Physics 595CL – Homework 3

Exercises 1 and 2, below, fill in missing steps in the proof of the Transport Theorem (or Liebnitz' Rule), used for the derivation of Planck's formula for blackbody spectral radiance, $B_\nu(T)$.

1) Let $A(t)$ be a 3×3 matrix, with nonzero determinant, whose entries are smooth functions of t. Prove:

$$\frac{d}{dt}[\det A(t)] = [\det A(t)]\mathrm{tr}(\dot{A}A^{-1}),$$

where $\dot{A} = \frac{d}{dt}A(t)$. You may use the fact that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for square matrices A and B (even if $AB \neq BA$). For extra credit, prove this formula, instead, for the more general case that $A(t)$ is an $n \times n$ matrix.

2) Let $F : \mathbb{R}^3 \to \mathbb{R}$ and $\vec{v} : \mathbb{R}^3 \to \mathbb{R}^3$ be smooth. Prove:

$$\nabla \cdot (F\vec{v}) = \vec{\nabla} F \cdot \vec{v} + F\,\nabla \cdot \vec{v}$$

3) Part c) of this exercise was used in lecture to derive the formula for $B(T)$, the blackbody radiance at temperature T, and for the derivation of the Stephan-Boltzmann law for blackbody radiation.

a) Find the Fourier series for $f(x) = x^2$ on $[-\pi, \pi]$,

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

i.e., find all a_n and b_n.

b) Use part a) and Parseval's Identity, which says,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2),$$

to find the exact sum of the series,

$$\zeta(4) = \sum_{n=1}^{\infty}\frac{1}{n^4}.$$

(This is the Riemann-zeta function evaluated at 4.)

c) Prove that

$$\int_0^\infty \frac{x^3}{e^x - 1}dx = \frac{\pi^4}{15}$$

by first showing that

$$\frac{1}{e^x - 1} = \sum_{n=1}^{\infty} e^{-nx}$$

for $x > 0$, and then using integration by parts along with part b).

4) Andrews, problem 3.1.

5) This problem uses the notation of the class lecture on NASA's GRACE satellite mission, as well as the sign conventions of climate science and geodesy (as opposed to physics).

a) Use Poisson's equation to derive Gauss' law for gravitation. Show:

$$\int\int_{\partial\Omega} \vec{a} \cdot \vec{n}\, dS = 4\pi G \int\int\int_{\Omega} \rho dV$$

b) Suppose that a sphere of mass M (such as a planet) has the property that the density $\rho = \rho(r)$ depends only on radial distance r, from the center of the sphere, and goes continuously to zero at the boundary of the sphere. Use part a) and symmetry arguments to deduce Newton's theorem, according to which the magnitude of the acceleration a of a test particle at a distance r from the center of a sphere is given by,

$$a = \|\vec{a}\| = \frac{GM}{r^2}$$

c) Explain why this formula does not hold for satellites in orbit around the Earth.