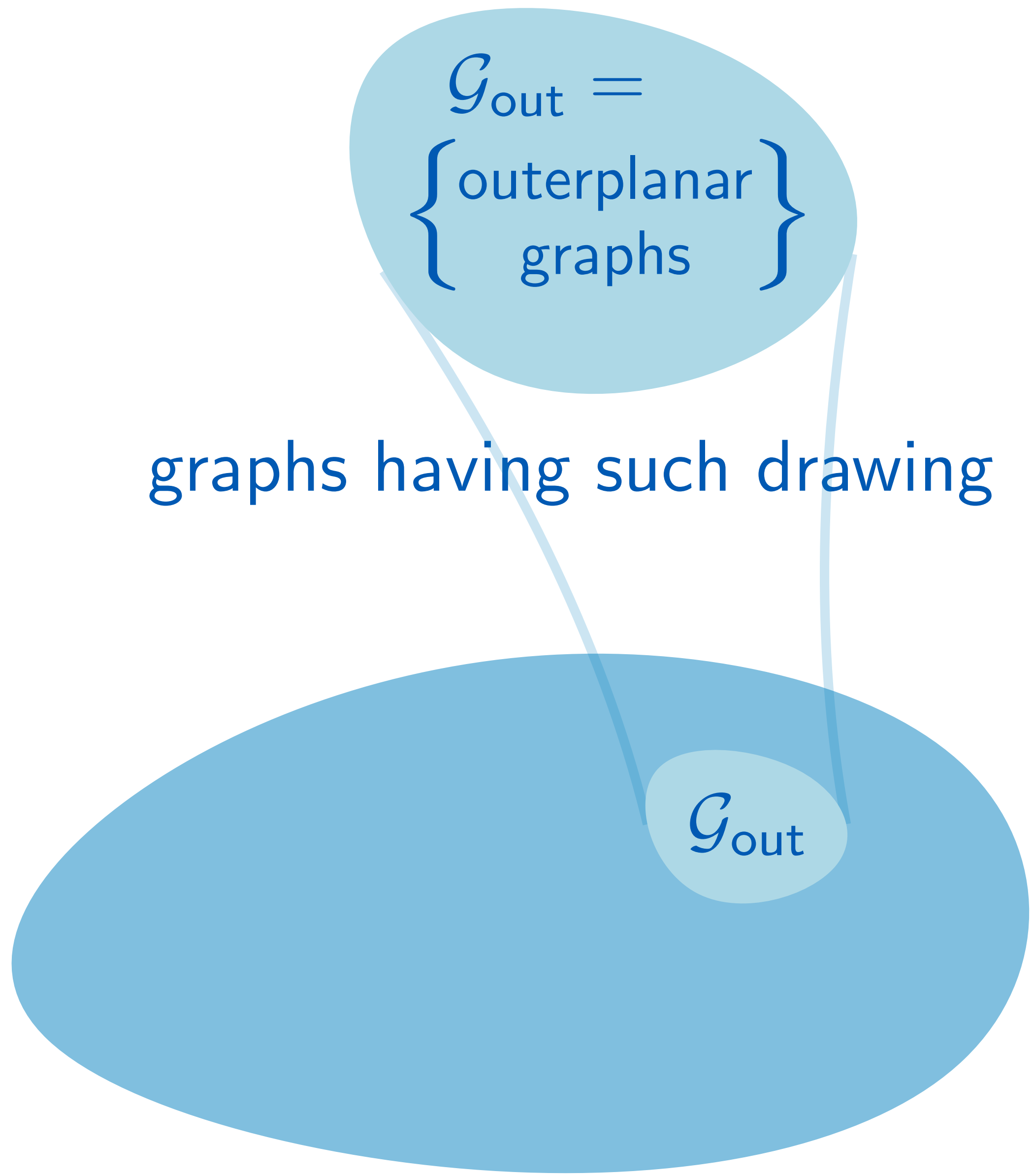
... but what if the graph has no such drawing?

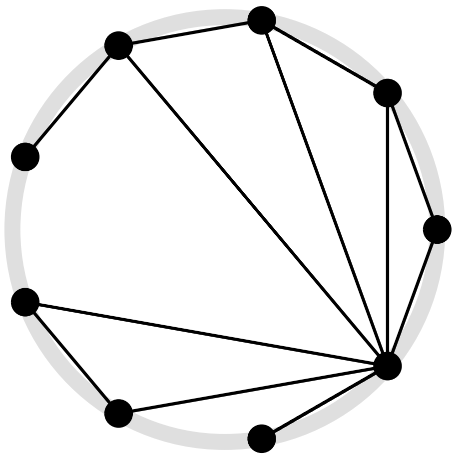


drawing style

- ▷ straight line
- ▷ crossing-free
- ▷ vertices on a circle

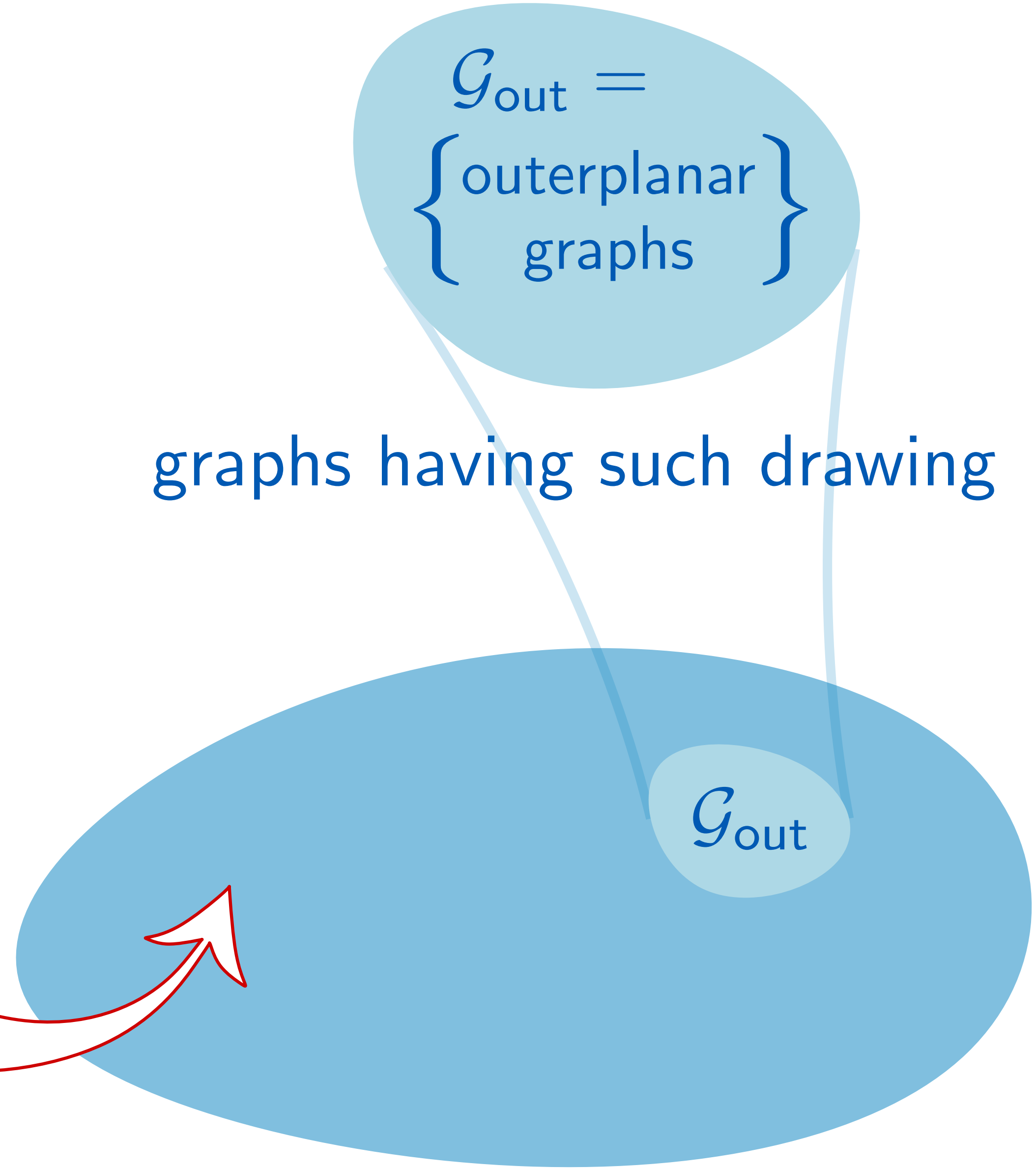


all graphs



drawing style

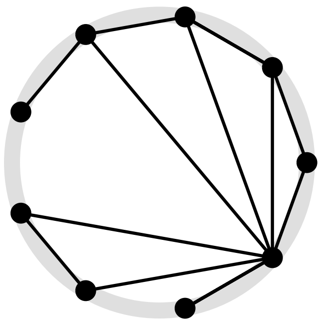
- ▷ straight line
- ▷ crossing-free
- ▷ vertices on a circle



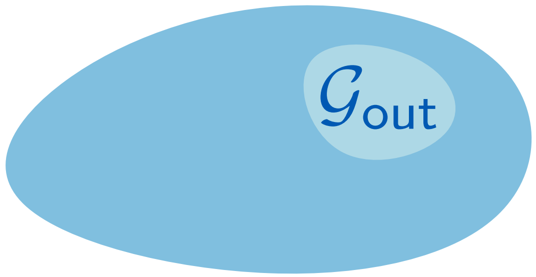
We want to draw these, too!

all graphs

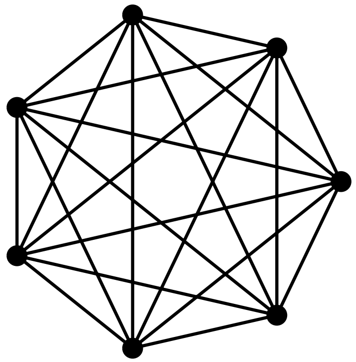
Problem



drawing style



$\mathcal{G} = \{\text{good graphs}\}$

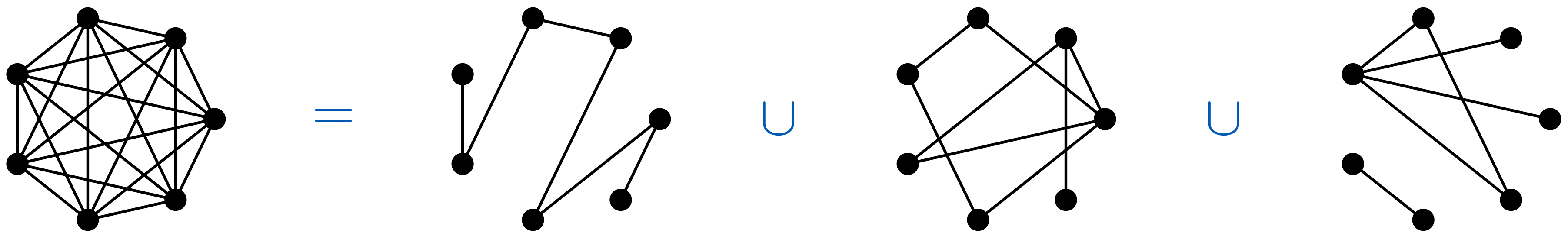


graph $H \notin \mathcal{G}$

Solution 1 | Solution 2 | Solution 3

Problem

drawing style $\mathcal{G} = \{\text{good graphs}\}$ graph $H \notin \mathcal{G}$



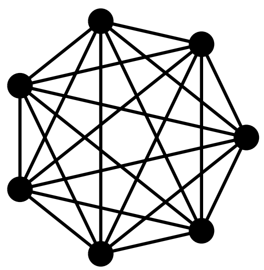
$H \notin \mathcal{G}$ $=$ $G_1 \in \mathcal{G}$ $\cup$ $G_2 \in \mathcal{G}$ $\cup$ $G_3 \in \mathcal{G}$

Split !

