FUNCTIONS FROM THE PLANE TO THE PLANE

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1. Introduction

This study was based in the article *Functions from $\mathbb{R}^2$ to $\mathbb{R}^2$: a study in nonlinearity* by Nicolau C. Saldanha and Carlos Tomei and in other works [2] and [3] of these authors together with Iaci P. Malta.

The goal of this study is to develop a global sense of how functions from the plane to the plane behave. In the case of functions from the real line to the real line, the standard practice is to compute critical points (i.e. maxima and minima) then use these points to infer geometric information about the function as a whole. We extend this approach to functions from the plane to the plane. Using topological tools we combine local theory of the function at singularities and regular points to obtain a global picture of how the function behaves. As we shall see the image of the critical set and the locations of its preimages are highly structured. This gives us detailed information concerning the number and location of all preimages of any given point. This article also provides a description of a method for finding the roots of such functions, which may be applied to solving sistems of two equations in two unknowns. We then compare the method described herein for obtaining roots to methods employed by such standard mathematical packages as Maple and Mathematica.

This paper is organized as follows: In Sections 2 and 3 contain several definitions and results that are prerequisite to our study. Also in Section 3 we prove lemmas that we need for Section 4, where we describe a method to find preimages of a given point.

2. Regular and Critical Points

For simplicity, we will assume that our maps from the plane to the plane are C^1, which means that its partial derivatives are continuous functions.

Definition 2.1. *A point p is said to be a* regular point *of a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ if the Jacobian Matrix $DF(p)$ is invertible. Its image is called a* regular value *of F. A point that is not regular is said to be* critical. *We shall denote the set of all critical points as C.*

Definition 2.2. *Given a point $p \in \mathbb{R}^2$ and positive real number ϵ, the set*

$$B_\epsilon(p) = \{q \mid d(q,p) = ||q - p|| < \epsilon\},$$

is called an open ball of radius ϵ and center p *or a* neighborhood of p.

Theorem 2.3. (Inverse Function Theorem) *Let p be a regular point of a C^1 function $F : \mathbb{R}^2 \to \mathbb{R}^2$. Then there exists a neighborhood U of p which is taken diffeomorphically to its image $F(U)$ (i.e. $F|_U$ is a C^1 bijective function that has a C^1 inverse $F^{-1} : F(U) \to U$).*

Observe that the statement above is equivalent to saying that if p is regular, then after a smooth changes of variables on appropriate neighborhoods of p and $F(p)$ the function F takes the form $\tilde{F}(x,y) = (x,y)$. More specifically, there exist neighborhoods U' and V' and diffeomorphisms $\phi : U \to U'$, $\psi : V = F(U) \to V'$ such that $F = \psi^{-1} \circ \tilde{F} \circ \phi$, where $\tilde{F} : U' \to V'$. Such a function, $\tilde{F}$, is termed *the normal form of F* at p.

The Inverse Function Theorem provides a useful tool to understand the local behavior of F at any regular point. We shall see presently that normal forms also exist at some types of critical points, and they may as well be used to understand the local behavior of F.

Definition 2.4. *A critical point p of F is a* fold point *(or simply a* fold*) if after smooth changes of variables on appropriate neighborhoods of p and $F(p)$ the function F takes the form $\tilde{F}(x,y) = (x,y^2)$. Similarly a critical point of F is a* cusp point *(*cusp*) if after smooth changes of variables on appropriate neighborhoods of p and $F(p)$ the function F takes the form $\tilde{F}(x,y) = (x, y^3 - xy)$.*

We can obtain a sense of how the function F behaves at folds and cusps by studying the following diagrams.

