Appendix C

The Lorenz Model

C.1 Introduction

In this appendix we show how the Lorenz model equations introduced in Chapter 1 are developed (derived is too strong a word) from the Navier–Stokes equation for fluid flow and the equation describing thermal energy diffusion. This development provides a prototype for the common process of finding approximate, but useful, model equations when we cannot solve the fundamental equations describing some physical situation.

The Lorenz model has become almost totemistic in the field of nonlinear dynamics. Unfortunately, most derivations of the Lorenz model equations leave so much to the reader that they are essentially useless for all but specialists in fluid dynamics. In this appendix, we hope to give a sufficiently complete account that readers of this text come away with a good understanding of both the physics content and the mathematical approximations that go into this widely cited model.

The Lorenz model describes the motion of a fluid under conditions of Rayleigh–Bénard flow: an incompressible fluid is contained in a cell which has a higher temperature T_w at the bottom and a lower temperature T_c at the top. The temperature difference $\delta T = T_w - T_c$ is taken to be the control parameter for the system. The geometry is shown in Fig. C.1.

Before launching into the formal treatment of Rayleigh–Bénard flow, we should develop some intuition about the conditions that cause convective flow to begin. In rough terms, when the temperature gradient between the top and bottom plates becomes sufficiently large, a small packet of fluid that happens to move up a bit will experience a net upward buoyant force because it has moved into a region of lower temperature and hence higher density: It is now less dense than its surroundings. If the upward force is sufficiently strong, the packet will move upward more quickly than its temperature can drop. (Since the packet is initially warmer than its surroundings, it will tend to loose thermal energy to its environment.) Then convective currents will begin to flow. On the other hand if the buoyant force is relatively weak, the temperature of the packet will drop before it can move a significant distance, and it remains stable in position.

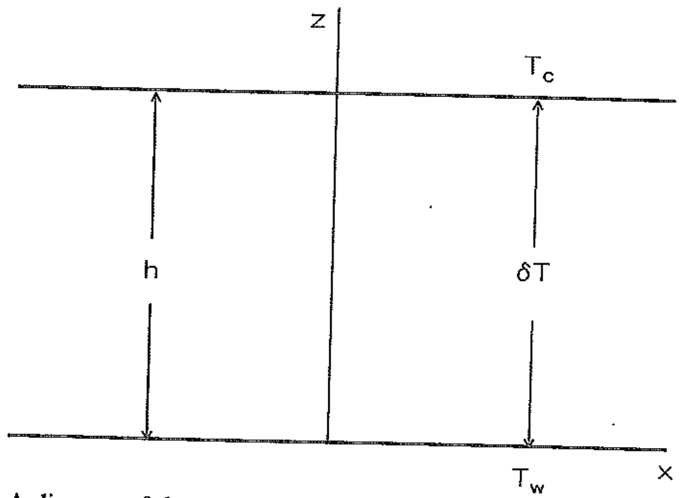

Fig. C.1. A diagram of the geometry for the Lorenz model. The system is infinite in extent in the horizontal direction and in the direction in and out of the page. $z = 0$ at the bottom plate.

We can be slightly more quantitative about this behavior by using our knowledge (gained in Chapter 11) about thermal energy diffusion and viscous forces in fluids. Imagine that the fluid is originally at rest. We want to see if this condition is stable. We begin by considering a small packet of fluid that finds itself displaced upward by a small amount Δz. The temperature in this new region is lower by the amount $\Delta T = (\delta T/h)\Delta z$. According to the thermal energy diffusion equation (Chapter 11), the rate of change of temperature is equal to the thermal diffusion coefficient D_T multiplied by the Laplacian of the temperature function. For this small displacement, we may approximate the Laplacian by

$$\nabla^2 T \approx \frac{\delta T}{h^2}\frac{\Delta z}{h} \qquad \text{(C.1-1)}$$

We then define a thermal relaxation time δt_T such that

$$\delta t_T \frac{dT}{dt} = \Delta T = \delta t_T D_T \nabla^2 T \qquad \text{(C.1-2)}$$

where the second equality follows from the thermal diffusion equation. Using our approximation for the Laplacian, we find that

$$\delta t_T = \frac{h^2}{D_T} \qquad \text{(C.1-3)}$$

Let us now consider the effect of the buoyant force on the packet of fluid. This bouyant force is proportional to the difference in density between the packet and its surroundings. This difference itself is proportional to the thermal expansion coefficient α (which gives the relative change in density per unit temperature change) and the temperature difference ΔT. Thus, we find for the bouyant force

$$F = \alpha \rho_o g \Delta T = \alpha \rho_o g \frac{\delta T}{h} \Delta z \tag{C.1-4}$$

where ρ_o is the original density of the fluid and g is the acceleration due to gravity.

