International Baccalaureate Diploma Programme, Mathematics SL

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Documents reviewed:

- *Mathematics SL* (includes curriculum guide, internal & external assessment descriptions and criteria) IBO Diploma Programme, Mathematics SL, International Baccalaureate Organization, Geneva, CH-1218, Switzerland (2004)
- *Mathematics HL* (includes curriculum guide, internal & external assessment descriptions and criteria) IBO Diploma Programme, Mathematics SL, International Baccalaureate Organization, Geneva, CH-1218, Switzerland (2004)
- *Diploma Programme Mathematics SL Information Booklet,* International Baccalaureate Organization, Geneva, CH-1218, Switzerland (2004)
- Two complete Mathematics SL past exams, Papers 1 and 2, and markings, May 2006
- One complete Mathematics HL exam, Papers 1, 2, and 3, and markings, May 2006
- Three Mathematics SL Type I (Investigation) Portfolio project assignments, one student solution with teacher markings, 2006
- Three Mathematics SL Type II (modeling) Portfolio project assignments, two student solutions with teacher markings, 2005, undated.
- IB Diploma Programme, *Vade Mecum* 2007, Section G, International Baccalaureate Organization, 2006

Background

Students in the Diploma Programme choose to enroll either in a Standard Level (SL) or Higher Level (HL) mathematics course. The course, "Mathematics SL" is designed for 16 to 19 year old students who "will expect to need a sound mathematical background as they prepare for future studies in subjects such as chemistry, economics, psychology and business administration."[1] The more demanding "Mathematics HL," requiring 240 hours, prepares students "expecting to include mathematics as a major component of their university studies, either as a subject in its own right or within courses such as physics, engineering and technology." Each course assumes student knowledge of a list of topics under the heading of "Presumed Knowledge," but teachers are given the flexibility to include topics in that list depending on the needs of their students.

The focus of this report is on Mathematics SL, and the grades are for Mathematics SL only. However, some discussion of Mathematics HL is also included because of the overlap of topics, and to set a broader context for the first course.

The International Baccalaureate Organization (IBO) lists "aims" and "objectives" for its mathematics courses. The aims include cultivating an appreciation of historical and multicultural perspectives of mathematics, cultural and historical contexts, and the universality of mathematics as a means of communication. Students are expected to "develop logical, critical and creative thinking," "employ and refine their powers of abstraction and generalization," and "appreciate the consequences arising from technological developments," among other aims. The more specific objectives include "read, interpret and solve a given problem using appropriate mathematical terms," "select and use appropriate mathematical strategies and techniques," and "use appropriate technological devices as mathematical tools."

There is no mention of proofs in either list, but technology plays a central role in the both the SL and HL curricula. The *Mathematics SL* and *Mathematics HL* curriculum guides both state, "Students are expected to have access to a graphic display calculator (GDC) at all times during the course."

A student enrolled in Mathematics SL is evaluated on the basis of an external examination, created by the IBO, and an internal portfolio consisting of two projects assigned and evaluated by the classroom teacher, but moderated by the IBO.

The SL external examination constitutes 80% of the student's evaluation for the course and consists of two parts, Paper 1 and Paper 2, weighted equally (the HL course also includes a Paper 3). Paper 1 of the SL program consists of 15 short-answer questions, and Paper 2 is comprised of 5 extended response questions. Students are given one hour and 30 minutes for each of the papers, and each paper may include questions about any part of the course syllabus. Students are required to have access to a graphing calculator for both papers with the following functionalities:

• draw graphs with any viewing window

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[1] There are actually two other programs at the Standard Level (SL): "Mathematical Studies SL," which "is designed to build confidence and encourage an appreciation of mathematics in students who do not anticipate a need for mathematics in their future studies," and "Further Mathematics SL" which goes into greater depth than "Mathematics SL."

• solve equations numerically
• add and multiply and find inverse matrices
• find a numerical derivative at a point
• find a numerical definite integral

For the Mathematics HL exams, the list also includes "finding p-value," for statistical tests. However, calculators with Computer Algebra Systems (CAS) are not permitted on the exams.

Content

The course syllabus for Mathematics SL consists of seven topics to be covered in 150 hours of class time:

Topic 1—Algebra
Topic 2—Functions and equations
Topic 3—Circular functions and trigonometry
Topic 4—Matrices
Topic 5—Vectors
Topic 6—Statistics and probability
Topic 7—Calculus

Mathematics HL includes the same seven topics, but in greater depth and with a broader range of subtopics. HL math students must also study all the sub-topics in one of the following options listed in the syllabus:

Topic 8—Statistics and probability
Topic 9—Sets, relations and groups
Topic 10—Series and differential equations
Topic 11—Discrete mathematics

For both the SL and HL courses, ten hours of the allotted course times are devoted to a portfolio which is described below.

Strangely missing from the syllabi and "Presumed Knowledge" for both the SL and HL courses is a systematic development of synthetic geometry with proofs. The geometry topics within "Presumed Knowledge" include the Pythagorean theorem and its converse – well and good – but the remaining items in the list are devoted to little more than area and perimeter, very elementary analytic geometry, or just vocabulary.