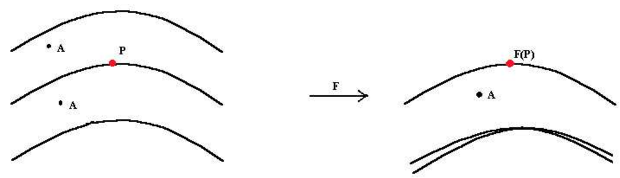

FIGURE 1. Fold

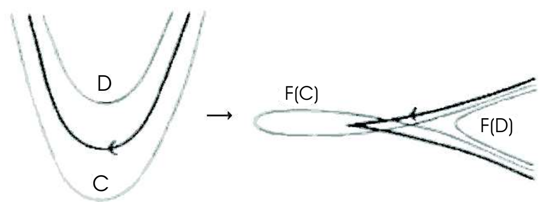

FIGURE 2. Cusp

We point out here that a function may have critical points other than folds and cups. However, roughly speaking, a theorem of Whitney (see [7]) states that most of the functions have only folds and cups as critical points. Such functions are called *excellent* and they are the functions that will be considered in this paper.

Theorem 2.5. (Implicit Function Theorem) *Suppose that* $f : \mathbb{R}^2 \to \mathbb{R}$ *is a* C^1 *function and that* $f(x_0, y_0) = 0$. *If*

$$\frac{\partial f}{\partial y}(x_0, y_0) \neq 0,$$

then there exist an open interval I *containing* x_0, *an open interval* I' *containing* y_0, *and a differentiable function* $g : I \to I'$ *so that* $f(x, g(x)) = 0$.

Applying the Implicit Function Theorem to the function $det(D\tilde{F})$, we easily see that for each point p of the critical set C that is a fold or a cusp, there exists a neighborhood U of p such that $U \cap C$ is the graph of a differentiable function g whose derivative at each point of its domain is non-zero. It follows that if F is an excellent function, then its critical set C is the union of regular simple curves.

Investigating the normal forms of cusps and folds a little further, we see that the $grad\ (det(D\tilde{F}))$ and $grad\ (det(DF))$ are both non-zero on these critical points. This implies that the Jacobian matrix $DF(p)$ has rank one and hence nullity one. The relationship between the Null Space N of $DF(p)$ and the line T (thought as a subspace of the plane) tangent to C at p, determines if p is a fold or a cusp. One can show that if $T \neq N$ then p is a fold. We omit the proof of this statement here, which can be found in [7]. Instead, we finish this section with the example below.

Example 2.6. *Consider the function* $F(x, y) = (x^2+x-y^2, 2xy-y)$. *We have*

$$DF(x, y) = \begin{pmatrix} 2x+1 & -2y \\ 2y & 2x-1 \end{pmatrix} \qquad \text{and} \qquad det(DF(x, y)) = 4\ (x^2+y^2)-1.$$

The set C is then the circle centered at the origin and of radius $1/2$. Writing
$x = \frac{1}{2}\cos t$ *and* $y = \frac{1}{2}\sin t$, *we find the Null Space of $DF(x, y)$ by solving*

$$\begin{pmatrix} \cos t+1 & -\sin t \\ \sin t & \cos t-1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Using the identity $\cos t = \cos^2(t/2) - \sin^2(t/2)$, *we conclude that* $(\sin(t/2), \cos(t/2))$ *spans the Null Space. The reader can easily verify that the points that are not folds are* $(1/2, 0)$ *and* $(-1/4, \pm\sqrt{3}/4)$, *since for these points we have* $N = T$. *These are the cusp points of* F.

3. Counting Preimages of Regular and Critical Points

We start by recalling some topological definitions.

Definition 3.1. *A set $U \subset \mathbb{R}^2$ is said to be* open *if given any point $p \in U$, there exists a positive ϵ so that the open ball $B_\epsilon(p) \subset U$. A set $C \subset \mathbb{R}^2$ is said to be* closed *if its complement is open and is said to be* bounded *if there exists an open ball B such that $C \subset B$.*

Proposition 3.2. *Let $\{U_\lambda\}$ be a collection of open sets of $\mathbb{R}^2$. Then*
(i) $\bigcup U_\lambda$ is an open set of $\mathbb{R}^2$.
(ii) Any finite intersection $U_{\lambda_1} \cap \cdots \cap U_{\lambda_n}$ is an open set of $\mathbb{R}^2$.