We assume that this bouyant force just balances the fluid viscous force; therefore, the packet moves with a constant velocity v_z. It then takes a time $\tau_d = \Delta z / v_z$ for the packet to be displaced through the distance Δz. As we learned in Chapter 11, the viscous force is equal to the viscosity of the fluid multiplied by the Laplacian of the velocity. Thus, we approximate the viscous force as

$$F_v = \mu \nabla^2 v_z \approx \mu \frac{v_z}{h^2} \tag{C.1-5}$$

where the right-most equality states our approximation for the Laplacian of v_z.

If we now require that the bouyant force be equal in magnitude to the viscous force, we find that v_z can be expressed as

$$v_z = \frac{\alpha \rho_o g h \delta T}{\mu} \Delta z \tag{C.1-6}$$

The displacement time is then given by

$$\tau_d = \frac{\mu}{\alpha \rho_o g h \delta T} \tag{C.1-7}$$

The original nonconvecting state is stable if the thermal diffusion time is less than the corresponding displacement time. If the thermal diffusion time is longer, then the fluid packet will continue to feel an upward force, and convection will continue. The important ratio is the ratio of the thermal diffusion time to the displacement time. This ratio is called the *Rayleigh number R* and takes the form

$$R = \frac{\alpha \rho_o g h^3 \delta T}{D_T \mu} \tag{C.1-8}$$

As we shall see, the Rayleigh number is indeed the critical parameter for Rayleigh–Bénard convection, but we need a more detailed calculation to tell us the actual value of the Rayleigh number at which convection begins.

C.2 The Navier–Stokes Equations

Because of the geometry assumed, the fluid flow can be taken to be two dimensional. Thus, we need consider only the x (horizontal) and z (vertical) components of the fluid velocity. The Navier–Stokes equations (see Chapter 11) for the x and z components of the fluid velocity are

$$\rho\frac{\partial v_z}{\partial t}+\rho\vec{v}\circ \boldsymbol{grad}\, v_z = -\rho g-\frac{\partial p}{\partial z}+\mu\nabla^2 v_z$$

$$\rho\frac{\partial v_x}{\partial t}+\rho\vec{v}\circ \boldsymbol{grad}\, v_x = -\frac{\partial p}{\partial x}+\mu\nabla^2 v_x \qquad \text{(C.2-1)}$$

In Eq. (C.2-1), ρ is the mass density of the fluid; g is the acceleration due to gravity; p is the fluid pressure, and μ is the fluid viscosity. Note that the acceleration due to gravity only affects the z component equation.

The temperature T of the fluid is described by the thermal diffusion equation (see Chapter 11), which takes the form

$$\frac{\partial T}{\partial t}+\vec{v}\circ \boldsymbol{grad}\, T = D_T\nabla^2 T \qquad \text{(C.2-2)}$$

where, as before, D_T is the thermal diffusion coefficient.

In the steady nonconvecting state (when the fluid is motionless) the temperature varies linearly from bottom to top:

$$T(x,z,t)=T_w-\frac{z}{h}\delta T \qquad \text{(C.2-3)}$$

For the purposes of our calculation, we will focus our attention on a function $\tau(x,z,t)$ that tells us how the temperature deviates from this linear behavior:

$$\tau(x,z,t)=T(x,z,t)-T_w+\frac{z}{h}\delta T \qquad \text{(C.2-4)}$$

If we use Eq. (C.2-4) in Eq. (C.2-2), we find that τ satisfies

$$\frac{\partial \tau}{\partial t} + \vec{v} \cdot \textbf{\textit{grad}}\,\tau - v_z \frac{\delta T}{h} = D_T \nabla^2 \tau \qquad \text{(C.2-5)}$$

We now need to take into account the variation of the fluid density with temperature. (It is this decrease of density with temperature that leads to a bouyant force, which initiates fluid convection.) We do this by writing the fluid density in terms of a power series expansion:

$$\rho(T) = \rho_o + \frac{\partial \rho}{\partial T}(T - T_w) + \ldots \qquad \text{(C.2-6)}$$

where ρ_o is the fluid density evaluated at T_w.

