On the isometric immersions of Kähler manifolds of nonnegative isotropic curvature

Francesco Mercuri* Maria Helena Noronha

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Abstract

We first study isometric immersions of Kähler manifolds of nonnegative isotropic curvature in Euclidean spaces with low codimension. We conclude that such manifolds have nonnegative sectional curvatures and are covered by the Riemannian product $S^{2s} \times \mathcal{P}^{2r} \times N \times \mathbf{C}^l$, where each factor is a Kähler manifold such that S^{2s} is a compact manifold and $\mathcal{P}^{2r}$ diffeomorphic to Euclidean space. We also study isometric immersions of Kähler manifolds of nonnegative isotropic curvature with flat normal bundle. We show that they are covered by the Riemannian product $S^{2s} \times \mathcal{P}^{2r} \times \mathbf{C}^l$, where S is a Riemannian product of spheres and $\mathcal{P}$ diffeomorphic to the Euclidean space.

1 Introduction

Let M be an n-dimensional Riemannian manifold with $n \geq 4$. For x in M we consider the complexified tangent space $T_xM \otimes \mathbf{C}$ and we extend the Riemannian metric $\langle\ ,\ \rangle$ to a complex bilinear form $(,)$. An element Z in $T_xM \otimes \mathbf{C}$ is said to be *isotropic* is $(Z, Z) = 0$. A two-plane $I \subset T_xM \otimes \mathbf{C}$ is called *totally isotropic* if $(Z, Z) = 0$ for any $Z \in I$. The Riemannian metric can also be extended to a Hermitian inner product, denoted by $\langle\langle\ ,\ \rangle\rangle$, so that

$$\langle\langle Z, W\rangle\rangle = (Z, \overline{W}), \text{ for } Z, W \in T_xM \otimes \mathbf{C}.$$

Let $\widetilde{\mathcal{R}}$ denote the complex linear extension of the curvature operator $\mathcal{R} : \Lambda^2 \to \Lambda^2$. Let I be a totally isotropic two-plane spanned by Z and W. The *isotropic curvature* of I, denoted by K_I, is defined as (see [6])

$$K_I = \frac{\langle\langle \widetilde{\mathcal{R}}(Z \wedge W), Z \wedge W\rangle\rangle}{||Z \wedge W||^2}.$$

In [7], the authors proved that a compact Kähler manifold of $K_I \geq 0$ is simply connected, its second Betti number is 1 and it has positive first Chern class. They also conjectured that it is biholomorphic to a Hermitian Symmetric Space of compact type. This conjecture is still wide open.

Recall that the complexified tangent bundle $TM \otimes \mathbf{C}$ of a Kähler manifold splits as $TM = T^{(1,0)} \otimes T^{(0,1)}$ and a vector in the holomorphic bundle $T^{(1,0)}$ is written as $Z = X - iJ(X)$, where J denote the parallel complex structure and $i^2 = -1$. The *bisectional curvature*, K_b, of a Kähler manifold is given by

$$K_b(Z, \bar{W}) = \frac{\langle\langle \widetilde{\mathcal{R}}(Z \wedge \bar{W}), Z \wedge \bar{W}\rangle\rangle}{||Z \wedge \bar{W}||^2}.$$

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In [8], compact Kähler manifolds of nonnegative bisectional curvatures were classified. They are covered by Riemannian products of Hermitian Symmetric Spaces of compact type with the complex space and with manifolds biholomorphic to the complex projective space.

It is obvious that if Z and $\bar{W}$ span a totally isotropic two-plane I, then $K_I = K_b(Z, \bar{W})$. However the nonnegativity of the isotropic curvature does not imply the nonnegativity of the bisectional curvature. Likewise, $K_b \geq 0 \not\Rightarrow K_I \geq 0$. This is easily seen by considering a compact manifold N with $K_b \geq 0$ but with some negative sectional curvature. The Riemannian product $M = N \times T^2$, where T^2 denotes the flat torus, is still a compact Kähler manifold of $K_b \geq 0$ but M does not have $K_I \geq 0$ since the latter would imply that the sectional curvatures of N are nonnegative (see proof of Proposition 2.1). The relationship between these two types of curvature, if exists, is not well understood.