The SL program offers exposure to a wide range of topics at the high school level and beginning university level mathematics, but sacrifices depth for breadth. The syllabus includes the Binomial Theorem, arithmetic and geometric series, law of sines, law of cosines, properties of dot product of vectors, binomial and normal distributions, expectations of discrete random variables, and topics in calculus.

However, missing from the SL curriculum is an introduction to complex numbers, the reciprocal trigonometric functions (secant, cosecant, cotangent), and all of the inverse trigonometric functions. Identities for trigonometric functions evaluated at the sum of two angles are missing, even though double angle formulas are included. Knowledge of these missing topics is normally assumed at the university level. Thus, these gaps in the SL curriculum constitute a serious deficit. Paradoxically, the term "one-to-one" is explicitly excluded from the SL curriculum, even though inverse functions are a part of the syllabus.

Little attention is given to mathematical proofs in the SL syllabus, and the reliance on technology is excessive. For example, for trigonometric functions, the SL syllabus instructs, "Double angle formulae can be established by simple geometrical diagrams and/or by use of a GDC [graphic display calculator]." Using a graphing calculator to show that the graphs of two functions appear to be the same can be used for motivation, but it does not replace a trigonometric proof.

The treatment of calculus in the SL course is thin. The definition of limit is not included, and the definition of derivative is used only to establish the derivatives of polynomial functions; other functions' derivatives are "justified by graphical considerations using a GDC." In particular, the derivative formulas for the three trigonometric functions studied in the course are used but evidently not derived. Missing too is any mention of the Mean Value Theorem or Riemann sums, even though students use the Fundamental Theorem of Calculus. Riemann sums are needed to explain the meaning of definite integrals, and the Mean Value Theorem is the single most important theoretical tool for justifying standard problem solving procedures in calculus. Implicit differentiation, a powerful analytical tool, is also not part of the SL syllabus. Points of inflection on the graph of a function are considered only in those cases where the second derivative exists (e.g. not for $y = x^{1/3}$). A good calculus course includes these topics.

The HL course also has some gaps in the more elementary topics. Its calculus syllabus excludes the Mean Value Theorem, Riemann sums, definition of limit, points of inflection where second derivative is not defined, and graphing functions by hand, as in the SL curriculum. However, the HL syllabus includes proof by induction, including a proof of De Moivre's Theorem as part of its treatment of complex numbers. It also includes a fuller treatment of trigonometry, including proofs of trigonometric identities, and row reduction of matrices.

External Exams: Papers 1 and 2

Students are not required to remember formulas for the external exams. Each examinee is provided with a clean copy of the *Mathematics SL Information Booklet* (or *Mathematics HL Information Booklet* for HL students). The *Mathematics SL Information Booklet* consists of four pages of formulas that students might need during the exams for Paper 1 and 2. Among them are the Binomial Theorem, the derivative of x^n and other basic functions, integral formulas, the quadratic formula, formulas for determinants of 2 by 2 and 3 by 3 matrices, dot product, mean, variance, formulas for area and volume, the distance formula, midpoint formula, and even fundamental properties of logarithms:

$$a^x = b \Leftrightarrow x = \log_a b, \qquad \log_a a^x = x = a^{\log_a x}, \qquad \log_b a = \frac{\log_c a}{\log_c b}$$

Access to such a list of formulas for the external exams is a weakness of the program. Students should be expected to understand and commit to memory all of these results.

Of the seven topics listed on the SL syllabus, statistics and probability is perhaps most heavily emphasized on the external exams that I reviewed. Of the 40 questions from both SL tests (including Papers 1 and 2), nine were probability or statistics problems. As an example, a problem on an SL Paper 1 exam provides students with frequency diagrams (with unlabeled axes) for four populations with the same range and size. Three graphical displays of unlabeled box-and-whisker plots and three graphical displays of unlabeled cumulative frequency diagrams are then given. The students are asked to associate each of these six graphical displays to one of the given frequency diagrams. The problem itself is well crafted, but the subject matter is highly specialized in the context of high school mathematics. Statistics and probability is overemphasized both in the syllabus and on the examinations.

There is little in the way of standard high school algebra on the SL exams, except for linear algebra which is well represented. Problems on the exams ask students to calculate dot products, norms, the vector from a given point to another point, equations of lines in terms of vectors and closely related topics. There are also problems involving matrices and systems of two or three linear equations. However, some of these problems are compromised by student access to calculators during the exams. For example, on an SL Paper 1 exam, the first part of one of the problems is the following:

The matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ -3 & 1 & -1 \\ 2 & -2 & 1 \end{pmatrix}$ has inverse $A^{-1} = \begin{pmatrix} -1 & -2 & -2 \\ 1 & 1 & 1 \\ a & 6 & b \end{pmatrix}$.

(a) Write down the value of
(i) *a*;
(ii) *b*.