Introducing the thermal expansion coefficient α, which is defined as

$$\alpha = -\frac{1}{\rho_o}\frac{\partial \rho}{\partial T} \qquad \text{(C.2-7)}$$

and using T-T_w from Eq. (C.2-4), we may write the density temperature variation as

$$\rho(T) = \rho_o - \alpha\rho_o\left[-\frac{z}{h}\delta T + \tau(x,z,t)\right] \qquad \text{(C.2-8)}$$

The fluid density ρ appears in several terms in the Navier–Stokes equations. The ***Boussinesq approximation***, widely used in fluid dynamics, says that we may ignore the density variation in all the terms except the one that involves the force due to gravity. This approximation reduces the v_z equation in Eq. (C.2-1) to

$$\rho_o \frac{\partial v_z}{\partial t} + \rho_o \vec{v} \cdot \textbf{\textit{grad}}\, v_z = -\rho_o g - \alpha\rho_o \frac{z}{h}\delta T - \frac{\partial p}{\partial z} + \alpha g \rho_o \tau(x,z,t) + \mu \nabla^2 v_z \qquad \text{(C.2-9)}$$

We then recognize that when the fluid is not convecting, the first three terms on the right-hand side of the previous equation must add to 0. Hence, we introduce an effective pressure gradient, which has the property of being equal to 0 when no fluid motion is present:

$$p' = p + \rho_o g z + \alpha\rho_o \frac{z^2}{2}\frac{\delta T}{h}$$

$$\frac{\partial p'}{\partial z} = \frac{\partial p}{\partial z} + \rho_o g + \alpha\rho_o \frac{z}{h}\delta T \qquad \text{(C.2-10)}$$

Finally, we use this effective pressure gradient in the Navier–Stokes equations and divide through by ρ_o to obtain

$$\frac{\partial v_z}{\partial t} + \vec{v} \circ \textbf{\textit{grad}}\, v_z = -\frac{1}{\rho_o}\frac{\partial p'}{\partial z} + \alpha\tau g + \nu\nabla^2 v_z$$

$$\frac{\partial v_x}{\partial t} + \vec{v} \circ \textbf{\textit{grad}}\, v_x = -\frac{1}{\rho_o}\frac{\partial p'}{\partial x} + \nu\nabla^2 v_x \qquad \text{(C.2-11)}$$

where $\nu = \mu/\rho_o$ is the so-called kinematic viscosity.

C.3 Dimensionless Variables

Our next step in the development of the Lorenz model is to express the Navier–Stokes equations Eq. (C.2-11) in terms of dimensionless variables. By using dimensionless variables, we can see which combinations of parameters are important in determining the behavior of the system. In addition, we generally remove the dependence on specific numerical values of the height h and temperature difference δT, and so on, thereby simplifying the eventual numerical solution of the equations.

First, we introduce a dimensionless time variable t'

$$t' = \frac{D_T}{h^2} t \qquad \text{(C.3-1)}$$

[You should recall from Eq. (C.1-3) (and from Chapter 11) that h^2/D_T is a typical time for thermal diffusion over the distance h.] In a similar fashion, we introduce dimensionless distance variables and a dimensionless temperature variable:

$$x' = \frac{x}{h} \qquad z' = \frac{z}{h} \qquad \tau' = \frac{\tau}{\delta T} \qquad \text{(C.3-2)}$$