Complete non-compact Kähler manifolds of $K_b \geq 0$ and complete non-compact Kähler manifolds of $K_I \geq 0$ are even less understood. In this paper we classify the ones with $K_I \geq 0$ that admit isometric immersions in Euclidean Space with codimension $p < n-1$, $2n$ being the manifold dimension. We also study Kähler submanifolds of Euclidean spaces and $K_I \geq 0$ and flat normal bundle.

Before stating the results we recall that if $f : M^n \to \mathbf{R}^{n+p}$ is an isometric immersion, the space

$$NR(x) = \{X \in T_xM | \alpha(X,Y) = 0, \forall Y \in T_xM\},$$

where α denotes the second fundamental form, is called the *Relative Nulllity Space*, and its dimension $\nu_f(x)$ is called the *index of relativity nullity*. In the following we use the notation $\bar{\nu}_f = min_{x \in M} \nu_f(x)$.

Our first result is the following theorem.

Theorem 1.1 *Let $f : M^{2n} \to \mathbf{R}^{2n+p}$ be an isometric immersion of a complete Kähler manifold of nonnegative isotropic curvature such that $p < n-1$. Then $M = M_1^{2m} \times \mathbf{C}^d$, $d \geq n-p$, $f = f_1 \times i$ and $\bar{\nu}_{f_1} = 0$. Moreover, the universal covering $\tilde{M}_1$ is the Riemannian product $S^{2s} \times \mathcal{P}^{2r} \times N$, where each factor is a Kähler manifold of nonnegative sectional curvature such that:*
(i) S^{2s} is a compact manifold.
(ii) $\mathcal{P}^{2r}$ is diffeomorphic to the Euclidean space.
(iii) N splits in a Riemannian product $\Sigma \times \mathcal{Q}$, where Σ is compact and $\mathcal{Q}$ is diffeomorphic to the Euclidean space.

It follows from the classification of compact manifolds with nonnegative bisectional curvature ([8]) that S^{2s} above is either an Hermitian Symmetric Space of compact type or biholomorphic to the Complex Projective Space. However, nothing can be said about the complex structure of the complete Kähler manifold $\mathcal{P}^{2r}$, which is diffeomorphic to Euclidean space, since the problem of classifying complete Kähler manifolds of positive (nonnegative) sectional curvature is still open.

We also obtain the following splitting theorem for complete Kähler manifolds of $K_b \geq 0$.

Theorem 1.2 *Let $f : M^{2n} \to \mathbf{R}^{2n+p}$ be an isometric immersion of a complete Kähler manifold of nonnegative bisectional curvature such that $p < n$. Then $M = M_1^{2m} \times \mathbf{C}^d$ and $f = f_1 \times i$, where $i : \mathbf{C}^d \to \mathbf{C}^d$, $d \geq n-p$. Moreover $\bar{\nu}_{f_1} = 0$ and $f_1 : M^{2m} \to \mathbf{R}^{2m+p}$ with $p \geq m$.*

In the last section of this paper we focus on isometric immersions that have flat normal bundle. Since these manifolds have pure curvature tensor (see Definition 4.1), their classification will follow from the following result.

Theorem 1.3 *Let M^{2n}, $n \geq 3$, be a complete Kähler manifold with pure curvature tensor and nonnegative isotropic curvature. Then M has nonnegative sectional curvatures and is covered by $S^{2s} \times \mathcal{P}^{2r}$, where S^{2s} splits in a Riemannian product of surfaces homeomorphic to the sphere and $\mathcal{P}^{2r}$ is a complete Kähler manifold diffeomorphic to the Euclidean space.*

We point out that the above result does not hold for $n = 2$, since the product of the sphere S^2 with the Hyperbolic plane is a Kähler manifold that admits a metric with pure curvature tensor and nonnegative isotropic curvature but does not have nonnegative sectional curvatures. Theorem 1.3 together with Hartman's Theorem [4] imply the following theorem.