This part of the problem is easily solved using the fact that the product of the above two matrices must be the identity matrix. However, an examinee is also free to type the entries of A into a calculator and then simply read off the entries of A^{-1} (including a and b) on the display, in which case the problem assesses nothing more than rote button pressing on a calculator. Both the HL and SL syllabi indicate that the one method to be examined for computing the inverse of a 3 by 3 matrix is use of a calculator, with explicit indication that no other method is expected. This is a poor use of technology.

Another SL Paper 1 problem that tests little beyond how to use a calculator is this one:

> Let $f(x) = 3 \sin 2x$, for $1 \leq x \leq 4$ and $g(x) = -5x^2 + 27x - 35$ for $1 \leq x \leq 4$. The graph of f is shown below.
>
> [The graph of f is displayed]
>
> (a) On the same diagram, sketch the graph of g.
> (b) One solution of $f(x) = g(x)$ is 1.89. Write down the other solution.
> (c) Let $h(x) = f(x) - g(x)$. Given that $h(x) > 0$ for $p < q$, write down the value of p and of q.

The main content of this problem is reading off the x-coordinates of the intersection of the two graphs on a calculator display.

Other problems do require some analytical skills, as for example, one that calls for the angular measure and radius of a sector of a circle, given a numerical length of arc subtending the angle, and given the numerical area of the sector. Problems in Paper 2 exams are broken into guided steps that require a demonstration of some algebraic skills as well. For the more rigorous HL course, the difficulty of the questions on the Paper 3 exam for Topics 8 through 11, listed above, is pitched at a university level, and goes well beyond the normal high school fare.

Overall, the external examinations for the SL course appear to represent the syllabus well (including the aforementioned overemphasis on statistics and probability), but too many of the problems are calculator based, and there is insufficient assessment of pencil and paper algebraic skills upon which much of a university curriculum depends.

Portfolio

A portfolio is required for both the SL and HL courses. For each course, the portfolio, constitutes 20% of a student's grade and ten hours of class time. It consists of two projects "representing the following two types of tasks: mathematical investigation, mathematical modeling." An example of mathematical investigation for the SL course, provided for this review, is an investigation of powers of a two by two symmetric matrix with k + 1 on the diagonal, and with off diagonal components both equal to k – 1. Students are given a list of tasks, including numerical calculations for specific examples, finding a general formula for the nth power, giving restrictions on k and n for their formulas, and finally explaining why their general formula is correct. The examples of mathematical modeling projects for the SL course that I examined require students to use the regression features of their calculators to find best-fit functions (polynomial or sinusoidal) for data that is provided, and then to use graphs to answer questions about the context for the data.

A detailed rubric is provided to teachers for evaluating the two projects. The rubrics for the SL and HL courses are the same, with a maximum of 29 points available for each project, with six criteria for each of the two projects. The criteria for the mathematical investigation are:

Criterion A	Use of notation and terminology
Criterion B	Communication
Criterion C	Mathematical process—searching for patterns
Criterion D	Results—generalization
Criterion E	Use of technology
Criterion F	Quality of work

The criteria for the mathematical modeling project are the same except that for Criterion C, the phrase, "searching for patterns" is replaced by "developing a model." "Criterion E: use of technology" is graded on a 3-point scale as follows:

0 The student uses a calculator or computer for only routine calculations.

1 The student attempts to use a calculator or computer in a manner that could enhance the development of the task.
2 The student makes limited use of a calculator or computer in a manner that enhances the development of the task.
3 The student makes full and resourceful use of a calculator or computer in a manner that significantly enhances the development of the task.

Full credit is evidently not available for projects of an algebraic or analytic nature which do not include the use of computing machines. Thus, for example, a correct proof of a general formula for the nth power of the matrices described above for the SL mathematical investigation, would not receive full credit unless it included superfluous calculator investigations.

Conclusions

The SL program is clearly laid out, and the syllabus and assessments are well aligned. Two strengths of the Mathematics SL course are its breadth of coverage and focus on problem solving skills. For those students for whom this is the last mathematics course ever to be taken, the curriculum is well chosen. It provides a glimpse into several parts of mathematics along with some practical skills, especially in the area of statistics.

On the other hand, if a student intends to take more mathematics courses at the university level, it is not clear how that student should be placed. What university mathematics courses have as prerequisites a small amount of calculus, but no exposure to complex numbers, almost no geometry, a spotty background in trigonometry, a smattering of linear algebra, and a good bit of statistics? Perhaps more statistics courses. The heavy reliance on calculators and virtually no memorization of formulas add to the deficits.

In fairness to the IBO program, highly motivated students follow the more rigorous HL syllabus. While far more complete, and mathematically advanced , as noted previously there are nevertheless some gaps even at this level. However, students with the ability to complete such a demanding curriculum are likely able to fill in missing topics on their own.

The Mathematics SL course shares many of the defects of mainstream U.S. high school mathematics programs, but unlike other programs its external examination prevents degeneration of the course below the level of its syllabus.

Grades for Mathematics SL

Clarity:	B
Content:	C
Rigor (Mathematical Reasoning):	D

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