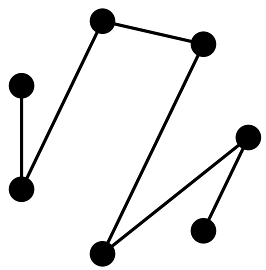
Solution 1

Problem

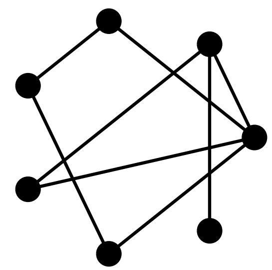
drawing style $\mathcal{G} = \{\text{good graphs}\}$ graph $H \notin \mathcal{G}$



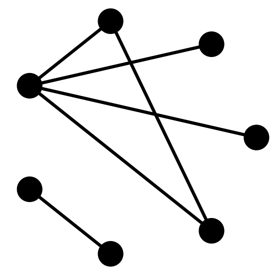
$=$



$\cup$

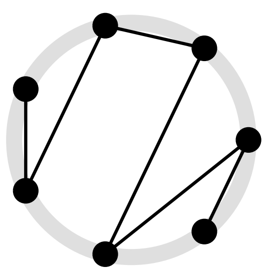


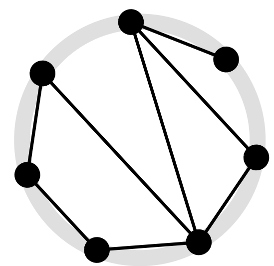
$\cup$

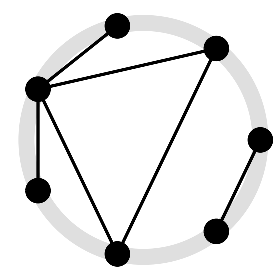


$H \notin \mathcal{G}$ $G_1 \in \mathcal{G}$ $G_2 \in \mathcal{G}$ $G_3 \in \mathcal{G}$

Split !



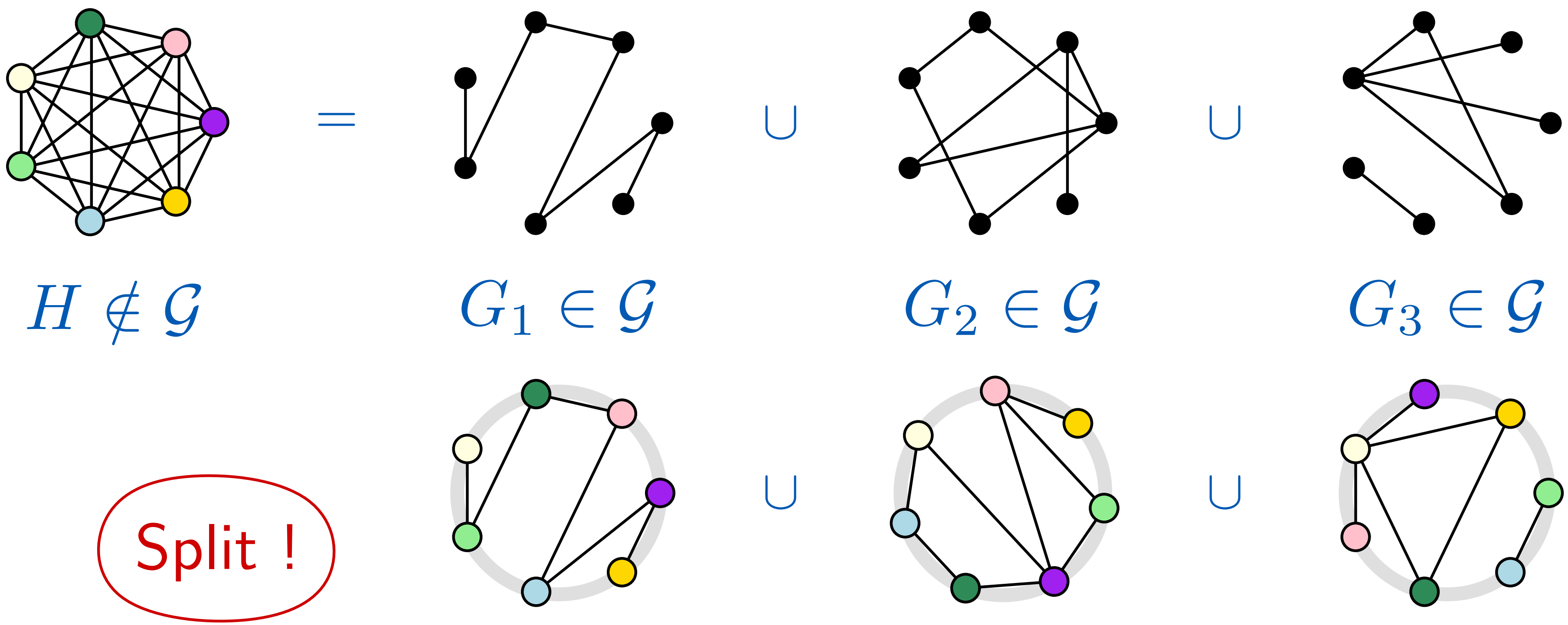




Solution 1

Problem

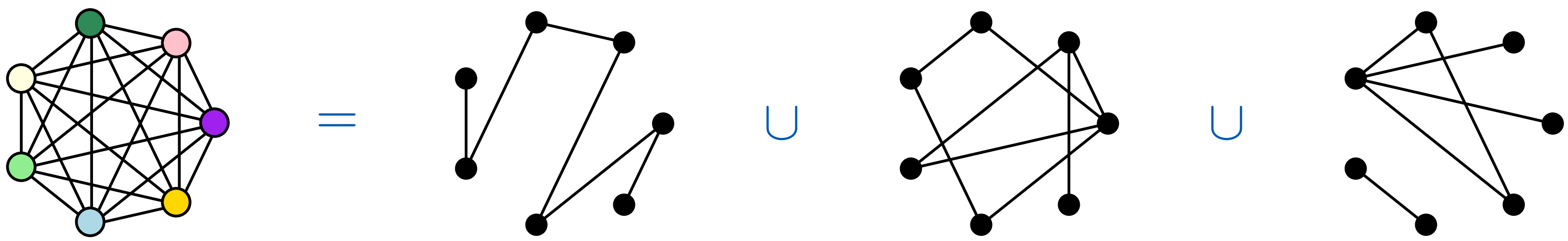
drawing style $\quad \mathcal{G} = \{\text{good graphs}\} \quad$ graph $H \notin \mathcal{G}$



Solution 1

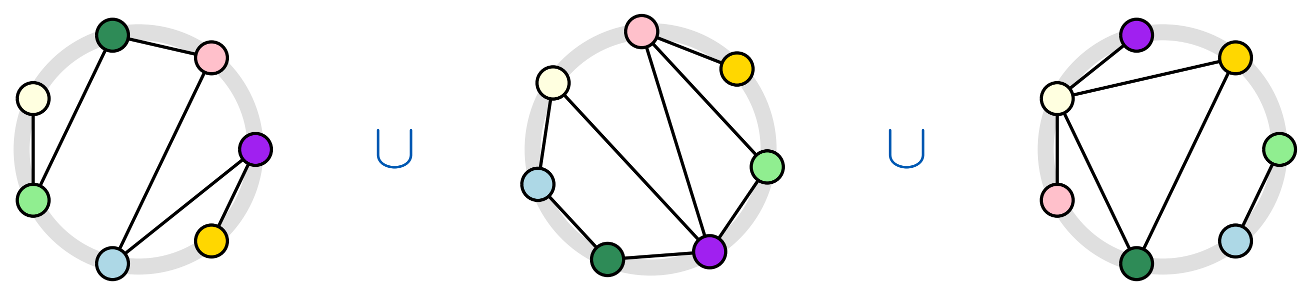
Problem

drawing style $\mathcal{G} = \{\text{good graphs}\}$ graph $H \notin \mathcal{G}$



$H \notin \mathcal{G}$ $G_1 \in \mathcal{G}$ $G_2 \in \mathcal{G}$ $G_3 \in \mathcal{G}$

Split !

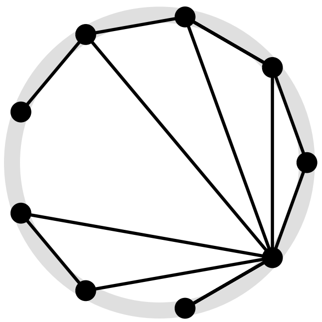


Cost = total # parts

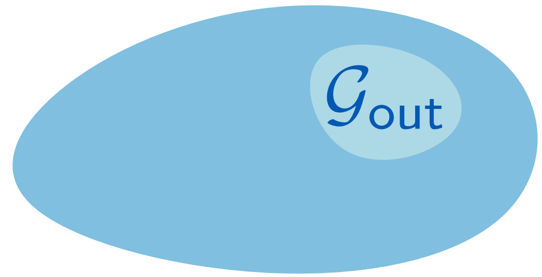
(denoted by $c_g^{\mathcal{G}}(H)$)

Solution 1

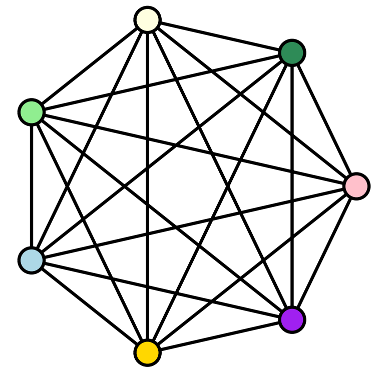
Problem



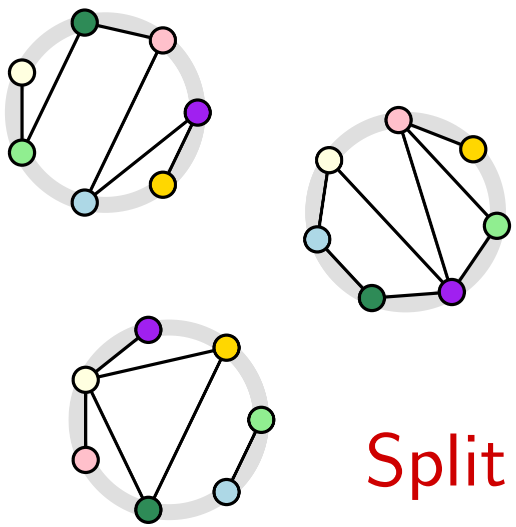
drawing style



$\mathcal{G} = \{\text{good graphs}\}$



graph $H \notin \mathcal{G}$

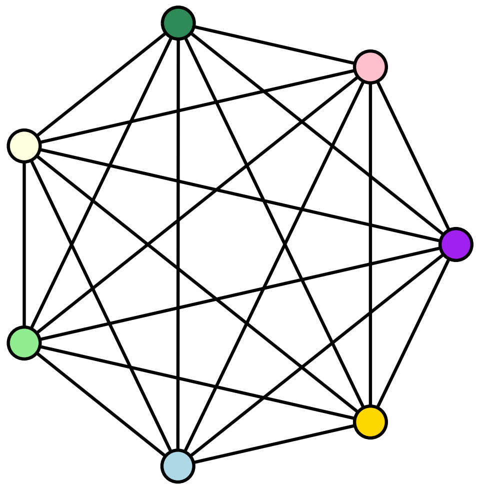


Split !