Theorem 1.4 *Let $f : M^{2n} \to \mathbf{R}^{2n+p}$, $n \geq 3$, be an isometric immersion of a complete Kähler manifold of nonnegative isotropic curvature and $R^{\perp} = 0$. Then $M = M_1^{2m} \times \mathbf{C}^d$, $f = f_1 \times i$ and $\bar{\nu}_{f_1} = 0$. Moreover, $\tilde{M}_1$ is Kähler manifold of nonnegative sectional curvatures and pure curvature tensor and hence as in Theorem 1.3.*

2 A splitting result for submanifolds of nonnegative isotropic curvature

We start this section by pointing out that if I is a totally isotropic two-plane, we find 4 orthonormal vectors $e_1, \ldots, e_4$ such that $Z = e_1 + ie_2$ and $W = e_3 + ie_4$ span I and

$$K_I = K(e_1, e_3) + K(e_1, e_4) + K(e_2, e_3) + K(e_2, e_4) + 2\langle R(e_1, e_2)e_3, e_4\rangle, \tag{1}$$

where $K(X, Y)$ denotes the secional curvature of the plane spanned by X and Y.

Proposition 2.1 *Let $f : M^n \to \mathbf{R}^{n+p}$ be an isometric immersion of a complete manifold of nonnegative isotropic curvature. If $\bar{\nu}_f \geq 1$ then $M = M_1^m \times \mathbf{R}^k$, where $k = \bar{\nu}_f$, and $f = f_1 \times i$, where $i : \mathbf{R}^k \to \mathbf{R}^k$ is the identity map and $\bar{\nu}_{f_1} = min_{x \in M_1} \nu_{f_1}(x) = 0$. Moreover, if $\bar{\nu}_f \geq 2$, then M_1 and thus M have nonnegative sectional curvatures.*

Proof Let U denote a unit vetor in the Relative Nullity Space. Then we consider the isotropic plane spanned by $Z = X + iV$ and $W = Y + iU$, where X, Y, V and U are orthonormal vectors. From equation (1) we get that

$$K(X, Y) + K(V, Y) \geq 0,$$

for all orthonormal vectors X, Y, and V, which implies that M has nonnegative Ricci curvature. Therefore the result follows from Hartman's Theorem [4]. Moreover, if $\bar{\nu}_f \geq 2$ we then take V in the Relative Nullity Space as well and we conclude that $K(X, Y) \geq 0$. ■

Corollary 2.2 *Let $f : M^n \to \mathbf{R}^{n+p}$ be an isometric immersion of a complete manifold of nonnegative isotropic curvature. If $k = \bar{\nu}_f \geq 2$, then $M = M_1^m \times \mathbf{R}^k$ and the universal covering $\tilde{M}_1$ splits in a Riemannian product of $S^l \times \mathcal{P}^{m-l}$, where S is the soul of M and $\mathcal{P}^{m-l}$ a complete manifold diffeomorphic to the Euclidean Space $\mathbf{R}^{m-l}$.*

Proof If $\tilde{M}_1$ is not compact, since $\tilde{M}_1$ has nonnegative sectional curvature, we use the Soul Theorem of Cheeger and Gromoll ([1]). If the sould is a point we have the result for $l = 0$. Since $\tilde{M}_1$, the soul cannot have dimension 1. If the codimension of the soul is 1, the result follows from the Soul Theorem. For the remainder cases, we consider two isotropic planes, one spanned by $Z = X + iY$ and $W = U + iV$ and a second spanned by $Z = X + iY$ and $\bar{W} = U - iV$, where X and Y are tangent to the soul and U and V are orthogonal to the soul. Since the sectional curvature of a plane determined by two vectors such that one is tangent and another is perpendicular to the soul vanishes, we obtain, from the definition of isotropic curvature for plane $\{Z, W\}$, that

$$K(X, U) + K(X, V) + K(Y, U) + K(Y, V) + 2\langle R(X, Y)U, V\rangle = 2\langle R(X, Y)U, V\rangle \geq 0.$$

Likerwise, for plane $\{Z, \bar{W}\}$ we get

$$K(X, U) + K(X, V) + K(Y, U) + K(Y, V) - 2\langle R(X, Y)U, V\rangle = -2\langle R(X, Y)U, V\rangle \geq 0.$$

We the have that $\langle R(X,Y)U,V\rangle = 0$, which applied to the Ricci equation of the totally geodesic immersion $i : S \to M$ gives $\langle R^{\perp}(X,Y)U,V\rangle = 0$.