The proof of this result can be found in any advanced calculus text.

Definition 3.3. *A set $C \subset \mathbb{R}^2$ is said to be* compact *if any open cover of C (a collection of open subsets whose union contains C) has a finite subcover, that is, the union of sets of a finite subcollection still contain C.*

The theorem below contains three important results on compact sets. Their proofs can be found in any analysis textbook for advanced undergraduates, for instance [5].

Theorem 3.4. *Let C be a subset of $\mathbb{R}^2$. Then*
(i) C is compact if and only if C is closed and bounded.
(ii) C is compact if and only if any sequence of points in C contains a subsequence that converges in C.
(iii) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be continuous. If C is compact, then so is $F(C)$.

Definition 3.5. *A continuous function $F : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be* proper *if the preimage $F^{-1}(C)$ of a compact set C is also compact.*

We now prove various lemmas, which will be useful in the next section.

Lemma 3.6. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a proper excellent function. Then any point in $\mathbb{R}^2$ has a finite number of preimages.*

Proof Let p be any point in the image $F(\mathbb{R}^2)$. Because the set $\{p\}$ is compact and F is proper, the set of the preimages of p, $F^{-1}(p) = \{q \mid F(q) = p\}$, is also compact. If $F^{-1}(p)$ is the null set, then the result is obvious. If not, let us suppose that $F^{-1}(p)$ is infinite. We then have a sequence $\{q_n\}$ in $F^{-1}(p)$ that, by Theorem

3.4 (ii), converges to a point $q \in F^{-1}(p)$. Therefore every neighborhood of q contains infinitely many preimages of p. This implies that q is not a regular point of F, since, by the Inverse Function Theorem, for any regular point a there exists a neighborhood of a which is taken diffeomorphically (in particular bijectively) to some neighborhood of $F(a)$. Therefore q would have to be a critical point. Recall that for any critical point b there exists a neighborhood V of b on which F behaves like the normal form of F at b. If b is a fold (respectively a cusp) the preimages of any point $(x_0, y_0) \in F(V)$ are the solutions to the system of equations $x_0 = x, y_0 = y^2$ (respectively $x_0 = x, y_0 = y^3 - xy$). It is clear that neither system can have infinitely many solutions. Hence any critical point has a neighborhood containing only finitely many preimages of p. It follows that $F^{-1}(p)$ is finite. $\square$

We point out that in the proof above, we used the fact that F is excellent only to guarantee that the points in $F(C)$ have a finite number of preimages. For points in $\mathbb{R}^2 - F(C)$, we need only the fact that F is proper.

Definition 3.7. *We say that a set A of points in the plane is* path connected *if any two points a and b in A may be joined by some path that is contained in A, that is, $\forall\ a, b \in A$ there exists a continuous map $f : [0,1] \to A$, such that $f(0) = a$ and $f(1) = b$. A* connected component *or* tile *of A is a path connected subset that is not contained in any larger path connected subset of A. By* adjacent tiles *we mean connected components which share a common boundary consisting of an arc and not simply a single point.*

Definition 3.8. *We say that a set A of points in the plane is* disconnected *if there exist open disjoint sets X and Y such that $A \subset (X \cup Y)$. A is* connected *if A is not disconnected. It can be shown that* path connected *implies* connected.