Solution 1 Solution 2 Solution 3

Problem

drawing style $\mathcal{G} = \{\text{good graphs}\}$ graph $H \notin \mathcal{G}$



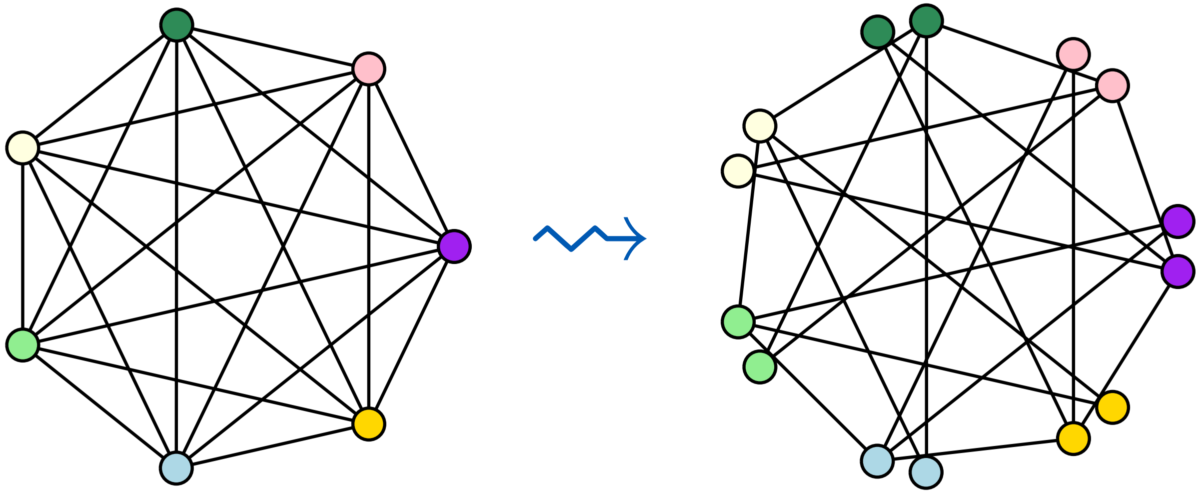
$H \notin \mathcal{G}$

Unfold !

Solution 3

Problem

drawing style $\mathcal{G} = \{\text{good graphs}\}$ graph $H \notin \mathcal{G}$



$H \notin \mathcal{G}$ $\rightsquigarrow$ $G \in \mathcal{G}$

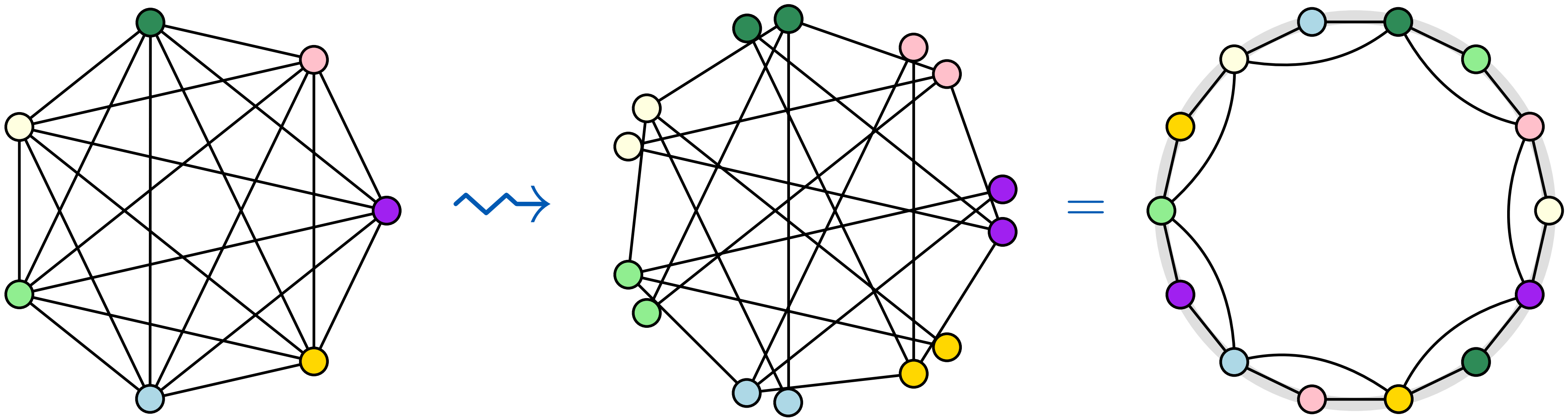
Unfold !

Solution 3

solution 3: unfold !

Problem

drawing style — $\mathcal{G} = \{\text{good graphs}\}$ — graph $H \notin \mathcal{G}$



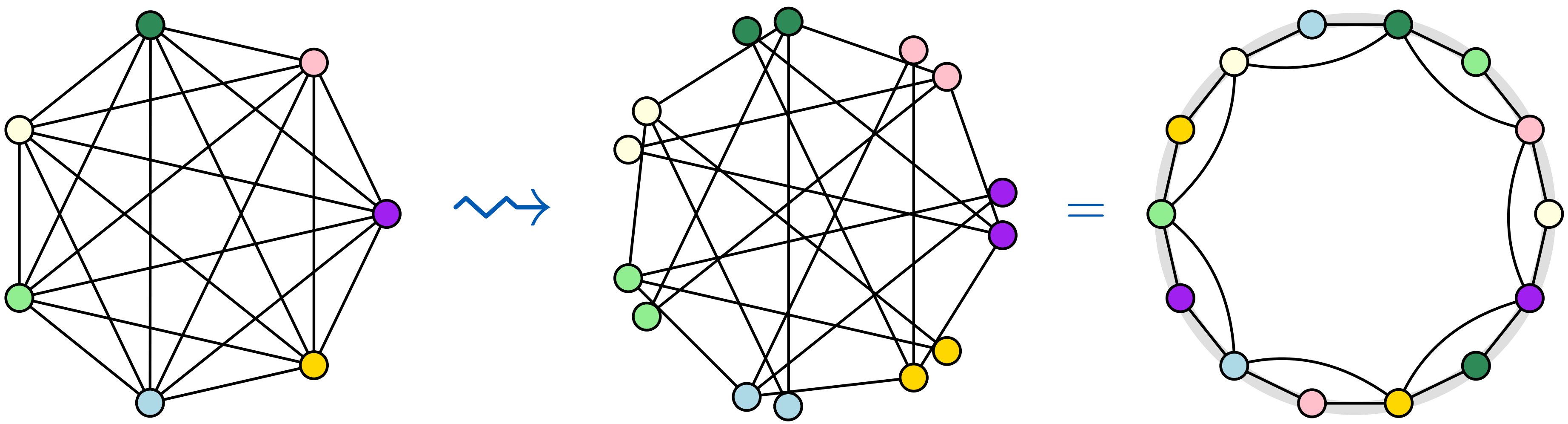
$H \notin \mathcal{G}$ $\qquad$ $G \in \mathcal{G}$

Unfold !

Solution 3

Problem

drawing style $\mathcal{G} = \{\text{good graphs}\}$ graph $H \notin \mathcal{G}$



$H \notin \mathcal{G}$ $G \in \mathcal{G}$

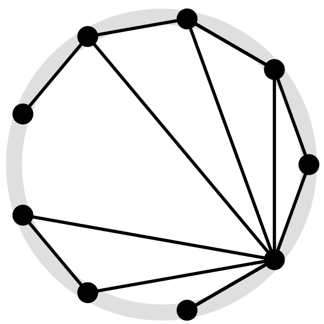
Unfold !

Cost = maximum # copies per vertex

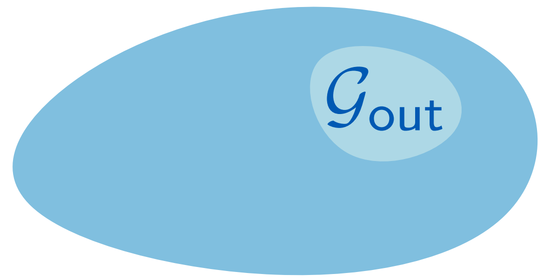
(denoted by $c_f^{\mathcal{G}}(H)$)

Solution 3

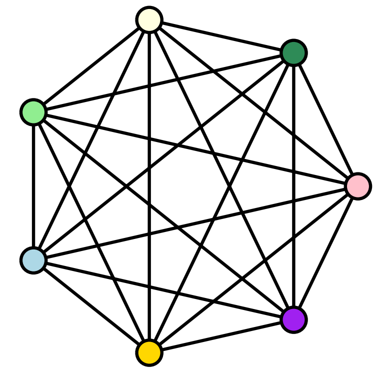
Problem



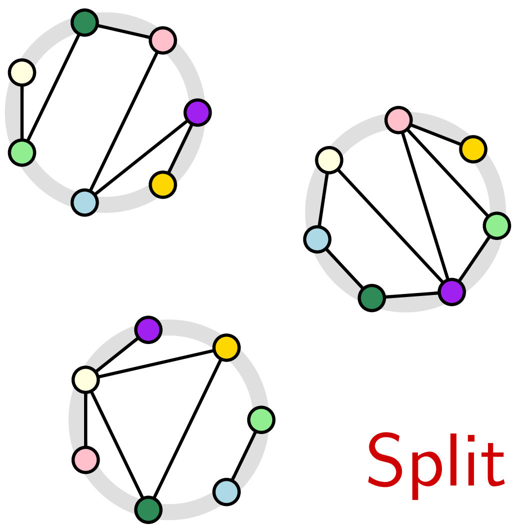
drawing style



$\mathcal{G} = \{\text{good graphs}\}$



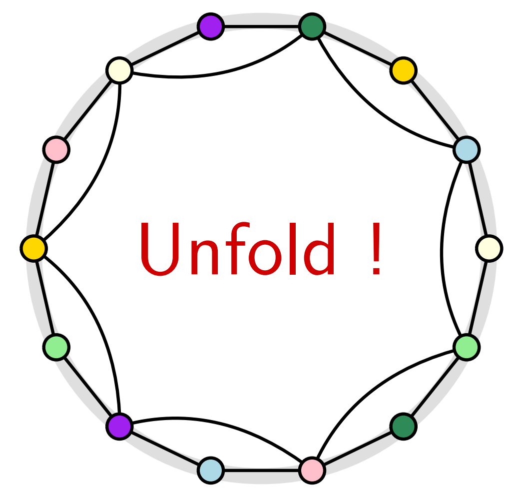
graph $H \notin \mathcal{G}$



Split !

