INTERNATIONAL RESEARCH EXPERIENCE FOR STUDENTS
UNIVERSIDADE ESTADUAL DE CAMPINAS - UNICAMP

ON THE COMPACT SUBMANIFOLDS OF CODIMENSION 3
WITH NONNEGATIVE CURVATURE

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Support: CNPq/NSF
CAMPINAS, SP - BRAZIL
JULY/2006
In this project, we’ve studied the holonomy algebra of a manifold immersed in $\mathbb{R}^{n+3}$.

Also, we applied our classification result in a problem related to covering spaces of a compact manifold with non-negative curvature.
We will denote by $\mathbb{R}^N_p$ the set of vectors that have origin at $p$.

**Definition**

A **vector field** in $\mathbb{R}^N$ is a map $X : \mathbb{R}^N \to \mathbb{R}^N_p$.

Identifying the vectors $\{e_i\}_{i=1}^N$ of the standard basis of $\mathbb{R}^N$ with their translates $\{e_i^p\}_{i=1}^N$ at the point $p$, we write

$$X(q) = \sum_{i=1}^{N} a_i(q)e_i.$$  

A vector field $X$ is said to be differentiable if the functions $a_i$’s are differentiable.
Covariant derivatives and connections

Definition

Let $X$ be a differentiable vector field in $\mathbb{R}^N$ and $c : [0, 1] \rightarrow \mathbb{R}^N$ a differentiable curve in $\mathbb{R}^N$. Setting $X(c(t)) = (x_1(t), \ldots, x_N(t))$, the covariant derivative of $X$ over $c(t)$ is given by

$$
\frac{DX}{dt} = (x'_1(t), \ldots, x'_N(t)).
$$

Definition

Given two vector fields $X$ and $Y$ in $\mathbb{R}^N$, we define the connection $\tilde{\nabla}_X Y$ as the vector field that at $p \in \mathbb{R}^N$ is given by the following procedure: Let $c(t)$ be a curve such that $c(0) = p$ and $c'(0) = X(p)$. Consider $Y$ defined over $c(t)$, $Y(c(t)) = (y_1(t), \ldots, y_N(t))$. Then,

$$
\tilde{\nabla}_X Y(p) = \frac{DY}{dt}(0) = (y'_1(t), \ldots, y'_N(t)).
$$
Definition

Let $\mathcal{M}$ be an topological Hausdorff space with a countable basis. Then $\mathcal{M}$ is said to be a **manifold** if there exists a natural number $n$ and for each point $p \in \mathcal{M}$ an open neighbourhood $U$ of $p$ and a continuous map $\phi : U \to \mathbb{R}^n$ which is a homeomorphism onto its image $\phi(U)$, an open subset of $\mathbb{R}^n$. The pair $(U, \phi)$ is called a **chart** on $\mathcal{M}$. The natural number $N$ is called the **dimension** of $\mathcal{M}$. To denote that the dimension of $\mathcal{M}$ is $n$ we write $\mathcal{M}^n$. 
Figure: Manifold: an easier definition
**Immersions and tangent space**

**Definition**

We say that a map \( f : \mathcal{M}^n \to \mathbb{R}^N \) is an **immersion** if for every \( p \in \mathcal{M} \), exists an open set \( W \subseteq \mathcal{M} \) (\( p \in W \)) and a map \( \phi : W \to U \subseteq \mathbb{R}^n \), \( U \) open in \( \mathbb{R}^n \), such that the jacobian matrix of \( f \circ \phi^{-1} : U \to \mathbb{R}^N \) has rank \( n \).

For a differentiable curve \( \alpha : (-\epsilon, \epsilon) \to \mathcal{M} \) such that \( \alpha(0) = p \), the vector \( \alpha'(t) \) is said to be a tangent vector to \( \mathcal{M} \) in the point \( p \).