Lemma 3.9. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a proper function with critical set C. The number of preimages in any tile of $\mathbb{R}^2 - F(C)$ is constant.*

Proof Let a and b be any two points in the same connected component of $\mathbb{R}^2 - F(C)$. Let γ be a path joining a to b which does not intersect C (the existence of such a path follows from the definition of a tile). Given a point c on γ, we consider its preimages, $s_1, \ldots, s_k$. Clearly all s_i's are regular points, otherwise c would not be in $I\!R^2 - F(C)$. Take disjoint neighborhoods U_i of the various s_i, which are taken diffeomorphically to open sets around c, by the Inverse Function Theorem. Let $T_i = F(U_i)$. It follows from Proposition 3.2 the finite intersection $T_c = \bigcap T_i$ is an open set containing c, and that all points in T have at least k preimages (one near each s_i). The collection of open subsets $\{T_c | c \in \gamma\}$ is an open cover of γ, which is a compact set. It follows that we can cover γ with a finite number of sets of the type T_c. Now we assume that a has k_1 preimages $p_1, ..., p_{k_1}$ and that b has k_2 preimages $q_1, ..., q_{k_2}$. Let T_0 denote the open set in the finite subcover containing a. Then all points in T_0 have at least k_1 preimages (one near each p_i). Consider a point $a_1 \in T_0 \cap \gamma$, which is distinct from a, and T_1, also in the finite subcover that contains a_1. Then all points in T_1 have at least k_1 preimages. Then we repeat the process for a point $a_2 \in T_1$ distinct from a_1. Because we have a finite collection of open sets T_i covering γ, proceeding in this manner finitely many times, we then arrive at the conclusion that all points on γ have at least k_1 preimages. Therefore $k_2 \geq k_1$. If we now begin the same process with b, it follows that all points on γ have at least k_2 preimages. This implies $k_1 = k_2$, and hence any two points in the same tile of $I\!R^2 - F(C)$ have the same number of preimages. □

Remark We point out that given T_i in the finite subcollection above, there exitst T_j in the subcollection such that $T_i \cap T_j \neq \emptyset$. In fact, if not, we would have $X = T_i$ and $Y = \cup_{j \neq i} T_j$ such that $A = X \cup Y$, where A is the set defined by the union of all T_i's. This would imply that A is disconnected contradicting that A is path connected.

Lemma 3.10. *Let $F : I\!R^2 \to I\!R^2$ be a proper excellent function. Then the number of preimages in adjacent connected components of $I\!R^2 - F(C)$ differ by two.*

Proof Consider two adjacent tiles of $I\!R^2 - F(C)$, A and B. Let w, on the boundary between A and B, be the image of a simple fold point, that is, w has only one critical preimage. Assume that w has $k + 1$ preimages and let p_0 denote the only critical preimage of w. Let $p_1, ..., p_k$ be the k preimages of w that are regular. The Inverse

Function Theorem implies the existence of disjoint open neighborhoods U_i of the various p_i's on which F is a diffeomorphism. Let $V_i = F(U_i)$. Then $V = \bigcap V_i$ is an open set containing w, so points in $A \cap V$ and points in $B \cap V$ have at least k preimages. It follows from the previous lemma that all points in A and all points in B have at least k preimages. Consider G an open set around p_0. Because p_0 is a fold, points on opposite sides of the critical curve near p_0 are mapped to the same tile of $\mathbb{R}^2 - F(C)$. It then follows that either $F(G) \subseteq A$ or $F(G) \subseteq B$, and that there exists a point $q \in F(G)$ which has two preimages in G. We may assume without loss that $F(G) \subseteq A$, and hence $q \in A$. The previous lemma implies that all points in A have at least $k+2$ preimages.

We now show that no point in B can have more than k preimages (the same argument may be used to demonstrate that no points in A may have more that $k+2$ preimages). Suppose points in B have more than k preimages. We then consider a sequence $\{q_j\}$, of regular points with more than k preimages converging to w (this fact follows from Sard's Theorem[1]. With the notation of the previous paragraph, for each q_j let $q_j^* \notin \bigcup U_i \cup G$ be one of the additional preimages. Since the set $\{w, q_1, \ldots, q_n, \ldots\}$ is compact and F is proper, a subsequence of $\{q_j^*\}$ must converge to some point $\tilde{q}$, which by the continuity of F and the uniqueness of the limit, must be a preimage of w. But then w has an additional preimage which contradicts the earlier assumption. □

The Lemmas above provide us with a useful tool for counting the number of preimages of any given point. Once we have identified the critical set, we may consider its image $F(C)$, and then divide $\mathbb{R}^2$ into regions in which points have a constant number of preimages. Furthermore we need only calculate the number of preimages in one such region in order to be sure of the number in all others. We finish this section by showing an example.