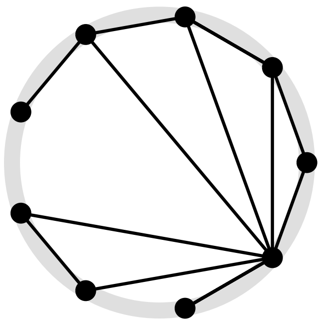
Solution 1

Solution 2

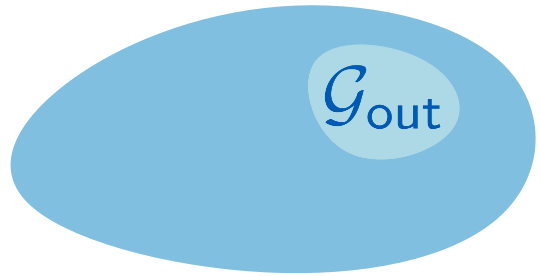


Solution 3

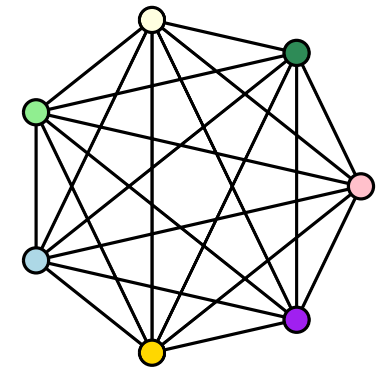
Problem



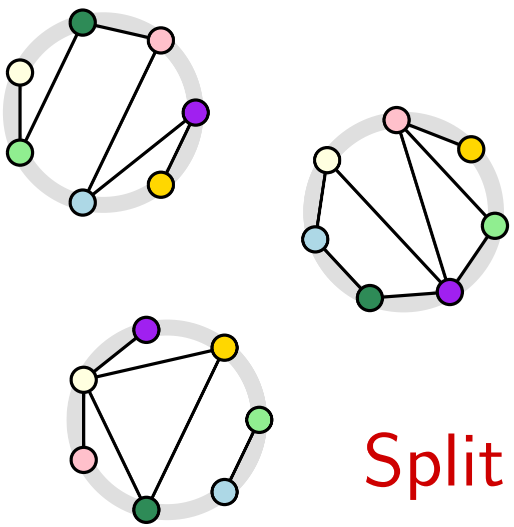
drawing style



$\mathcal{G} = \{\text{good graphs}\}$



graph $H \notin \mathcal{G}$

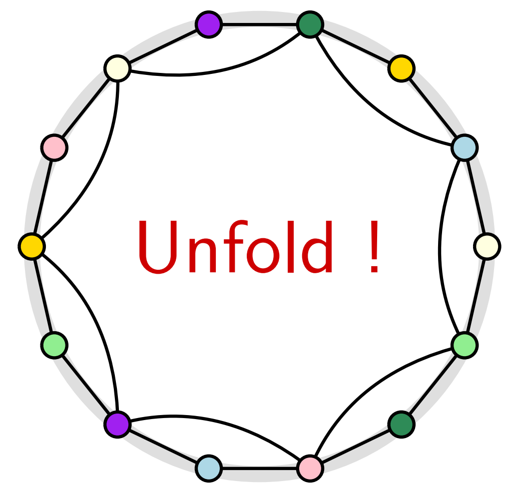


Split !

Solution 1

“interpolation”

Solution 2



Solution 3

Problem

drawing style $\quad \mathcal{G} = \{\text{good graphs}\} \quad$ graph $H \notin \mathcal{G}$

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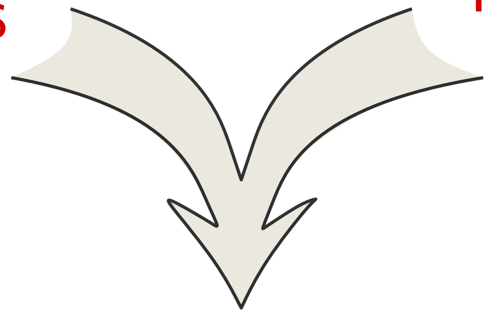
copies for same vertex
in distinct components
(as for solution 1)



few copies per vertex
(as for solution 3)

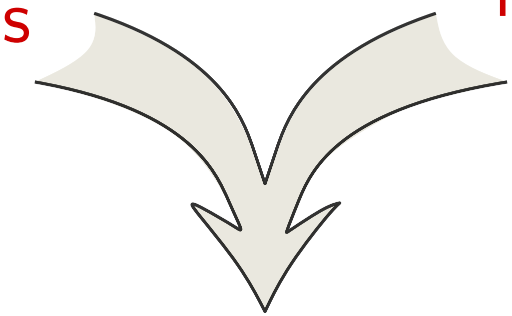
Problem

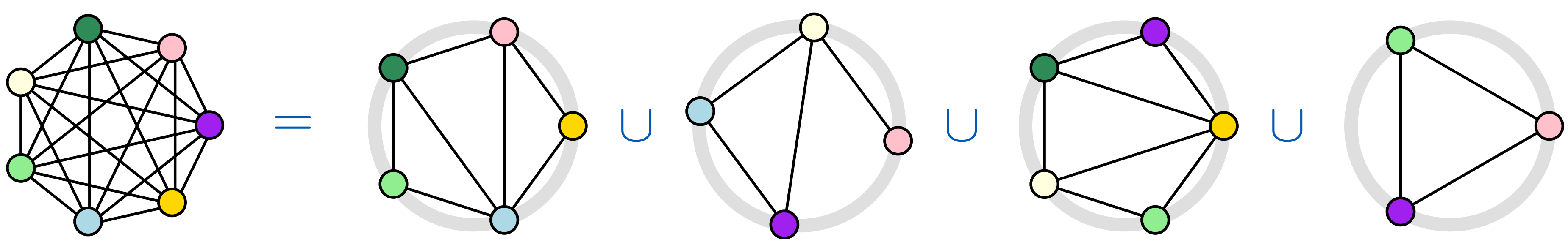
drawing style $\mathcal{G} = \{\text{good graphs}\}$ graph $H \notin \mathcal{G}$

---

copies for same vertex in distinct components (as for solution 1)

few copies per vertex (as for solution 3)





$H \notin \mathcal{G}$ $=$ $G_1 \in \mathcal{G}$ $\cup$ $G_2 \in \mathcal{G}$ $\cup$ $G_3 \in \mathcal{G}$ $\cup$ $G_4 \in \mathcal{G}$

solution 2: interpolation

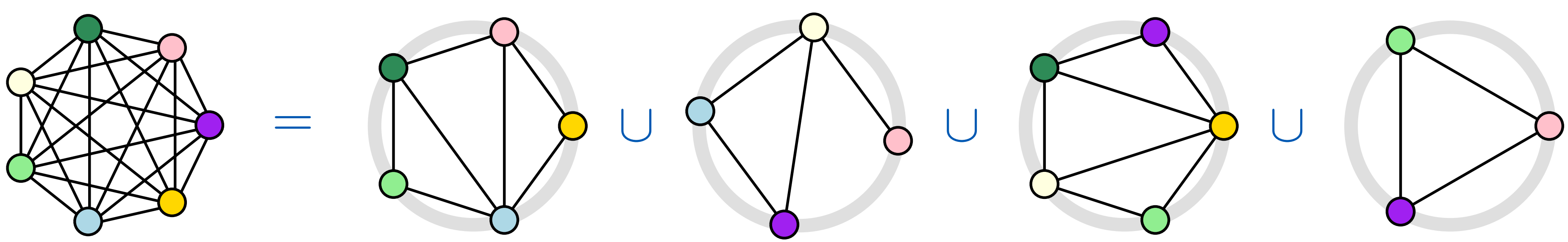
Problem

drawing style $\quad \mathcal{G} = \{\text{good graphs}\} \quad$ graph $H \notin \mathcal{G}$

---

copies for same vertex
in distinct components
(as for solution 1)

few copies per vertex
(as for solution 3)

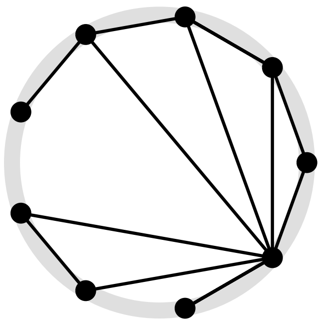


$H \notin \mathcal{G} \qquad G_1 \in \mathcal{G} \qquad G_2 \in \mathcal{G} \qquad G_3 \in \mathcal{G} \qquad G_4 \in \mathcal{G}$

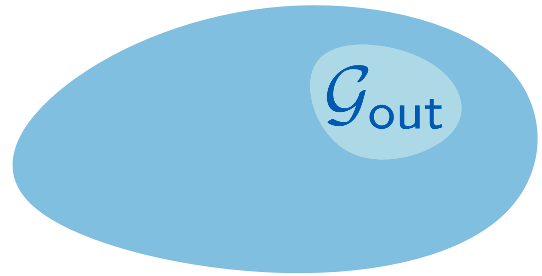
Cost $=$ maximum $\#$ copies per vertex

(denoted by $c_\ell^{\mathcal{G}}(H)$)

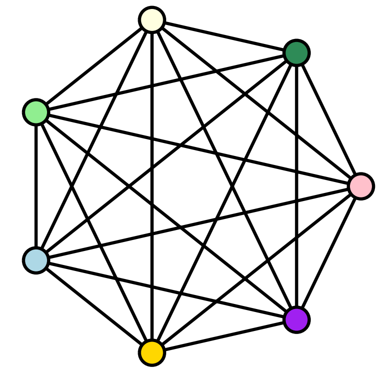
Problem



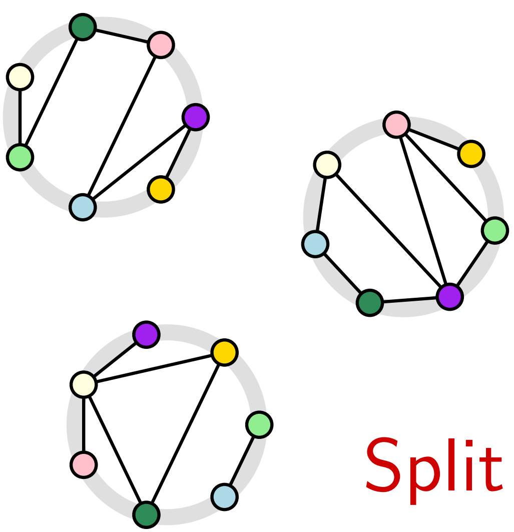
drawing style



$\mathcal{G} = \{\text{good graphs}\}$

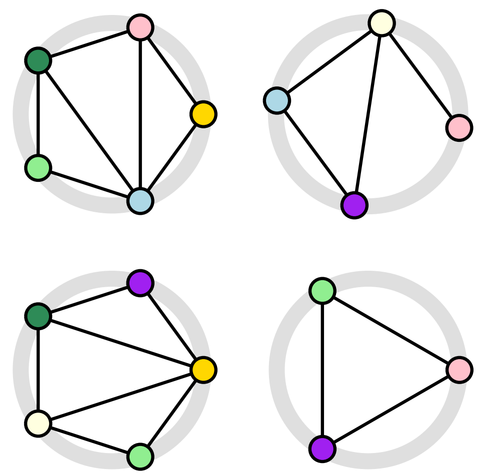


graph $H \notin \mathcal{G}$



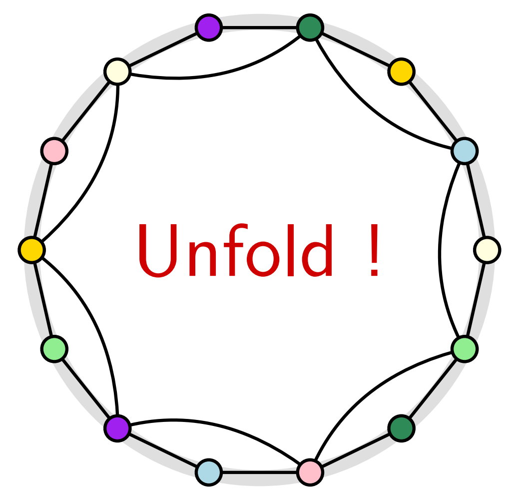
Split !