Now if M is simply connected, so is S that together with $\langle R^{\perp}(X,Y)U,V\rangle = 0$ implies that the holonomy group of the normal bundle NS is trivial. The result now follows from Strake's theorem ([9]).$\blacksquare$

3 Kähler submanifolds of nonnegative isotropic curvature

Recall that if J denotes the parallel complex structure of a Kähler manifold then for al vectors X, Y and U we have

$$R(JX,JY)U = R(X,Y)U \qquad \text{and} \qquad R(X,Y)JU = JR(X,Y)U.$$

These properties imply that

$$K_b(Z,\bar{W}) = K(X,Y) + K(X,JY) = K(X,Y) + K(JX,Y),$$

for $Z = X - iJX$ and $\bar{W} = Y + iJY$, X and Y orthonormais.

Proof of Theorem 1.2

It follows from the Fwu's lemma in [2] (on page 100) that for each $x \in M$ there exists a J-invariant subspace $N_x \subset T_xM$ of dimension $2(n-p)$ and an orthogonal transformation $\tilde{J} : T_x^{\perp}M \to T_x^{\perp}M$ such that

(i) $\alpha(X,Jn) = \tilde{J}\alpha(X,n)$, for all $X \in T_xM$ and all $n \in N_x$.

(ii) $\tilde{J}^2\alpha(X,n) = -\alpha(X,n)$.

We will show that under the hypothesis of $K_b \geq 0$, the subspace N_x is a subspace of the Relative Nullity Space. In fact, since $K_b \geq 0$, in particular, implies nonnegative holomorphic curvature, we have

$$K(n,Jn) = \langle\alpha(n,n),\alpha(Jn,Jn)\rangle - \langle\alpha(n,Jn),\alpha(n,Jn)\rangle = -2|\alpha(n,n)|^2 \geq 0,$$

and thus $\alpha(n,n) = \alpha(Jn,Jn) = \alpha(n,Jn) = 0$. Now let X be a unit vector orthogonal to n and Jn. Using that $K_b \geq 0$ we get

$$K(X,n) + K(X,Jn) = \langle\alpha(X,X),\alpha(n,n)\rangle + \langle\alpha(X,X),\alpha(Jn,Jn)\rangle - |\alpha(X,n)|^2 - |\alpha(X,Jn)|^2 \geq 0,$$

which implies that $\alpha(X,n) = 0$, $\forall\, X \perp n, Jn$.

Since $K_b \geq 0$ also implies nonnegative Ricci curvature we obtain that the desired splitting by Hartman's theorem. In addition, the complex structure J preserves the Riemannian splitting and since it is parallel, we conclude that M_1 is a Kähler manifold. $\blacksquare$

Proof of Theorem 1.1

Since $p < n$, we apply Fwu's lemma quoted above. Since $Z = X - iJX$ and $W = n + iJn$ span a totally isotropic 2-plane, and $K_I(Z,W) = 2K(X,n) + 2K(X,Jn)$ we get using the Gauss equation that

$$K_I(Z,W) = 2\Big[\langle\alpha(X,X),\alpha(n,n)\rangle + \langle\alpha(X,X),\alpha(Jn,Jn)\rangle - |\alpha(X,n)|^2 - |\alpha(X,Jn)|^2\Big] \geq 0, \quad (2)$$

which then implies $\alpha(X,n) = 0$, $\forall\, X \perp n, Jn$.

Since $p < n-1$ we have that $dimN \geq 4$ and we have totally isotropic two-planes contained in N. We consider $Z = n_1 + in_2$ and $W = Jn_2 + iJn_1$. Therefore, from equation (1) we obtain

$$K(n_1,Jn_1) + K(n_2,Jn_2) + K(n_1,Jn_2) + K(n_2,Jn_1) + 2\langle R(n_1,n_2)Jn_2,Jn_1\rangle \geq 0.$$

Using the Gauss equation, and the facts $\langle R(n_1, n_2)Jn_2, Jn_1\rangle = \langle R(n_1, n_2)n_2, n_1\rangle$, $K(n_1, Jn_2) = K(Jn_1, n_2)$, and $\langle \alpha(n_1, n_1), \alpha(Jn_2, Jn_2)\rangle = -\langle \alpha(n_1, n_1), \alpha(n_2, n_2)\rangle$, we conclude that

$$-2|\alpha(n_1, n_1)|^2 - 2|\alpha(n_2, n_2)|^2 \geq 0,$$

implying then that $\alpha(n, n) = 0$, $\forall\, n \in N$.