Example Consider the function

$$F(x, y) = (x^2 + x - y^2, 2xy - y).$$

[1]It states that every neighborhood of any point of the plane contains regular values of F (see [4], page 11).

wich we presented in Section 2. The critical set C and its image $F(C)$ are shown in the following diagram (we can easily find them with *Mathematica*)

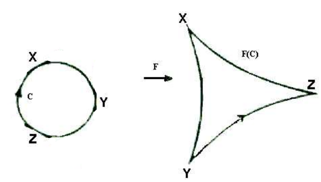

FIGURE 3. Critical set and image F(C)

Observe that this function can be expressed in complex notation as $F(z) = z^2 + \bar{z}$. Therefore, for a point q far outside of $F(C)$ the behavior of F is very much like that of z^2. We then expect, and it is indeed the case, that a point far outside of the critical set like q would have two preimages. Given that fact it is easy to find good initial conditions for Newton's method to solve for the preimages of q, we conclude that the number of preimages of any point outside $F(C)$ is two. From the lemmas above, we know that when we trespass $F(C)$ the number of preimages will increase by two.

4. Finding preimages and the flower

Definition 4.1. *A function $F : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be* nice *if the three following properties hold:*
(i) F is proper.
(ii) Any point q in $F(C)$ has at most two critical preimages.
(iii) If q has two critical preimages p_1 and p_2 then both are folds and the tangent lines to $F(C)$ at q corresponding to p_1 and p_2 are distinct.

We shall restrict our attention to functions which are both nice and excellent.

In his works in singularity theory ([6] and [7]) Whitney has shown that for excellent functions the set of all regular points $\mathbb{R}^2 - C$ is an open dense subset of the plane. This is equivalent to saying that any point in $I\!R^2 - C$ has a neighborhood contained in $I\!R^2 - C$ and every neighborhood of any point in $I\!R^2$ contains at least one point of $I\!R^2 - C$. The set C itself consists of a discrete set of cusps joined by smooth arcs of folds. Whitney also demonstrated that excellent function from the plane to the plane are abundant. Roughly speaking, most of the excellent functions are also nice.

We will now show that a natural way to understand the global behavior of nice excellent functions is to study the preimages of $F(C)$, in other words, $F^{-1}(F(C))$, which we shall refer to as the *flower* of F. The *tiles of the flower* are then the various path connected components of $F(F^{-1}(C))$.

We describe a method to obtain the preimages of any given point. The method starts by picking a point for which the computation of its preimages is a trivial task and drawing a path from this point to the given point. We then use various continuation methods (Newton-Kantorovich's method, for example) to find the inversion of this path. The preimages of this path lead to the preimages of the desired point.

We will illustrate the technique described herein with a simple example, using the function

$$F(x, y) = (x^2 + x - y^2, 2xy - y).$$

whose critical points, folds and cusps, we found in Section 2. The flower and the image of the critical set are shown below.

Since we have the critical set, applying F we will obtain its image $F(C)$. Let X, Y and Z be the cusps that we found in example 2.6. Take A and B points in $F(C)$ close to the image of X, that for simplicity we are also denoting by X. We know that there exist preimages A and B close to X in the critical set C. By the normal form of cusps, there exist another two preimages also near X. Since the new preimages of A and B are regular points, we can use the Newton's method that will give us an arc that contains the new preimages of X, A and Y and another one that contain preimages of X, B and Z. Proceding in the same way for the cusps Y and Z we find the following flower

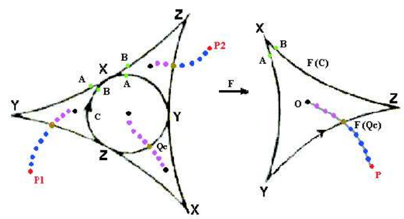

FIGURE 4. Preimages of P

This function can be expressed in complex notation as $f(z) = z^2 + \bar{z}$, which implies that points outside of $F(C)$ have two preimages while points inside have four.