Solution 1



Interpolate !

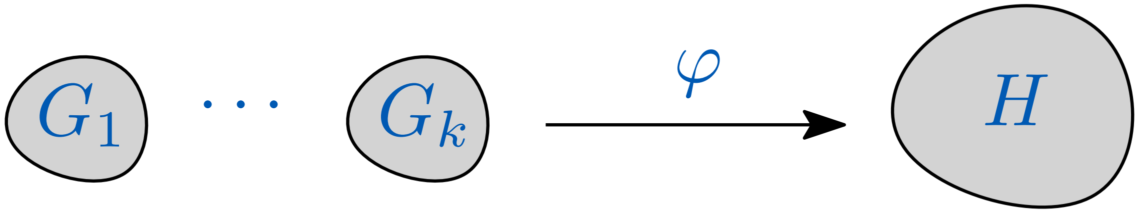
Solution 2



Solution 3

$\mathcal{G}$-cover of H

- edge-surjective homomorphism $\varphi : G_1 \dot\cup \cdots \dot\cup G_k \to H$ where $G_1, \ldots, G_k \in \mathcal{G}$

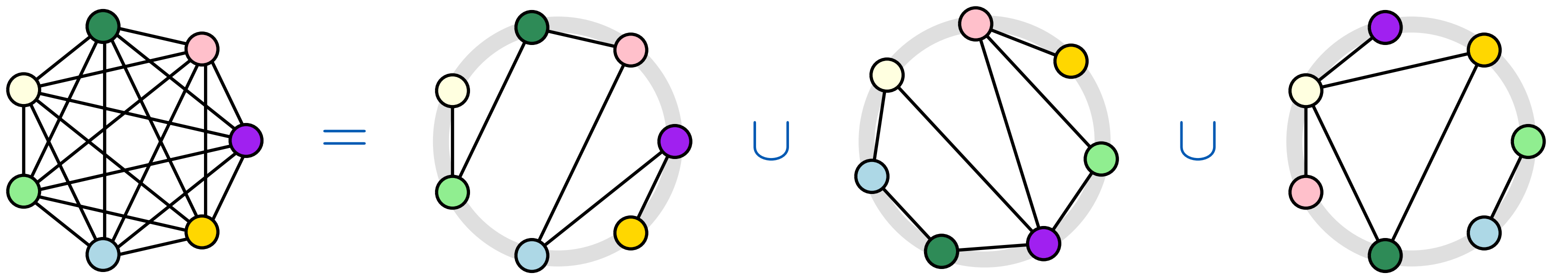


- size of φ is k (# good graphs used)
- φ injective if φ restricted to G_i injective for all i ($\varphi(G_i)$ is subgraph in H)

$\mathcal{G}$-cover of H

- global $\mathcal{G}$-covering number of H

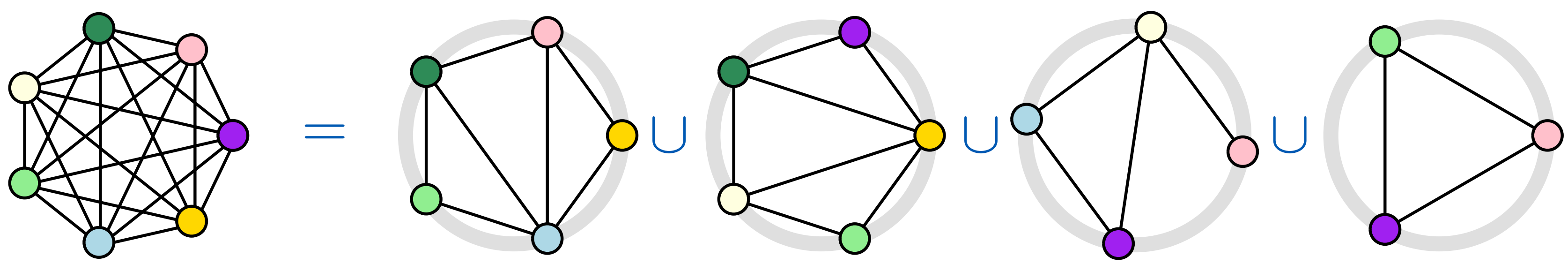
  $c_g^{\mathcal{G}}(H) = \min\{\text{size of } \varphi \mid \varphi \text{ injective } \mathcal{G}\text{-cover of } H\}$



"split H into few $\mathcal{G}$-graphs"

$\mathcal{G}$-cover of H

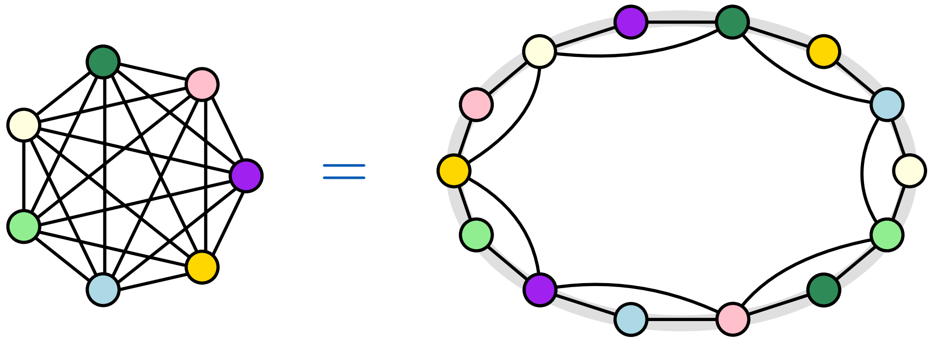
- global $\mathcal{G}$-covering number of H
  $c_g^{\mathcal{G}}(H) = \min\{\text{size of } \varphi \mid \varphi \text{ injective } \mathcal{G}\text{-cover of } H\}$
- local $\mathcal{G}$-covering number of H
  $c_\ell^{\mathcal{G}}(H) = \min\{\max |\varphi^{-1}(v)| \mid \varphi \text{ injective } \mathcal{G}\text{-cover of } H\}$



"each vertex in few $\mathcal{G}$-graphs"

$\mathcal{G}$-cover of H

- global $\mathcal{G}$-covering number of H
  $c_g^{\mathcal{G}}(H) = \min\{\text{size of } \varphi \mid \varphi \text{ injective } \mathcal{G}\text{-cover of } H\}$
- local $\mathcal{G}$-covering number of H
  $c_\ell^{\mathcal{G}}(H) = \min\{\max |\varphi^{-1}(v)| \mid \varphi \text{ injective } \mathcal{G}\text{-cover of } H\}$
- folded $\mathcal{G}$-covering number of H
  $c_f^{\mathcal{G}}(H) = \min\{\max |\varphi^{-1}(v)| \mid \varphi\ \mathcal{G}\text{-cover of } H\}$

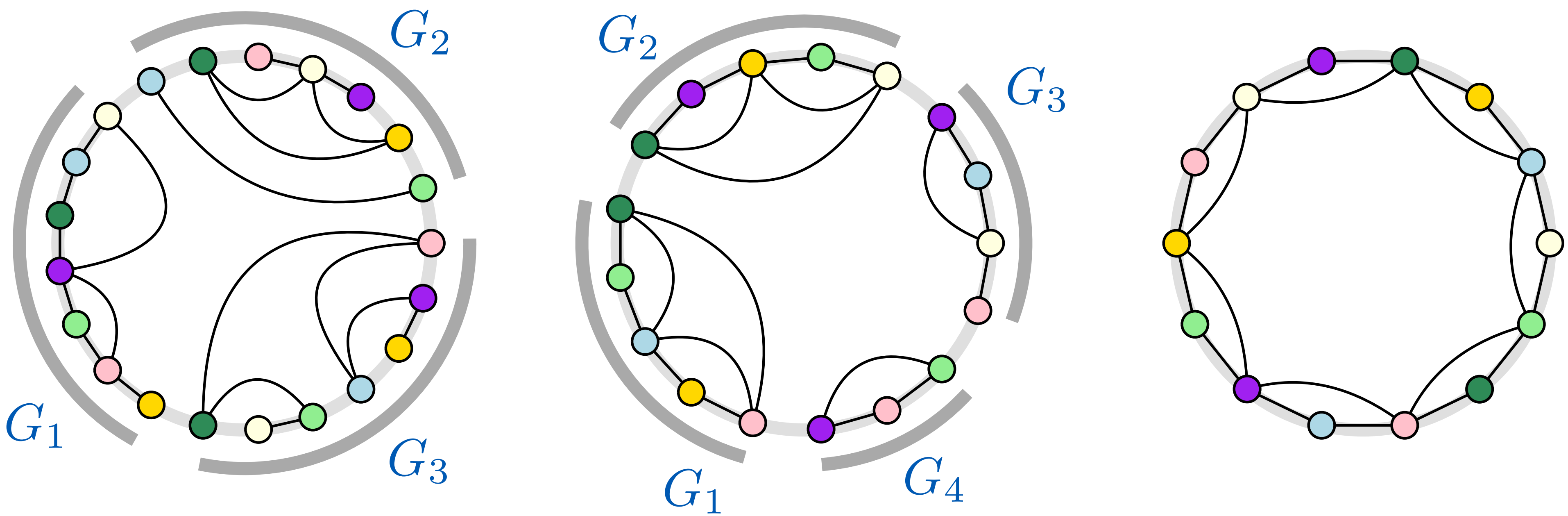


"few copies of each vertex"

Assumption: $\mathcal{G}$ = {good graphs} is closed under taking disjoint unions

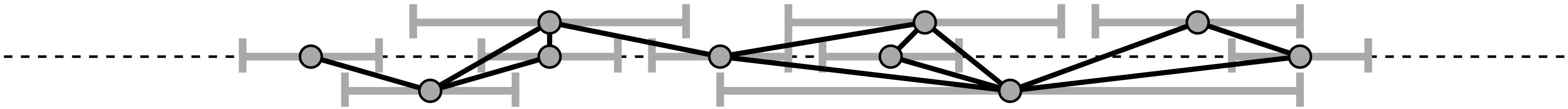
Lem. For all classes $\mathcal{G}$ and all graphs H we have

$$c_g^{\mathcal{G}}(H) \geq c_\ell^{\mathcal{G}}(H) \geq c_f^{\mathcal{G}}(H).$$



"global ≥ local ≥ folded"

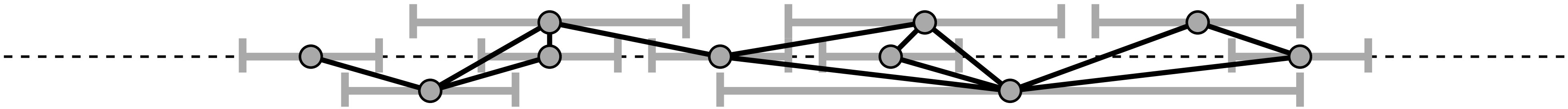
- drawing style: intersection graphs of intervals