We can also define a dimensionless velocity using the dimensionless distance and dimensionless time variables. For example, the x component of the dimensionless velocity is

$$v_x{}' = \frac{dx'}{dt'} = \frac{D_T}{h} v_x \tag{C.3-3}$$

Finally, the Laplacian operator can also be expressed in terms of the new variables with the replacement

$$\nabla'^2 = h^2 \nabla^2 \tag{C.3-4}$$

If we use these new variables in the Navier–Stokes equations (C.2-11) and multipy through by $h^3/(\nu D_T)$, we arrive at

$$\frac{D_T}{\nu}\left[\frac{\partial v'_z}{\partial t'} + \vec{v}' \cdot \textbf{grad}' v'_z\right] = -\frac{h^2}{\nu D_T \rho_o}\frac{\partial p'}{\partial z'} + \frac{\alpha \delta T g h^3}{\nu D_T}\tau' + \nabla'^2 v'_z \tag{C.3-5}$$

$$\frac{D_T}{\nu}\left[\frac{\partial v'_x}{\partial t'} + \vec{v}' \cdot \textbf{grad}' v'_x\right] = -\frac{h^2}{\nu D_T \rho_o}\frac{\partial p'}{\partial x'} + \nabla'^2 v'_x \tag{C.3-6}$$

We recognize that certain dimensionless ratios of parameters appear in the equations. First, the ***Prandtl number*** σ gives the ratio of kinematic viscosity to the thermal diffusion coefficient:

$$\sigma = \frac{\nu}{D_T} \tag{C.3-7}$$

The Prandtl number measures the relative importance of viscosity (dissipation of mechanical energy due to the shearing of the fluid flow) compared to thermal diffusion, the dissipation of energy by thermal energy (heat) flow. The Prandtl number is about equal to 7 for water at room temperature.

The ***Rayleigh number*** R tells us the balance between the tendency for a packet of fluid to rise due to the buoyant force associated with thermal expansion relative to the dissipation of energy due to viscosity and thermal diffusion. R is defined as the combination

$$R = \frac{\alpha g h^3}{\nu D_T}\delta T \tag{C.3-8}$$

The Rayleigh number is a dimensionless measure of the temperature difference between the bottom and top of the cell. In most Rayleigh–Bénard experiments, the Rayleigh number is the control parameter, which we adjust by changing that temperature difference.

Finally, we introduce a dimensionless pressure variable Π defined as

$$\Pi = \frac{p' h^2}{\nu \rho_o D_T} \tag{C.3-9}$$

We now use all these dimensionless quantities to write the Navier–Stokes equations and the thermal diffusion equation in the following form, in which, for the sake of simpler typesetting, we have dropped the primes (but we remember that all the variables are dimensionless):

$$\frac{1}{\sigma}\left[\frac{\partial v_z}{\partial t} + \vec{v} \circ \boldsymbol{grad}\, v_z\right] = -\frac{\partial \Pi}{\partial z} + R\tau + \nabla^2 v_z$$

$$\frac{1}{\sigma}\left[\frac{\partial v_x}{\partial t} + \vec{v} \circ \boldsymbol{grad}\, v_x\right] = -\frac{\partial \Pi}{\partial x} + \nabla^2 v_x$$

$$\frac{\partial \tau}{\partial t} + \vec{v} \circ \boldsymbol{grad}\, \tau - v_z = \nabla^2 \tau \tag{C.3-10}$$

We should point out that in introducing the dimensionless variables and dimensionless parameters, we have not changed the physics content of the equations, nor have we introduced any mathematical approximations.

C.4 The Streamfunction

As we discussed in Chapter 11, for two dimensional fluid flows, we may introduce a streamfunction $\Psi(x,z,t)$, which carries all the information about the fluid flow. The actual fluid velocity components are obtained by taking partial derivatives of the streamfunction:

$$v_x = -\frac{\partial \Psi(x,z,t)}{\partial z} \qquad v_z = \frac{\partial \Psi(x,z,t)}{\partial x} \tag{C.4-1}$$

(We are free to place the minus sign on either of the velocity components. The sign choice made here gives us the conventional signs in the Lorenz model equations.) We now use the streamfunction in the thermal diffusion equation:

$$\frac{\partial \tau}{\partial t} - \frac{\partial \Psi}{\partial z}\frac{\partial \tau}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \tau}{\partial z} - \frac{\partial \Psi}{\partial x} = \nabla^2 \tau \tag{C.4-2}$$

in which we have expanded the *grad* term explicitly in terms of components. (Mathematically experienced readers may recognize the middle two terms on the left-hand side of the previous equation as the Jacobian determinant of the functions Ψ and τ with respect to the variables x and z.)