**Definition**

For a point \( p \in \mathcal{M} \), we define the **tangent space** \( T_p \mathcal{M} \) as follows:

\[
T_p \mathcal{M} = \{ \alpha'(0) \mid \alpha : (-\epsilon, \epsilon) \to \mathcal{M}, \ \alpha(0) = p \}.
\]

The number \( N - n \) is called **codimension** of the immersion.
**Figure:** The (green) tangent space of the (orange) manifold is the set of (yellow) tangent vectors of (black) curves in the point $p$. 
**Definition**

A (tangent) vector field in $\mathcal{M}$ is a map $X$ that associates to each point $p \in \mathcal{M}$ a vector $X(p) \in T_p \mathcal{M}$.

Given two vector fields on $\mathcal{M}$, $X$ and $Y$, we can extend them to $\tilde{X}$ and $\tilde{Y}$ vector fields in $\mathbb{R}^N$, then define a connection in $\mathcal{M}$ as

$$\nabla_X Y(p) = (\tilde{\nabla}_{\tilde{X}} \tilde{Y}(p))^T,$$

where $(\cdot)^T$ denotes the orthogonal projection onto $T_p \mathcal{M}$. So, we have:

$$\tilde{\nabla}_{\tilde{X}} Y = \nabla_X Y + (\tilde{\nabla}_{\tilde{X}} \tilde{Y})^N,$$

with $(\cdot)^N$ the orthogonal projection onto $T_p^\perp \mathcal{M}$.

We denote $(\tilde{\nabla}_{\tilde{X}} \tilde{Y})^N$ by $\alpha(X, Y)$ and call it the second fundamental form of the immersion. Note that $\alpha$ is a symmetric bilinear form.
Normal field and shape operators

Definition

A normal vector field in $\mathcal{M}$ is a map $\xi$ that associates to each point $p \in \mathcal{M}$ a vector $\xi(p) \in T_p^\perp \mathcal{M}$.

Given a tangent vector field in $\mathcal{M}$ and a normal vector field $\xi$, we define the normal connection as

$$\nabla^\perp_X \xi = (\tilde{\nabla}_X \xi)^N.$$

The tangent component of $\tilde{\nabla}_X \xi$ defines an operator

$$A_\xi : T_p \mathcal{M} \to T_p \mathcal{M},$$

called shape operator (or Weingarten operator), given by

$$A_\xi(X) = -(\tilde{\nabla}_X \xi)^T.$$
Some identities

The shape operator is symmetric. Now we’ll present some important identities.

The Gauss formula:

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X, Y)$$  \hspace{1cm} (1)

$$\langle A_\xi X, Y \rangle = \langle X, A_\xi Y \rangle = \langle \xi, \alpha(X, Y) \rangle$$ \hspace{1cm} (2)
Curvature tensor

Definition

Let $\mathcal{X} (\mathcal{M})$ denote the set of vectors fields in $\mathcal{M}$. For $X, Y, Z \in \mathcal{X} (\mathcal{M})$ we define the **curvature tensor** as

$$\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

and the **sectional curvature** of the plane spanned by $X$ and $Y$ as

$$k(X, Y) = \frac{\langle \mathcal{R}(X,Y)Y, X \rangle}{\|X \wedge Y\|^2}.$$
Using the equations 1 and 2, we can express the curvature tensor and the sectional curvature as

\[ R(X, Y)Z = A_{\alpha(Y,Z)}X - A_{\alpha(X,Z)}Y \]  

(3)

\[ k(X, Y) = \langle \alpha(X, X), \alpha(Y, Y) \rangle - \langle \alpha(X, Y), \alpha(X, Y) \rangle^2 \]  

(4)

The equation 3 is known as Gauss Equation, and it implies the equation 4.
Fundamental theorem of submanifolds

Let

\[(D_X\alpha)(Y, Z) = \nabla^\perp_X \alpha(Y, Z) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z)\]

\[(D_Y\alpha)(X, Z) = \nabla^\perp_Y \alpha(X, Y) - \alpha(\nabla_Y X, Z) - \alpha(X, \nabla_Y Z).\]