- good graphs: $\mathcal{I} = \{$interval graphs$\}$

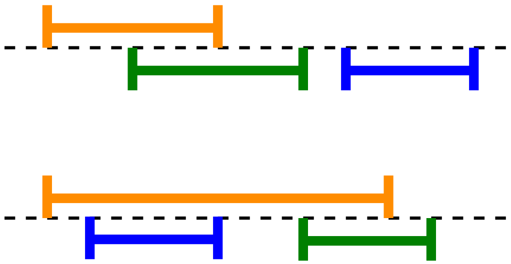
global	local	folded

- ▷ drawing style: intersection graphs of intervals



- ▷ good graphs: $\mathcal{I} = \{\text{interval graphs}\}$

global	local	folded
$c_g^{\mathcal{I}}(H)$ (track number)		



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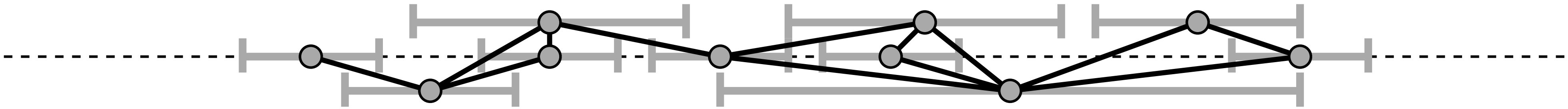
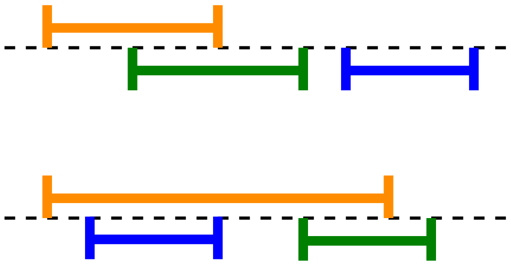
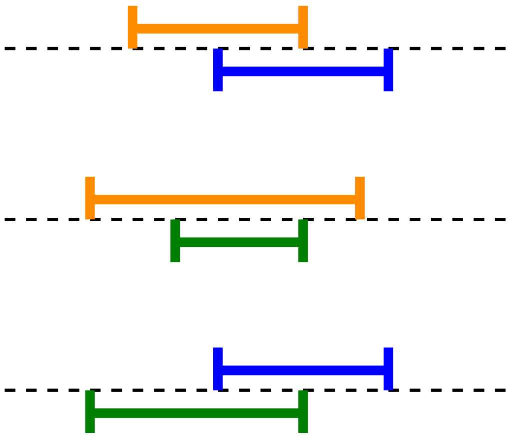


▷ good graphs: $\mathcal{I} = \{$interval graphs$\}$

global	local	folded
$c_g^{\mathcal{I}}(H)$ (track number)	$c_\ell^{\mathcal{I}}(H)$	





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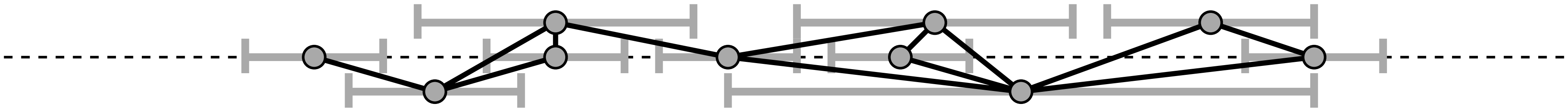

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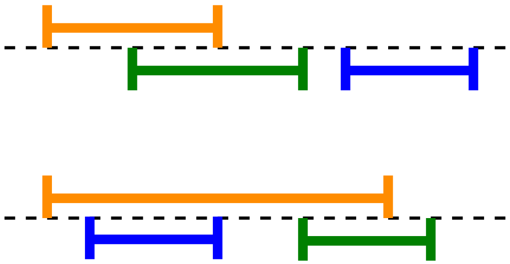
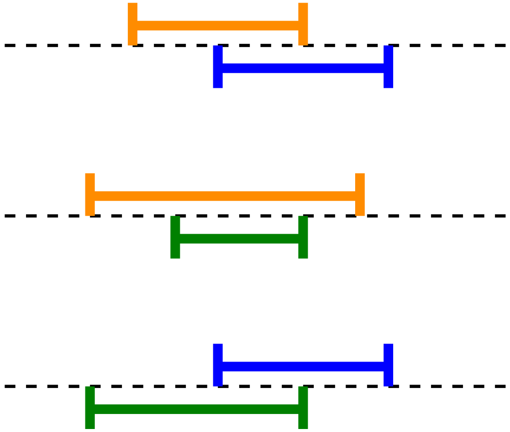
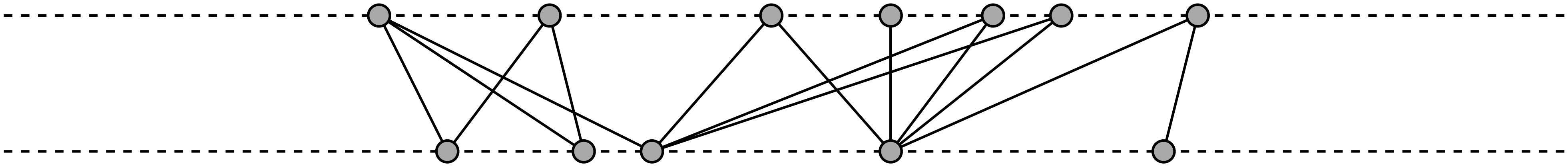
global	local	folded
$c_g^{\mathcal{I}}(H)$	$c_\ell^{\mathcal{I}}(H)$	$c_f^{\mathcal{I}}(H)$
(track number)		(interval number)





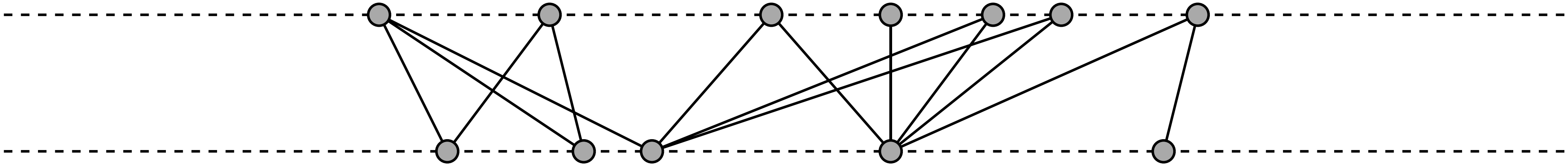
- drawing style: 2-layer drawings



- good graphs: $\mathcal{B} = \{$bipartite graphs$\}$

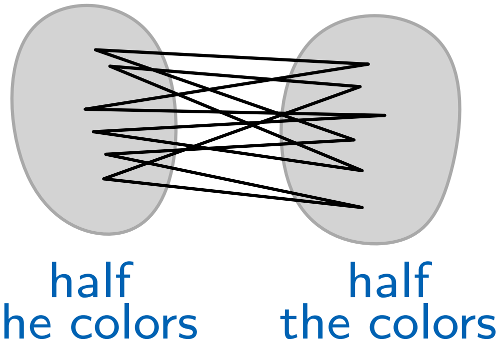
global	local	folded

- drawing style: 2-layer drawings



- good graphs: $\mathcal{B} = \{$bipartite graphs$\}$

global	local	folded
$c_g^{\mathcal{B}}(H)$		
$= \lceil \log(\chi(H)) \rceil$		



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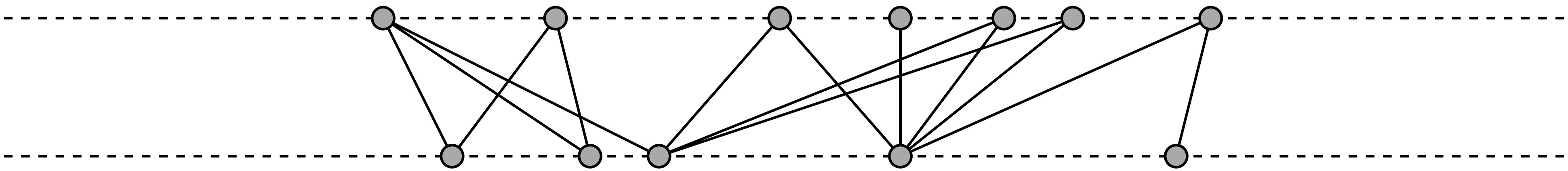

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$c_g^{\mathcal{B}}(H)$	$c_\ell^{\mathcal{B}}(H)$	

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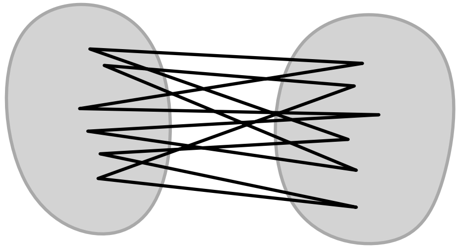
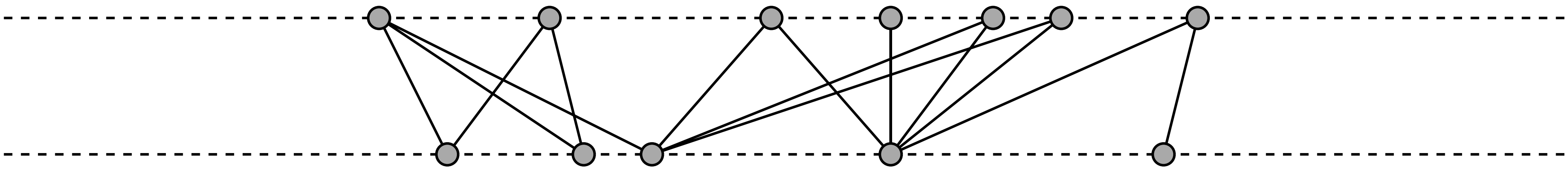


half the colors

half the colors

Open: $= c_g^{\mathcal{B}}(H)$?

