Math 595MP – Assignment 3

Problem 1. Starting with the Planck function $B_\lambda(T)$ (which gives the power per unit area per unit solid angle per unit wavelength λ) derive the the explicit expression for the black body radiance in terms of T and frequency ν given by,

$$B_\nu(T) = \frac{2h\nu^3}{c^2[\exp(h\nu/k_BT) - 1]}.$$

Hint: See the formula for $B_\lambda(T)$ from lecture or in Prob 1, pg 23 of the textbook, and recall from lecture that $B_\lambda(T)$ and $B_\nu(T)$ must be related by,

$$\int_{\lambda_1}^{\lambda_2} B_\lambda(T)d\lambda = \int_{\nu_2}^{\nu_1} B_\nu(T)d\nu$$

where $\nu_i\lambda_i = c$ for $i = 1, 2$.

Problem 2. Derive the Stephan–Boltzmann Law $F = \sigma T^4$ for Blackbody radiation by completing the following steps:

a) Find the Fourier series for $f(x) = x^2$ on $[-\pi, \pi]$,

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

i.e., find all a_n and b_n.

b) Use part a) and Parseval's Identity, which says,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2),$$

to find the exact sum of the series,

$$\zeta(4) = \sum_{n=1}^{\infty}\frac{1}{n^4}.$$

(This is the Riemann-zeta function evaluated at 4.)

c) Prove that

$$\int_0^\infty \frac{x^3}{e^x - 1}dx = \frac{\pi^4}{15}$$

by first showing that

$$\frac{1}{e^x - 1} = \sum_{n=1}^{\infty} e^{-nx}$$

for $x > 0$, and then using integration by parts along with part b).

d) Complete the calculation from lecture,

$$F = \pi\int_0^\infty B_\lambda(T)d\lambda = \sigma T^4$$

To do this, first notice from problem 1, $F = \pi\int_0^\infty B_\nu(T)d\nu$. Calculate this integral by making the substitution, $x = h\nu/k_BT$, use part c), and deduce that the Stephan-Boltzmann constant σ is given by,

$$\sigma = \frac{2\pi^5 k_B^4}{15h^3c^2}.$$