Codazzi Equation:

\[(D_X\alpha)(Y, Z) = (D_Y\alpha)(X, Z) \quad (5)\]

Ricci Equation:

\[\langle R^\perp(X, Y)\xi_1, \xi_2 \rangle = \langle [A_{\xi_1}, A_{\xi_2}](X), Y \rangle \quad (6)\]

Theorem

If \(\mathcal{M}^d\) is a \(d\)-dimensional manifold, then \(\mathcal{M}\) may be isometrically and locally embedded in a \(D\)-dimensional space \(\mathcal{M}^D\) if and only if its metric, twisting vector and the extrinsic curvature satisfy the Gauss, Codazzi and Ricci equations.
Let $\mathcal{V}$ be an $\mathbb{F}$-vector space. Given two vectors $u, v \in \mathcal{V}$, we define $u \wedge v : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ as

$$(u \wedge v)(a, b) = u^*(a)v^*(b) - u^*(b)v^*(a),$$

for $a, b \in \mathcal{V}$, where $u^*$ and $v^*$ are the functionals associated with $u$ and $v$.

**Definition**

The **exterior algebra** of $\mathcal{V}$, denoted by $\bigwedge^2(\mathcal{V})$, is the set of all possible wedge products $u \wedge v$, for $u, v \in \mathcal{V}$. 

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The exterior algebra is a $\mathbb{F}$-vector space, with the usual sum and scalar product. Moreover, $\bigwedge^2(V)$ is a Lie algebra, with the bracket defined by

$$[x \wedge y, z \wedge w] = \langle x, z \rangle y \wedge w + \langle y, w \rangle x \wedge z - \langle x, w \rangle y \wedge z - \langle y, z \rangle x \wedge w,$$

for $x \wedge y, z \wedge w \in \bigwedge^2(V)$

We can identify $\bigwedge^2(V)$ with $o(V)$, the algebra of skew-symmetric operators on $V$ in this way: for $x \wedge y \in \bigwedge^2(V)$ we associate the operator $(x \wedge y) : V \to V$ such that, for $z \in V$, $(x \wedge y)(z) = \langle x, z \rangle y - \langle y, z \rangle x$. 
We’ll start the transition from geometry to algebra.

**Definition**

Let $\mathcal{M}$ be a manifold and $p \in \mathcal{M}$. We define the **curvature operator** as the operator $R : \bigwedge^2 (T_p \mathcal{M}) \to \bigwedge^2 (T_p \mathcal{M})$ implicit given by

$$\langle R(X \wedge Y)Y, X \rangle = \langle R(X \wedge Y), X \wedge Y \rangle.$$

The curvature operator in $p$ may be also given as

$$R(X \wedge Y) = \sum_{\xi} A_{\xi}(X) \wedge A_{\xi}(Y), \quad (7)$$

for $\xi$ the normal fields in $p$ (in other words, $A_{\xi}$ are the components of 2nd fundamental form). As our spaces are finite-dimensional, we adopt the notation \( \{S_i\}_i \) for \( \{A_{\xi_i}\}_i \).
Curvature operator

Some definitions

Given a symmetric operator $S$ we denote by $D(S)$ the subspace which is simultaneously the range of $S$, the orthogonal complement of the kernel of $S$, and the span of the nonnullity eigenvectors of $S$.

For a immersed manifold $\mathcal{M}$, if $p$ is its codimension, the relative curvature subspace at $m$ is $k(m) = D(S_1) + \ldots + D(S_p)$.

The Lie group consisting of parallel translations of $\mathcal{M}$ around loops based at $m$ is called the holonomy group at $m$. The holonomy groups at different points are isomorphic. We shall denote its Lie algebra $h_m$. In fact, $h_m$ is spanned by the parallel translates to $m$ of the curvature transformations at all of the points of $\mathcal{M}$.