- drawing style: 2-layer drawings



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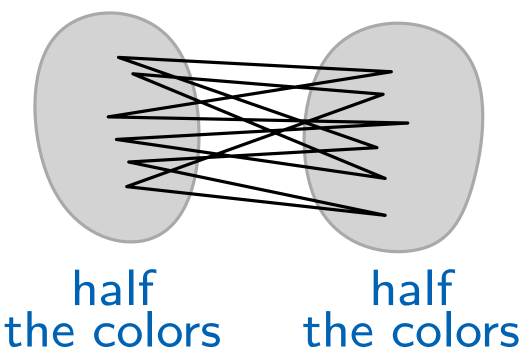
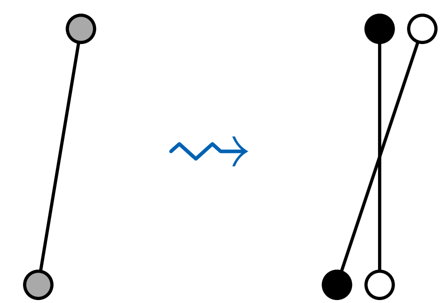
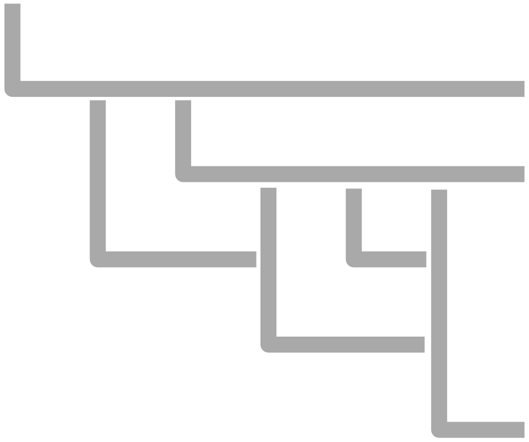
global	local	folded
$c_g^{\mathcal{B}}(H)$	$c_\ell^{\mathcal{B}}(H)$	$c_f^{\mathcal{B}}(H)$
$= \lceil \log(\chi(H)) \rceil$	**Open:** $= c_g^{\mathcal{B}}(H)$?	≤ 2

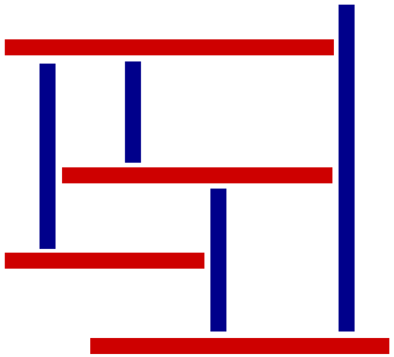




- ▷ drawing style: contact graphs of **L**-shapes
- ▷ good graphs: $\mathcal{L} = \{$**L**-graphs$\}$



- ▷ drawing style: contact graphs of **2-DIR**-segments
- ▷ good graphs: {planar bipartite graphs}



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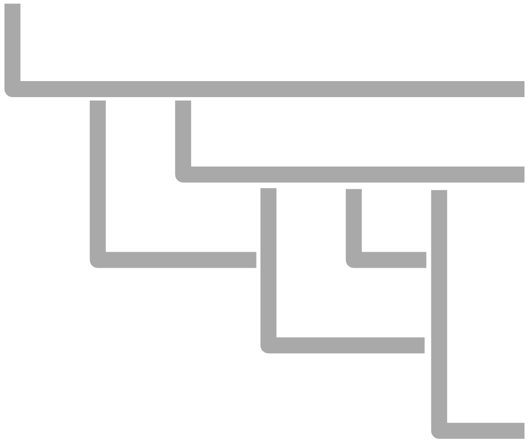

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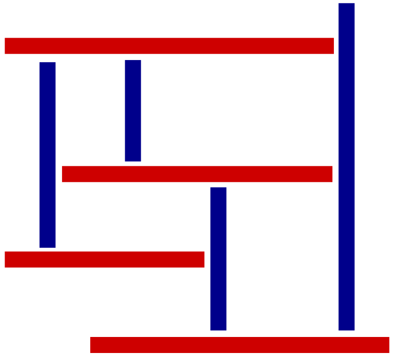

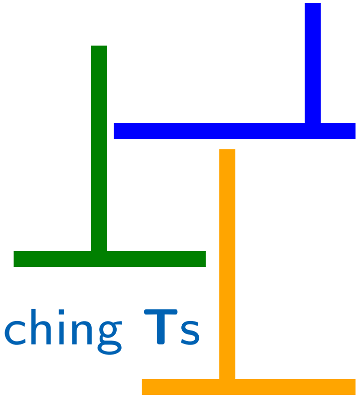

touching **T**s

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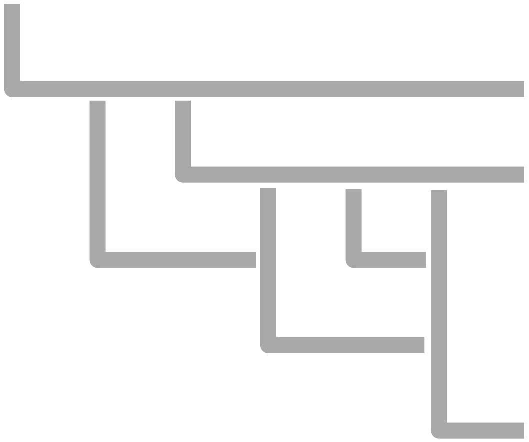
- ▷ good graphs: $\mathcal{L} = \{$**L**-graphs$\}$



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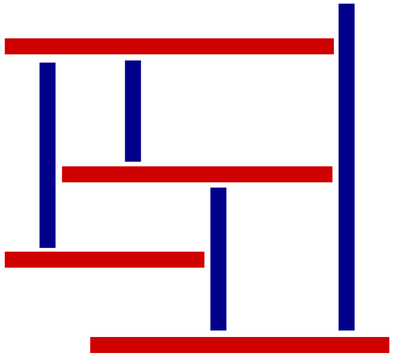
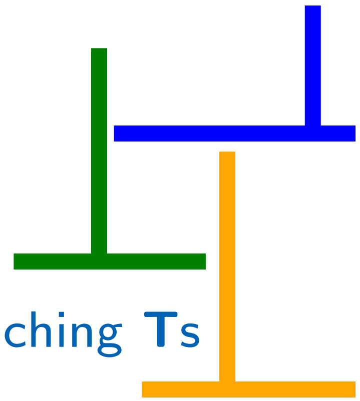
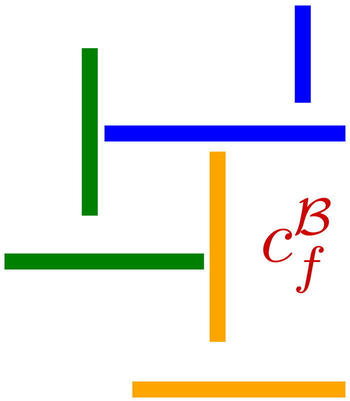




touching **T**s

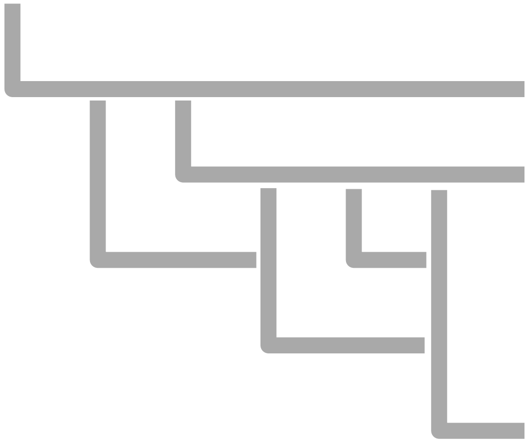


$c_f^{\mathcal{B}} \leq 2$

more examples

▷ drawing style: contact graphs of **L**-shapes

▷ good graphs: $\mathcal{L} = \{$**L**-graphs$\}$



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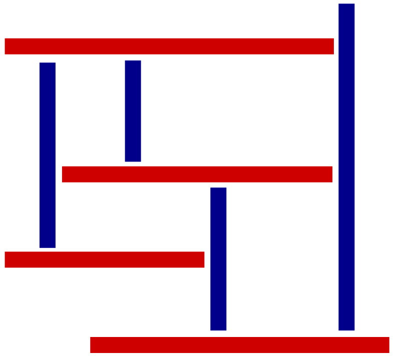

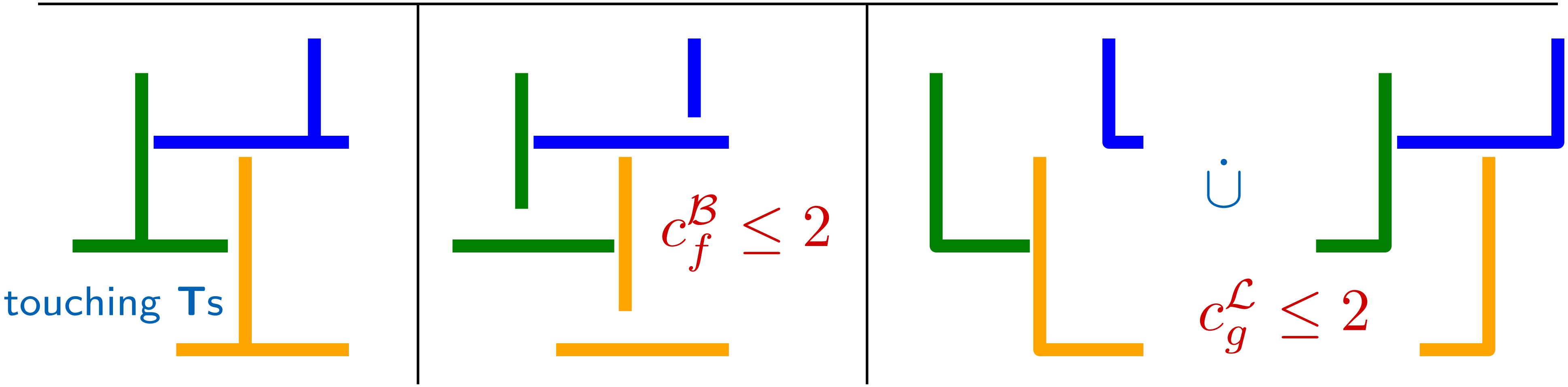

$$c_g^{\mathcal{G}}(H) \geq c_\ell^{\mathcal{G}}(H) \geq c_f^{\mathcal{G}}(H).$$

By how much can these differ?

For $\mathcal{B} = \{$bipartite graphs$\}$ and any graph H we have

$$c_g^{\mathcal{B}}(H) = \lceil \log(\chi(H)) \rceil \qquad \text{and} \qquad c_f^{\mathcal{B}}(H) \leq 2.$$

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Thm. For any class $\mathcal{G}$ not containing arbitrarily large complete graphs, and any r there is a graph H with $c_\ell^{\mathcal{G}}(H) \geq r$.

[Bläsius-Stumpf-U. '15]

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[Bläsius-Stumpf-U. ’15]

Thm. For any class $\mathcal{G}$ that is closed under topological minors there is a function f s.t.

$$c_g^{\mathcal{G}}(H) \leq f(c_f^{\mathcal{G}}(H)) \text{ for all graphs } H.$$

[Bläsius-Stumpf-U. ’15]

Conj. (**L**inear **A**rboricity **C**onjecture)

For $\mathcal{G} = \{$unions of paths$\}$ and any graph H we have

$$c_g^{\mathcal{G}}(H) \in \left\{ \left\lceil \frac{\Delta(H)}{2} \right\rceil, \left\lceil \frac{\Delta(H)+1}{2} \right\rceil \right\}.$$

[Akiyama et al. '80]

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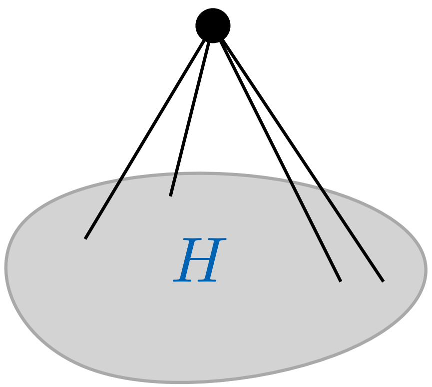
Proof. v connected to odd-degree vertices



Eulerian tour in $H \oplus v$ decomposes into folded paths after removing v.