The fluid flow equations can also be written in terms of the streamfunction. Unfortunately, the equations become algebraically messy before some order emerges. The v_z equation becomes

$$\frac{1}{\sigma}\left[\frac{\partial^2\Psi}{\partial t\partial x}-\frac{\partial\Psi}{\partial z}\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial\Psi}{\partial x}\frac{\partial^2\Psi}{\partial z\partial x}\right]=-\frac{\partial\Pi}{\partial z}+R\tau+\nabla^2\frac{\partial\Psi}{\partial x} \qquad \text{(C.4-3)}$$

The v_x equation becomes

$$\frac{1}{\sigma}\left[-\frac{\partial^2\Psi}{\partial t\partial z}+\frac{\partial\Psi}{\partial z}\frac{\partial^2\Psi}{\partial x\partial z}-\frac{\partial\Psi}{\partial x}\frac{\partial^2\Psi}{\partial z\partial x}\right]=-\frac{\partial\Pi}{\partial x}-\nabla^2\frac{\partial\Psi}{\partial z} \qquad \text{(C.4-4)}$$

If we now take $\partial/\partial x$ of Eq. (C.4-3) and subtract from it $\partial/\partial z$ of Eq. (C.4-4), the pressure terms drop out, and we have

$$\frac{1}{\sigma}\left[+\frac{\partial}{\partial t}(\nabla^2\Psi)-\frac{\partial}{\partial z}\left\{\frac{\partial\Psi}{\partial z}\frac{\partial^2\Psi}{\partial x\partial z}-\frac{\partial\Psi}{\partial x}\frac{\partial^2\Psi}{\partial z^2}\right\}-\frac{\partial}{\partial x}\left\{\frac{\partial\Psi}{\partial z}\frac{\partial^2\Psi}{\partial x^2}-\frac{\partial\Psi}{\partial x}\frac{\partial^2\Psi}{\partial z\partial x}\right\}\right]$$

$$=R\frac{\partial\tau}{\partial x}+\nabla^4\Psi \qquad \text{(C.4-5)}$$

Eq. (C.4-2) and the rather formidable looking Eq. (C.4-5) contain all the information on the fluid flow.

C.5 Fourier Expansion, Galerkin Truncation, and Boundary Conditions

Obviously, we face a very difficult task in trying to solve the partial differential equations that describe our model system. For partial differential equations, the usual practice is to look for solutions that can be written as products of functions, each of which depends on only one of the independent variables x, z, t. Since we have a rectangular geometry, we expect to be able to find a solution of the form

$$\Psi(x,z,t)=\sum_{m,n}e^{\omega_{m,n}t}\{A_m\cos\lambda_m z+B_m\sin\lambda_m z\}$$

$$\text{x}\{C_n\cos\lambda_n x+D_n\sin\lambda_n x\} \qquad \text{(C.5-1)}$$

where the λs are the wavelengths of the various Fourier spatial modes and $\omega_{m,n}$ are the corresponding frequencies. We would, of course, have a similar equation for τ , the temperature variable. (Appendix A contains a concise introduction to Fourier analysis.)

As we saw in Chapter 11, the standard procedure consists of using this sine and cosine expansion in the original partial differential equations to develop a corresponding set of (coupled) ordinary differential equations. This procedure will lead to an infinite set of ordinary differential equations. To make progress, we must somehow reduce this infinite set to a finite set of equations. This truncation process is known as the ***Galerkin procedure***.

For the Lorenz model, we look at the boundary conditions that must be satisfied by streamfunction and the temperature deviation function and choose a very limited set of sine and cosine terms that will satisfy these boundary conditions. It is hard to justify this truncation a priori, but numerical solutions of a larger set of equations seem to indicate (SAL62) that the truncated form captures most of the dynamics over at least a limited range of parameter values.