Denote by $r(m)$ the Lie algebra generated by the curvature transformations at $m$; thus $r(m)$ is a subalgebra of $h_m$ and it is clear that $r(m) \subseteq \bigwedge^2(k(m))$ for every $m$. 
We will present a (partial) classification theorem for $r(m)$ of immersed manifolds with codimension 3.

Just to fix our notations, let $S_1$, $S_2$, $S_3$ be the components of the 2nd fundamental form of $\mathcal{M}$ at the point $m$ and:

$$R(x \wedge y) = S_1(x) \wedge S_1(y) + S_2(x) \wedge S_2(y) + S_3(x) \wedge S_3(y),$$

$$\mathcal{V}_1 = D(S_1), \quad \mathcal{V}_2 = D(S_2), \quad \mathcal{V}_3 = D(S_3),$$

$$k(m) = \mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3,$$

$$r(m) = \text{algebra generated by } \text{Im}(R).$$
Our manifolds satisfy the following relations: if $Z_i$ are the sets defined by

$$Z_i = \{ v \in V \mid v \in V_j, \ j \neq i \},$$

then we assume that

$$Z_i \neq 0, \ i = 1, 2, 3.$$  \hfill (8)

**Figure:** We need one element in $V_i$ which is not on any other $V_j$. 

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**Equation**

$\star$
The classification for $r(m)$ will be given after some lemmas. We will work with the cases $\mathcal{V} \neq \mathcal{V}_1$ and $S_1$ has nonzero signature.

**Lemma**

$\bigwedge^2(\mathcal{V})$ is generated by $\mathcal{V}_1 \wedge \mathcal{V}_2 + \mathcal{V}_1 \wedge \mathcal{V}_3$.

**Lemma**

If $v \neq 0$ is a vector in $\mathcal{V}$, then $v \wedge \mathcal{V}$ generates $\bigwedge^2(\mathcal{V})$.

**Lemma**

If $v_1 \in \mathcal{V}_1$, $v_2 \in \mathcal{V}_2$ and $v_3 \in \mathcal{V}_3$, all of them nonzero with $\mathcal{V}_1 \cap \mathcal{V}_2 \neq 0$ and $\mathcal{V}_1 \cap \mathcal{V}_3 \neq 0$, then $\bigwedge^2(\mathcal{V})$ is generated by $v_1 \wedge \mathcal{V}_1 + v_2 \wedge \mathcal{V}_2 + v_3 \wedge \mathcal{V}_3$. 
Some lemmas

**Lemma**

If $v_i \in \mathcal{V}_i$, $i = 1, 2, 3$ are vectors such that $v_1 \wedge (v_2 + v_3) \neq 0$ then $\bigwedge^2(\mathcal{V})$ is generated by $\bigwedge^2(\mathcal{V}_1) + \bigwedge^2(\mathcal{V}_2) + \bigwedge^2(\mathcal{V}_3) + v_1 \wedge v_2 + v_1 \wedge v_3$.

**Lemma**

If $\mathcal{V}_i \perp \mathcal{V}_j$ for every $i, j = 1, 2, 3$, if dim $\mathcal{V}_1$ or dim $(\mathcal{V}_2 + \mathcal{V}_3)$ do not equals 2, and if dim $\mathcal{V}_2$ or dim $\mathcal{V}_3$ do not equals 2, then $X = \bigwedge^2(\mathcal{V}_1) + \bigwedge^2(\mathcal{V}_2) + \bigwedge^2(\mathcal{V}_3)$ is a proper subalgebra of $\bigwedge^2(\mathcal{V})$.