From Corollary 2.2 we get that M is Riemannian product $M_1 \times \mathbf{C}^d$ where M_1 has nonnegative sectional curvature. Again, because the complex structure J preserves the Riemannian splitting and is parallel, we conclude that M_1 is a Kähler manifold. Moreover,the universal covering $\tilde{M}_1$ splits in a Riemannian product of $\Sigma^l \times \mathcal{P}^{2m-l}$, where Σ is the soul of M and $\mathcal{P}^{m-l}$ a complete manifold diffeomorphic to the Euclidean Space $\mathbf{R}^{m-l}$.

Now we consider two orthogonal distributions defined on $\tilde{M}_1$: $J(T\Sigma)$ and $J(T^\perp\Sigma)$. Since $T\Sigma$ and $T^\perp\Sigma$ are parallel and involutive distributions and J is parallel we conclude that $J(T\Sigma)$ and $J(T^\perp\Sigma)$ define parallel and involutive distributions and therefore $\tilde{M}_1$ also splits in a Riemannian product of two other manifolds $N_1 \times N_2$, with $TN_1 = J(T\Sigma)$ and $TN_2 = J(T^\perp\Sigma)$. Since each N_i has nonnegative sectional curvatures, each one splits in a Riemannian product $N_i = \Sigma_i \times \mathcal{P}_i$, where Σ_i is the soul of N_i. Therefore $\Sigma = \Sigma_1 \times \Sigma_2$.

Now observe that if a vector $X \in T\Sigma_1$, since $X \in J(T\Sigma)$, $J(X) \in T\Sigma$ and, in particular, $J(X) \in T\Sigma_1$. Similarly, we obtain that $J(T\mathcal{P}_2)$ is invariant by J. It follows that Σ_1, $\mathcal{P}_2$, and $N = \Sigma_2 \times \mathcal{P}_1$ are Kähler manifolds and $\tilde{M}_1 = \Sigma_1 \times \mathcal{P}_2 \times N$. ■

4 Kähler Submanifolds of Flat Normal Bundle

We start this section by proving a preliminary lemma for manifolds whose curvature tensor satisfies the condition defined below:

Definition 4.1 *A Riemannian manifold is said to have* pure curvature tensor *if for every $x \in M$ there is an orthonormal basis $\{e_1, \ldots, e_n\}$ such that the two forms $e_i \wedge e_j$ are eigenvectors of the curvature operator $\mathcal{R}$.*

Notice that this definition is equivalent to saying that there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ for which $\langle R(e_i, e_j)e_k, e_m\rangle = 0$, whenever the set $\{i, j, k, m\}$ has more than two elements. We call this basis an *$\mathcal{R}$-basis*. Observe also that the Ricci equation implies that submanifolds of Space Forms with flat normal bundle have pure curvature tensor.

Lemma 4.2 *Let M^{2n} be a Kähler manifold of pure curvature tensor. Then there exists an $\mathcal{R}$-basis $\{e_1, \ldots, e_n\}$ such that the plane spanned by e_{2j-1}, e_{2j} is invariant by the complex structure J for all $j = 1, \ldots, n$. Moreover, if e_k is orthogonal to e_{2j-1} and e_{2j}, then*

$$K(e_k, e_{2j-1}) = K(e_k, e_{2j}) = 0. \tag{3}$$

Proof Since $J^2 = -I$, J splits the tangent space in n planes invariant by J. Therefore, if all sectional curvatures are zero, any orthonormal basis is an $\mathcal{R}$-basis and the result is obvious.

Let us then suppose that we reorder an $\mathcal{R}$-basis $\{e_1, \ldots, e_n\}$ so that $K(e_1, e_2) \neq 0$. We have

$$R(e_1, e_2)J(e_2) = J(R(e_1, e_2)e_2) = J(K(e_1, e_2)e_1) = K(e_1, e_2)J(e_1),$$

where the first equality was implied by the fact that J is parallel and the second by the purity of the tensor. On the other hand, we also have

$$R(e_1, e_2)J(e_2) = \langle J(e_2), e_1\rangle R(e_1, e_2)e_1 = -\langle J(e_2), e_1\rangle K(e_1, e_2)e_2.$$

Since $K(e_1,e_2) \neq 0$ we obtain that $J(e_1) = e_2$.