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□

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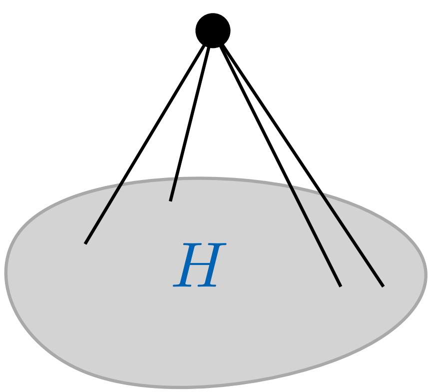

Eulerian tour in $H \oplus v$ decomposes into folded paths after removing v.

$$\Longrightarrow c_f^{\mathcal{G}}(H) \le \left\lceil \frac{\Delta(H)+1}{2} \right\rceil$$

□

Open: … for $\mathcal{G} = \{$caterpillars$\}$ and $\mathcal{G} = \{$interval graphs$\}$.

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$$c_g^{\mathcal{B}}(H) = \lceil \log(\chi(H)) \rceil.$$

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Conj. (Hadwiger)
If H is K_n-minor free, then $\chi(H) < n$.

Open: If H is K_n-minor free, then $c_\ell^{\mathcal{B}}(H) \leq \log(n)$.

Let $\mathcal{I} = \{$interval graphs$\}$.

Thm. For any planar graph H we have $c_g^{\mathcal{I}}(H) \leq 4$, and this is tight. [Gonçalves '07]

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Thm. For $\mathcal{S} = \{\text{star forests}\} \subset \mathcal{I}$ and any graph H we have

$$c_\ell^{\mathcal{S}}(H) \leq \max\left\{ \left\lceil \frac{|E(H')|}{|V(H')|} \right\rceil \mid H' \subseteq H \right\} + 1$$

[Knauer, Ueckerdt '15]

Cor. For planar H we have $c_\ell^{\mathcal{I}}(H) \leq c_\ell^{\mathcal{S}}(H) \leq 4$.

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Cor. For planar H we have $c_\ell^{\mathcal{I}}(H) \leq c_\ell^{\mathcal{S}}(H) \leq 4$.

Thm. For any planar graph H we have $c_f^{\mathcal{I}}(H) \leq 3$, and this is tight.

[Scheinerman-West '83]

Open: Is it true that for planar H we have $c_\ell^{\mathcal{I}}(H) \leq 3$?

Thm. For $\mathcal{G} = \{$outerplanar graphs$\}$ and any planar graph H we have $c_g^{\mathcal{G}}(H) \leq 2$. [Gonçalves '05]

($40+$ page proof)

Open: Find shorter proof for $c_f^{\mathcal{G}}(H) \leq 2$.

Let $\mathcal{P} = \{\text{planar graphs}\}$.

$$X_g^{\mathcal{P}}(m) \stackrel{\text{def}}{=} \max\{\chi(H) \mid c_g^{\mathcal{P}}(H) \leq m\}$$

$$X_\ell^{\mathcal{P}}(m) \stackrel{\text{def}}{=} \max\{\chi(H) \mid c_\ell^{\mathcal{P}}(H) \leq m\}$$

$$X_f^{\mathcal{P}}(m) \stackrel{\text{def}}{=} \max\{\chi(H) \mid c_f^{\mathcal{P}}(H) \leq m\}$$

(maximum chromatic number among all graphs with $c_\star^{\mathcal{P}} \leq m$)

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Thm. (Earth-Moon Coloring Problem)
We have $6m - 2 \leq X_g^{\mathcal{P}}(m) \leq 6m$ for $m \geq 3$.

[Heawood '90, Beineke-Harary '65]

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We have $X_f^{\mathcal{P}}(m) = 6m$.
[Jackson-Ringel '84]

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We have $X_f^{\mathcal{P}}(m) = 6m$.

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Open: Find bounds on $X_\ell^{\mathcal{P}}(m)$.

Conj. For $\mathcal{G} = \{$complete bipartite graphs$\}$ and any graph H we have $\chi(H) \leq c_g^{\mathcal{G}}(H) + 1$. [Alon-Saks-Seymour]

(It is easy to find examples with $\chi(H) = c_g^{\mathcal{G}}(H) + 1$.)

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Thm. The conjecture is false.
There exists H with $\chi(H) \geq \Omega(c_g^{\mathcal{G}}(H)^{\frac{6}{5}})$. [Huang-Sudakov '12]

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Open: Is there a function f such that $\chi(H) \leq f(c_f^{\mathcal{G}}(H))$?

computational complexity

class* $\mathcal{G}$	$c_g^{\mathcal{G}}(H)$	$c_\ell^{\mathcal{G}}(H)$	$c_f^{\mathcal{G}}(H)$
K_2	NPC	P	
stars	NPC	P	
trees	P		
bipartite	NPC	NPC	P
interval graphs	NPC	NPC	NPC
planar	NPC	open	NPC
outerplanar	open	open	open

* closure under disjoint unions

decreasing difficulty?

Introduction

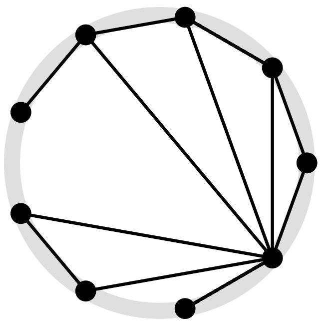
Definitions and Examples

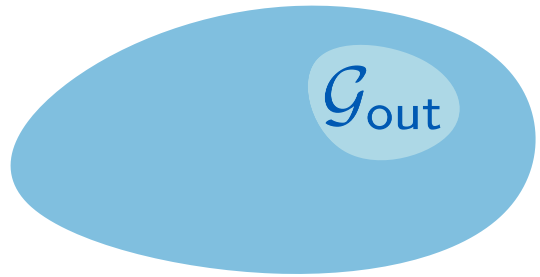
Questions

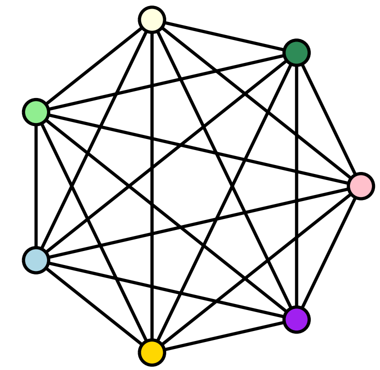
- separability
- evidence for open problems
- stronger results, shorter proofs
- extremal parameters
- computational complexity
- . . .

▷ Summary ◁

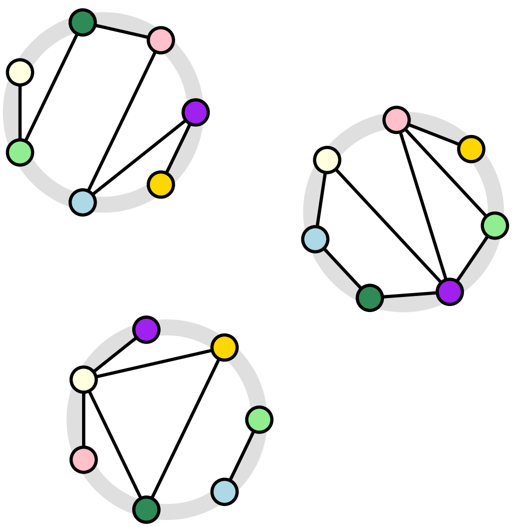
summary

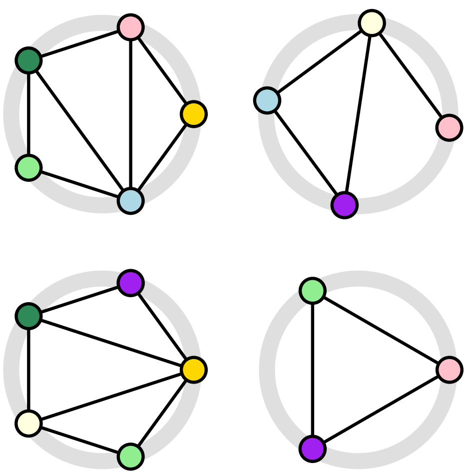


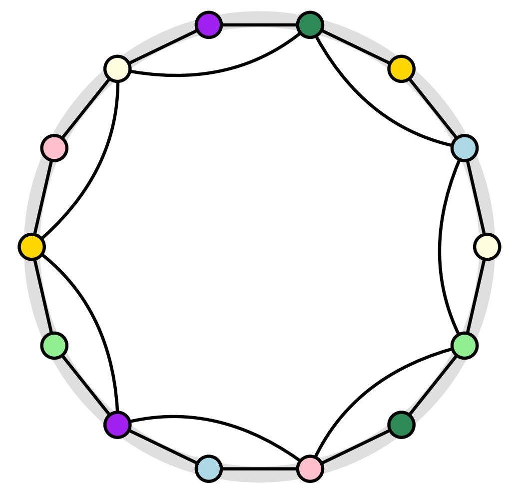




drawing style $\quad$ $\mathcal{G} = \{\text{good graphs}\}$ $\quad$ graph $H \notin \mathcal{G}$







Global: $c_g^{\mathcal{G}}(H)$ $\quad$ Local: $c_\ell^{\mathcal{G}}(H)$ $\quad$ Folded: $c_f^{\mathcal{G}}(H)$

<table>
<tr><th>$\mathcal{G}$ = {good graphs}
(drawing style)</th><th>graph $H \notin \mathcal{G}$</th><th>question</th></tr>
<tr><td rowspan="2">$\mathcal{B}$ = {bipartite graphs}</td><td>any</td><td>$c_\ell^{\mathcal{B}}(H) = c_g^{\mathcal{B}}(H)$</td></tr>
<tr><td>K_n-minor free</td><td>$c_\ell^{\mathcal{B}}(H) \leq \log(n)$</td></tr>
<tr><td rowspan="2">{outerplanar graphs}</td><td>K_n</td><td>$c_f^{\mathcal{G}}(H), c_\ell^{\mathcal{G}}(H), c_g^{\mathcal{G}}(H)$</td></tr>
<tr><td rowspan="2">planar</td><td>$c_f^{\mathcal{G}}(H) \leq 2$</td></tr>
<tr><td rowspan="2">$\mathcal{I}$ = {interval graphs}</td><td>$c_\ell^{\mathcal{I}}(H) \leq 3$</td></tr>
<tr><td>any</td><td>$c_g^{\mathcal{G}}(H) \leq \left\lceil \frac{\Delta(H)+1}{2} \right\rceil$</td></tr>
<tr><td>{complete bipartite graphs}</td><td>any</td><td>$\chi(H) \leq f(c_f^{\mathcal{G}}(H))$</td></tr>
<tr><td>$\mathcal{P}$ = {planar graphs}</td><td>$c_\ell^{\mathcal{P}}(H) \leq m$</td><td>$\chi(H)$</td></tr>
</table>

class* $\mathcal{G}$	$c_g^{\mathcal{G}}(H)$	$c_\ell^{\mathcal{G}}(H)$	$c_f^{\mathcal{G}}(H)$
K_2	NPC	P	
stars	NPC	P	
trees	P		
bipartite	NPC	NPC	P
interval graphs	NPC	NPC	NPC
planar	NPC	open	NPC
outerplanar	open	open	open

* closure under disjoint unions

decreasing difficulty?