**Lemma**

If $\mathcal{V}_1 \cap (\mathcal{V}_2 + \mathcal{V}_3) = 0$, $\mathcal{V}_2 \cap \mathcal{V}_3 = 0$, $\mathcal{V}_1 \neq (\mathcal{V}_2 + \mathcal{V}_3) \perp$, $\mathcal{V}_2 \not\in \mathcal{V}_3 \perp$ and dim $\mathcal{V}_1 > 2$, dim $\mathcal{V}_2 > 2$, dim $\mathcal{V}_3 > 1$, then $\bigwedge^2(\mathcal{V})$ is generated by $\bigwedge^2(\mathcal{V}_1) + \bigwedge^2(\mathcal{V}_2) + \bigwedge^2(\mathcal{V}_3)$. 
Lemma

The range of $R$ contains

$$S_1(V_2 + V_3)^\perp \wedge V_1 + S_2(V_1 + V_3)^\perp \wedge V_2 + S_3(V_1 + V_2)^\perp \wedge V_3.$$ 

If $V_i \cap (V_j + V_k)$ for $i, k \neq i$, then the range of $R$ equals

$$\bigwedge^2(V_1) + \bigwedge^2(V_2) + \bigwedge^2(V_3).$$

Lemma

The subalgebra generated by $\bigwedge^2(V_1) + \bigwedge^2(V_2) + \bigwedge^2(V_3)$ is $r(m)$. 
Theorem

Let $\mathcal{M}$ be a manifold, $m \in \mathcal{M}$ and $r(m)$ the Lie algebra generated by the image of curvature operator $R : T_m\mathcal{M} \rightarrow T_m\mathcal{M}$, as defined in 7. There are only the following options for $r(m)$:

$$r(m) = \begin{cases} 
  o(n) \\
  o(k) \oplus u(2) \\
  o(k) \oplus o(n - k) \\
  o(k_1) \oplus o(k_2) \oplus o(n - k_1 - k_2)
\end{cases}$$
Covering spaces

**Definition**

Let $\mathcal{M}$ be a manifold. We say that a pair $(E, p), p : E \to \mathcal{M},$ is a **covering space** of $\mathcal{M}$ if for every $m \in \mathcal{M}$ has an open set $m \in U \subseteq \mathcal{M}$ such that $p^{-1}(U)$ is a disjoint union of open sets $\omega_i$ in $E,$ each of which is mapped homeomorphically onto $U$ by $p.$

**Definition**

We say that a manifold $\mathcal{M}$ has **nonnegative sectional curvature** if for every point $p \in \mathcal{M}$ and every plane $\{X, Y\},$ where $X$ and $Y$ are vector fields in $T_p\mathcal{M},$ we have $k(X, Y) \geq 0.$
Let $\sigma : [0, 1] \to X$ and $\tau : [0, 1] \to X$ be paths in a topological space $X$ with the same endpoints, $\sigma(0) = \tau(0) = x_0$, $\sigma(1) = \tau(1) = x_1$. We say that $\sigma$ and $\tau$ are **homotopic** if there is a continuous map $F : [0, 1] \times [0, 1] \to X$ such that

i) $F(s, 0) = \sigma(s)$

ii) $F(s, 1) = \tau(s)$

iii) $F(0, t) = x_0$

iv) $F(1, t) = x_1$
Consider now a topological space $X$ and $x_0 \in X$. If $c$ is a loop at $x_0$, let $[c]$ denote the equivalence class of all loops at $x_0$ that are homotopic to $c$.

Let $\Pi_1(X, x_0)$ be the group of all homotopy classes of loops at $x_0$, with the composition operation.

**Theorem**

If $X$ is path connected, then for all $x, y \in X$, $\Pi_1(X, x) \cong \Pi_1(X, y)$. 

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Thanks to uniqueness (up to isomorphism), we can define the **fundamental group of** $X$ and denote it by $\Pi_1(X)$.

**Definition**

We say that a space $X$ is **simply connected** if $\Pi_1(X) = \{1\}$, that is, every loop is homotopically trivial.