The boundary conditions for the temperature deviation function are simple. Since τ represents the deviation from the linear temperature gradient and since the temperatures at the upper and lower surfaces are fixed, we must have

$$\tau = 0 \ \text{ at } z = 0, 1 \tag{C.5-2}$$

For the streamfunction, we look first at the boundary conditions on the velocity components. We assume that at the top and bottom surfaces the vertical component of the velocity v_z must be 0. We also assume that we can neglect the shear forces at the top and bottom surfaces. As we saw in Chapter 11, these forces are proportional to the gradient of the tangential velocity component; therefore, this condition translates into having $\partial v_x / \partial z = 0$ at $z = 0$ and $z = 1$. For the Lorenz model, these conditions are satisfied by the following *ansatz* for the streamfunction and temperature deviation function:

$$\Psi(x, z, t) = \psi(t)\sin(\pi z)\sin(ax)$$
$$\tau(x, z, t) = \mathrm{T}_1(t)\sin(\pi z)\cos(ax) - \mathrm{T}_2(t)\sin(2\pi z) \tag{C.5-3}$$

where the parameter a is to be determined. As we shall see, this choice of functions not only satisfies the boundary conditions, but it also greatly simplifies the resulting equations.

The particular form of the spatial part of the streamfunction Ψ models the convective rolls observed when the fluid begins to convect. You may easily check this by calculating the velocity components from Eq. (C.4-1). The form for the temperature deviation function has two parts. The first, T_1, gives the temperature difference between the upward and downward moving parts of a convective cell. The second, T_2, gives the deviation from the linear temperature variation in the center of a convective cell as a function of vertical position z. (The minus sign in front of the T_2 term is chosen so that T_2 is positive: The temperature in the fluid must lie between T_w and T_c.)

C.6 Final Form of the Lorenz Equations

We now substitute the assumed forms for the streamfunction and the temperature deviation function into Eqs. (C.4-2) and (C.4-5). As we do so, we find that most terms simplify. For example, we have

$$\nabla^2\Psi = -(a^2+\pi^2)\Psi$$

$$\nabla^4\Psi = +(a^2+\pi^2)^2\Psi \qquad \text{(C.6-1)}$$

The net result is that some of the complicated expressions that arise from $\vec{v} \cdot \textbf{\textit{grad}}\, v$ terms disappear, and we are left with

$$-\frac{d\psi(t)}{dt}(a^2+\pi^2)\sin\pi z \sin ax =$$

$$-\sigma R\,\mathrm{T}_1(t)\sin\pi z \sin ax$$

$$+\sigma(a^2+\pi^2)^2\psi(t)\sin\pi z \sin ax \qquad \text{(C.6-2)}$$

The only way the previous equation can hold for all values of x and z is for the coefficients of the sine terms to satisfy

$$\frac{d\psi(t)}{dt} = \frac{\sigma R}{\pi^2+a^2}\mathrm{T}_1(t) - \sigma(\pi^2+a^2)\psi(t) \qquad \text{(C.6-3)}$$

The temperature deviation equation is a bit more complicated. It takes the form

$$\dot{T}_1 \sin \pi z \cos ax - \dot{T}_2 \sin 2\pi z + ((\pi^2 + a^2)T_1 \sin \pi z \cos ax$$

$$-4\pi^2 T_2 \sin 2\pi z - a\psi \sin \pi z \cos ax$$

$$= -[\pi\psi \cos \pi z \sin ax]\,[aT_1 \sin \pi z \sin ax]$$

$$-[a\psi \sin \pi z \cos ax]\,[\pi T_1 \cos \pi z \cos ax]$$

$$+[\psi \sin \pi z \cos ax]\,[2\pi T_2 \cos 2\pi z] \qquad \text{(C.6-4)}$$

We first collect all those terms which involve $\sin \pi z \cos ax$. We note that the last of these terms in Eq. (C.6-4) is$2a\pi\psi T_2 \sin \pi z \cos ax \cos 2\pi z$. Using standard trigonometric identities, this term can be written as the following combination of sines and cosines: $\left(-\frac{1}{2}\sin \pi z + \frac{1}{2}\sin 3\pi z\right)\cos ax$. The $\sin 3\pi z$ term has a spatial dependence more rapid than allowed by our *ansatz*; so, we drop that term. We may then equate the coefficients of the terms in Eq. (C.6-4) involving $\sin \pi z \cos ax$ to obtain

$$\dot{T}_1 = a\psi - (\pi^2 + a^2)T_1 - \pi a \psi T_2 \qquad \text{(C.6-5)}$$

All the other terms in the temperature deviation equation are multiplied by $\sin 2\pi z$ factors. Again, equating the coefficients, we find