Say we want to find all preimages of the point O shown in the figure. From the flower we already know that inside our critical curve we have four preimages, one in each of the tiles of the flower. We first compute the pre-images of a point that we denote by P far enough from of $F(C)$. Since for large values of $|z|$ the function F behaves like z^2, each of the two complex square roots P is a good initial condition for the Newton's method. Using then Newton's method for each of them we find the preimages of P, $P1$ and $P2$. Now construct a path δ from P to O, that crosses $F(C)$ transversally at the image of a simple fold. An easy way to accomplish this is to pick a point Q_c which is a simple fold, and choose δ such that it intersects $F(C)$ at $F(Q_c)$.

Choose a point in δ near P such that this point is not in $F(C)$. By the inverse function theorem we know its preimages must lie near the two preimages of P, namely, $P1$ and $P2$. We again use use Newton's method twice, one using the initial condition $P1$ and another, using $P2$ as initial condition. We obtain the two preimages of the new point. Repeating this process we can obtain the preimages of all points on δ up to the point where δ intersects $F(C)$. The preimages of points on δ outside of $F(C)$ will be two paths, one starting at $P1$ and another one starting at $P2$. Both paths lead to a point on the flower, which is the other preimage of $F(Q_c)$.

The part of δ inside of $F(C)$ has four preimages: two are the continuations of the paths from $P1$ and $P2$ leading to preimages of A in tiles of the flower outside of C. The other two are paths from Q_c leading to the other two preimages of O, one inside C and the other outside C. To continue the first two paths it is possible to use Newton's method as before since all points on these paths are regular (they intersect the flower, but not C itself). To find the two new paths we use the normal form of F at a fold. We know that points in opposite sides of C are mapped in the same tile by F. To choose these points, we can take the two vectors $p_+ = Q_c + sv$ and $p_- = Q_c - sv$ that for small values of s and $v \in ker(DF(P_c))$. [2] lie in opposite sides of the critical curve. Once we have found points on these paths near Q_c we are again at regular points. Hence we can use Newton's method again to compute the rest of the path. These paths lead to the four preimages of O.

Remark The standard numerical approach used by programs like Maple and Mathematica also uses Newton's method. For that, these programs select hundreds of initial conditions at random. Some converge to roots, while others do not. If you are only entitled to use finitely many initial conditions there is no guarantee that you will find all roots.

One of the values of the method described herein is that we know the number of preimages of any given point before we begin searching for them. Another value is that it uses the global knowledge of the function in order to choose initial conditions. Furthermore, once the critical set for some function $F(x, y)$ has been found it becomes a simple task to solve the equation $F(x, y) = (a, b)$ for virtually any constants a and b. While the other approach would treat solving $F(x, y) = (a_1, b_1)$ and $F(x, y) = (a_2, b_2)$ as two distinct problems. Perhaps the most valuable feature of this method is that we understand the mathematical theory behind the method, instead of just getting an answer from a software.

[2]We see that this vectors are very good aproach for Q_c because the Taylor series give us

$$F(Q_c+sv) = F(Q_c)+DF(Q_c)sv+R(s) = F(Q_c)+sDF(Q_c)v+R(s) = F(Q_c)+R(s)$$

where $\frac{R(s)}{|sv|} \to 0$ when $s \to 0$

We finish this paper by applying this method to a more sophisticated example that can be found more detailed in [3].