**Theorem**

*Every connected manifold has a covering that is simply connected. Such space is unique (up to diffeomorphisms) and is called universal covering.*

**Theorem**

*Let $\mathcal{M}$ be a compact manifold with a compact universal covering. Then $\Pi_1(\mathcal{M})$ is finite.*
Theorem (Cheeger-Gronoll)

Let $\mathcal{M}$ be a compact manifold such that $k \geq 0$. Then the universal covering $\tilde{\mathcal{M}}$ is given by

$$\tilde{\mathcal{M}} = \tilde{\mathcal{M}} \times \mathbb{R}^l,$$

where $\tilde{\mathcal{M}}$ is a compact manifold.

Let $\mathcal{M}$ be a compact manifold, $k \geq 0$, $(\tilde{\mathcal{M}}, p)$ a covering space and $f$ an immersion.

$$\tilde{\mathcal{M}} \rightarrow^p \mathcal{M} \rightarrow^f \mathbb{R}^{n+3}$$

Then $\tilde{f} = f \circ p : \tilde{\mathcal{M}} \rightarrow \mathbb{R}^{n+3}$ is an immersion.
Cheeger-Gronoll and consequences

Theorem

exists \( x \in \mathcal{M} \) such that \( \dim N(x) = 0 \), where

\[
N(x) = \{ X \in T_p\mathcal{M} \mid \alpha(X, Y) = 0, \ \forall \ Y \in T_p\mathcal{M} \}.
\]

So, for \( \tilde{f} \), there exists \( \tilde{x} \in \tilde{\mathcal{M}} \) such that \( \dim N(\tilde{x}) = 0 \) or, in other words, \( \dim \mathcal{V} = n \).

In this case, we can apply our classification theorem.
As we have proved, the options for \( r(\tilde{x}) \) are

\[
r(\tilde{x}) = \begin{cases} 
  o(n) \\
  o(k) \oplus u(2) \\
  o(k) \oplus o(n - k) \\
  o(k_1) \oplus o(k_2) \oplus o(n - k_1 - k_2) 
\end{cases}
\]

But by Cheeger-Gronoll theorem we know that \( r(\tilde{x}) \subseteq o(n - l) \oplus o(l) \).
Let us analyse the options for \( l \).
If \( l = 0 \), then
\[
\tilde{M} = \tilde{M} \times \mathbb{R}^0.
\]
So \( M \) and \( \tilde{M} \) are compact and \( \Pi(M) \) is finite.

If \( l = 1 \) then we have the following options:
- if \( n = 5 \), then \( o(n - l) \) becomes \( o(4) \), and \( u(2) \subseteq o(4) \). So, \( r(\tilde{x}) \) may be \( u(2) \).
- the other options are
  - \( o(1) \oplus o(n - 1) \)
  - \( o(1) \oplus o(k) \oplus o(n - 1 - k) \)

If \( l = 2 \) we have \( r(\tilde{x}) \subseteq o(n - 2) \). The only options where \( o(n - 2) \) appears is when \( r(\tilde{x}) = o(1) \oplus o(1) \oplus o(n - 2) \).
For $l \geq 3$, our classification theorem says that we can’t have nothing newer. So, we have the following:

**Theorem**

Let $\mathcal{M}^n$ a compact manifold immersed in $\mathbb{R}^{n+3}$, with $k \geq 0$ and $\tilde{\mathcal{M}}$ as a covering space. Then, either $\Pi_1(\mathcal{M})$ is finite or

$$\tilde{\mathcal{M}} = \tilde{\mathcal{M}}^{n-l} \times \mathbb{R}^l,$$

for $l \leq 2$. 
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The research done on this article was performed during July 2006 while participating in IRES at IMECC/Unicamp, funded by National Science Foundation (USA) and CNPq (Brasil).

The authors of the paper would like to thanks our advisor Maria Helena Noronha and the rest of the mathematics department at the State University of Campinas for their hospitality and valuable help in this project. Specially, we also wish to thank Helena Lopes for the organization of the event.