Now if $e_k \perp e_1, e_2$ we have $R(e_1,e_k)J(e_k) = 0$, since $J(e_k) \neq e_1, e_k$. Therefore

$$0 = R(e_1,e_k)J(e_k) = J(R(e_1,e_k)e_k) = K(e_1,e_k)J(e_1),$$

implying that $K(e_1,e_k) = 0$. Likewise we obtain $K(e_2,e_k) = 0$

By repeating this procedure for another sectional curvature $K(e_i,e_j) \neq 0$ we complete the proof of the Lemma.$\blacksquare$

Proof of Theorem 1.3

Let $\{e_1,\ldots,e_n\}$ be an $\mathcal{R}$-basis as in Lemma 4.2. Let us consider the isotropic plane spanned by

$$Z = e_{2j-1} + ie_k \quad \text{and} \quad W = e_{2j} + ie_l,$$

where $e_k, e_l \perp e_{2j-1}, e_{2j}$ and $J(e_k) \neq e_l$. It follows from Equation 1 and Lemma 4.2 that

$$K_I = K(e_{2j-1}, e_{2j}) \geq 0.$$

Now observe that in the case of pure curvature tensor, the sectional curvatures of planes spanned by the vectors in the $\mathcal{R}$-basis are the eigenvalues of the curvature operator. Therefore we conclude that $\mathcal{R}$ is a nonnegative operator.

Let us suppose that M is compact. Irreducible Compact manifolds of pure nonnegative operator were classified in [3]. Their holonomy algebra is the orthogonal algebra $o(m)$, m being the dimension of the manifold. It follows that our M is locally reducible, since the only possible case for a Kähler manifold would be a surface of nonnegative curvature. We also conclude that if the universal covering of M is still compact, then $\tilde{M}$ is a Riemannian product of surfaces homeomorphic to spheres. If $\tilde{M}$ is not compact, as in the proof of Corollary 2.2 and Theorem 1.1, we get that $\tilde{M}$ splits in a Riemannian product of $\Sigma^l \times \mathcal{P}^{2n-l}$, where Σ is the soul of M and $\mathcal{P}^{2n-l}$ a complete manifold diffeomorphic to the Euclidean Space $\mathbf{R}^{2n-l}$.

If M is complete and non-compact we also obtain that the same splitting $\Sigma^l \times \mathcal{P}^{2n-l}$. With the same arguments and notation used in the proof of Theorem 1.1, we get that $\tilde{M} = \Sigma_1 \times \mathcal{P}_2 \times N$, where each factor is a Kähler manifold. We show next that in the case of pure crvature tensor, $J(T\Sigma) = \Sigma$, that is, $\tilde{M} = \Sigma \times \mathcal{P}_2$. In fact, if not, the manifold $N = \Sigma_2 \times \mathcal{P}_1$ is such that $J(T\Sigma_2) = T(\mathcal{P}_1)$. This implies that the holomorphic curvatures $K_h(X, JX) = 0$ for all $X \in T\Sigma_2$ and all $X \in T(\mathcal{P}_1)$. Using Lemma 4.2, it is straightforward to verify that this implies that all holomorphic curvatures of N would be zero and therefore N would be holomorphically isometric to the complex space $\mathbf{C}^l$. ([5], Theorem 7.9). In this case the soul of N would be a point and hence $dimT\Sigma_2 = dimT\mathcal{P}_1 = 0$.

We then have $\tilde{M} = \Sigma \times \mathcal{P}_2$, where Σ is a compact Kähler manifold of nonnegative sectional curvatures and pure curvature operator. Therefore, Σ must be a product of spheres.$\blacksquare$

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Mercuri : IMECC-UNICAMP, Universidade Estadual de Campinas,
13081-970, Campinas, SP, Brasil; mercuri@ime.unicamp.br

Noronha: Department of Mathematics, California State University Northridge,
Northridge, CA, 91330-8313, USA; maria.noronha@csun.edu