$$\dot{T}_2 = \frac{\pi a}{2}\psi T_1 - 4\pi^2 T_2 \qquad \text{(C.6-6)}$$

To arrive at the standard form of the Lorenz equations, we now make a few straightforward change of variables. First, we once again change the time variable by introducing a new variable $t'' = (\pi^2 + a^2)t'$. We then make the following substitutions:

$$X(t) = \frac{a\pi}{(\pi^2 + a^2)\sqrt{2}}\psi(t)$$

$$Y(t) = \frac{r\pi}{\sqrt{2}}T_1(t)$$

$$Z(t) = \pi r T_2(t) \qquad \text{(C.6-7)}$$

where r is the so-called reduced Rayleigh number:

$$r = \frac{a^2}{(a^2+\pi^2)^3} R \tag{C.6-8}$$

We also introduce a new parameter b defined as

$$b = \frac{4\pi^2}{a^2+\pi^2} \tag{C.6-9}$$

With all these substitutions and with the replacement of σ with p for the Prandtl number, we finally arrive at the standard form of the Lorenz equations:

$$\begin{aligned} \dot{X} &= p(Y-X) \\ \dot{Y} &= rX - XZ - Y \\ \dot{Z} &= XY - bZ \end{aligned} \tag{C.6-10}$$

At this point we should pause to note one important aspect of the relationship between the Lorenz model and the reality of fluid flow. The truncation of the sine–cosine expansion means that the Lorenz model allows for only one spatial mode in the x direction with "wavelength" $2\pi/a$. If the actual fluid motion takes on more complex spatial structure, as it will if the temperature difference between top and bottom plates becomes too large, then the Lorenz equations no longer provide a useful model of the dynamics.

Let us also take note of where nonlinearity enters the Lorenz model. We see from Eq. (C.6-10) that the product terms XZ and XY are the only nonlinear terms. These express a coupling between the fluid motion (represented by X, proportional to the streamfunction) and the temperature deviation (represented by Y and Z, proportional to T_1 and T_2, respectively. The Lorenz model does not include, because of the choice of spatial mode functions, the usual $\vec{v} \cdot \textbf{\textit{grad}}\, v$ nonlinearity from the Navier–Stokes equation.

C.7 Stability Analysis of the Nonconvective State

The parameter a is determined by examining the conditions on the stability of the nonconvective state. The nonconvective state has $\psi = 0$ and $\tau = 0$ and hence corresponds to $X, Y, Z = 0$. If we let x, y, and z represent the values of X, Y, and Z near this fixed point, and drop all nonlinear terms from the Lorenz equations, the dynamics near the fixed point is modeled by the following *linear* differential equations:

$$\begin{aligned} \dot{x} &= p(x-y) \\ \dot{y} &= rx - y \\ \dot{z} &= -bz \end{aligned} \tag{C.7-1}$$

Note that $z(t)$ is exponentially damped since the parameter b is positive. Thus, we need consider only the x and y equations. Using our now familiar results from Section 3.11, we see that the nonconvective fixed point becomes unstable when $r > 1$. Returning to the original Rayleigh number, we see that the condition is

$$R \geq \frac{(\pi^2 + a^2)^3}{a^2} \tag{C.7-2}$$

We choose the parameter a to be the value that gives the lowest Rayleigh number for the beginning of convection. In a sense, the system selects the wavelength $2\pi/a$ by setting up a convection pattern with the wavelength $2\pi/a$ at the lowest possible Rayleigh number. This condition yields $a = \pi/\sqrt{2}$. Hence, the Rayleigh number at which convection begins is $R = 27\pi^4/4$. The parameter b is then equal to 8/3, the value used in most analyses of the Lorenz model.

C.8 Further Reading

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[Bergé, Pomeau, Vidal, 1984], Appendix D, contains a slightly different development of the Lorenz model equations, and in addition, provides more details on the how the dynamics evolve as the reduced Rayleigh number r changes.

[Sparrow, 1983] gives a detailed treatment of the Lorenz model and its behavior.

S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Dover, New York, 1984). Chapter II. A wide-ranging discussion of the physics and mathematics of Rayleigh-Benard convection along with many historical references.

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