Let us consider the function

$$F(x,y) = (-6x^4-6x^2y^2+xy^3+6y^4-x, \frac{25}{24}x^4+x^3y+x^2y^2+\frac{1}{6}xy^3-y^4-y)$$

The critical set of F (a circle) and its image, $F(C)$ are show in the figure below:

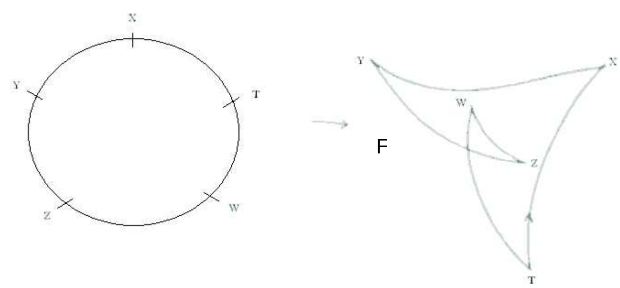

FIGURE 5. Critical set of F

Computing the Critical set of F we see that there are five cusps (listed as Z, Y, X, T and W in the figure). By considering the images of curves near the cusps we determine that Z, Y, X and T are effective outside the critical curve. While W is actually inside the critical curve.

From the normal form of F at a cusp we obtain reasonable initial conditions for Newton's method to determine the location of preimages of points on $F(C)$ near a cusp. Finding the preimages of the rest of the critical set, we have the flower showed in the next figure.

Obviously the critical set is part of the flower. The tiles for the flower (resp. for $F(C)$) are labelled X_i (resp. Y_j). Clearly each tile has some set of piecewise smooth arcs as a bound. For those tiles of

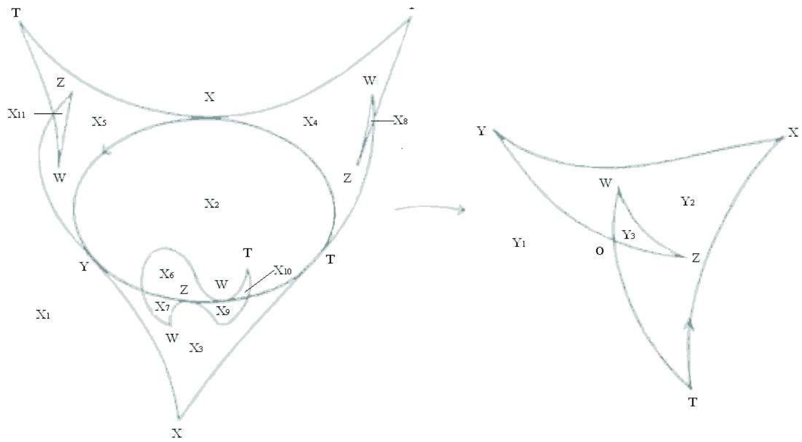

FIGURE 6. The flower

the flower which are bounded by a single closed curve the last proposition applies.

It is easy to compute the preimages of a point with a large norm using Newton's method. From this we determine that points in Y_1 have two preimages. Furthermore since F is a proper function and all tiles X_i $(i > 1)$ are closed topological disks and Y_1 is not, the preimages of points in Y_1 must lie in X_1. By last lemma we know the number of preimages of points in adjacent tiles differ by two. From knowledge obtained in computing the critical set we already know the effective sides of the folds. The number of preimages increase by two as we move from Y_1 to Y_2 and again when moving from Y_2 to Y_3. It is then apparent that points in Y_2 and Y_3 have four and six preimages respectively.

The flower indeed provides us with a detailed geometric description of the behavior of F. Using the flower it is easy to see both how many preimages a given point has, and in what regions of $\mathbb{R}^2$ those preimages lie. For example, points in X_{11} lie in a closed topological disk bounded by preimages of the arcs joining O, Z and W. Since all points in X_{11} are regular and F restrict to X_{11} is injective, the

Inverse Function theorem tells us that points inside this region are mapped homeomorphically to points in Y_3, the closed topological disk bounded by the arcs joining Z, W and O. Following this reasoning it is clear that any point in Y_3 has one preimage in each of the tiles $X_6, X_7, X_8, X_9, X_{10}$ and X_{11}, for a total of six preimages.

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