CLOSED AND EXACT DIFFERENTIAL FORMS IN $\mathbf{R}^n$

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Abstract. We show in this paper that if every closed 1-form defined on a domain of $\mathbf{R}^2$ is exact then such a domain is simply connected. We also show that this result does not hold in dimensions 3 and 4.

1. Introduction

It is well known that every closed 1-form defined on a simply connected domain Ω of $\mathbf{R}^n$ is exact (see Section 2 for the definitions). This statement poses the following question: *Suppose we have a domain Ω of $\mathbf{R}^n$ with the property that every 1-closed form is exact. Is this domains simply connected?* We answer this question in the affirmative for $n = 2$ and give counterexamples for $n = 3, 4$.

The paper is organized as follows: Section 2 contains some definitions and results that will be used in the later sections, section 3 proves the case for $n = 2$, section 4 defines the Real Projective Plane that will be used to construct the counterexample for $n = 4$ and sections 5 and 6 present the counterexamples for cases $n = 4$ and $n = 3$ respectively.

2. Background: 1- Forms on Simply Connected Domains

We start this section by recalling some definitions of multivariable calculus.

Let Ω be a *domain* of $\mathbf{R}^n$, that is, a path connected open set of $\mathbf{R}^n$. Let $f : \Omega \subset \mathbf{R}^n \to \mathbf{R}$ be a *smooth function*, that is, a function whose all partial derivatives of all orders are continuous.

The *gradient* of f is defined as

$$\nabla(f) = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}),$$

and the directional derivative in the direction of a vector $v = (v_1, \dots, v_n)$ is given by

$$df(v) = \nabla(f) \cdot v = \frac{\partial}{\partial x_1} v_1 + \cdots + \frac{\partial}{\partial x_n} v_n.$$

Observe that if we denote by x_i the *projection map*

$$x_i : \mathbf{R}^n \to \mathbf{R}$$

given by $(x_1, \dots, x_n) \mapsto x_i$, we have

$$\nabla(x_i) = (0, \dots, 1, \dots, 0) \quad \Rightarrow \quad dx_i(v) = v_i.$$

It then follows that

$$df(v) = \frac{\partial f}{\partial x_1} dx_1(v) + \cdots + \frac{\partial f}{\partial x_n} dx_n(v). \tag{1}$$

We know that for each point $p \in \Omega$, df and the dx_i's are linear maps from $\mathbf{R}^n$ to $\mathbf{R}$, that is, they are elements of the dual space $(\mathbf{R}^n)^*$. Therefore equation (1) shows that, as elements of $(\mathbf{R}^n)^*$, df is written as the linear combination

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n.$$

This motivates the following definition.

Definition 2.1. *A differential* 1*-form* ω *defined on a domain* Ω *is a map that to each point* $p \in \Omega$ *assigns* $\omega(p) \in (\mathbf{R}^n)^*$ *given by*

$$\omega(p) = a_1(p)dx_1 + \cdots + a_n(p)dx_n,$$

such that each $a_i : \Omega \subset \mathbf{R}^n \to \mathbf{R}$ *is a smooth function.*

Example The 1-form

$$\omega = -\frac{y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y$$

defined on $\Omega = \mathbb{R}^2 - (0,0)$.

Definition 2.2. *A differential* 1*-form* ω *defined on a domain* Ω *is said to be* closed *if*

$$\frac{\partial a_i}{\partial x_j}(p) = \frac{\partial a_j}{\partial x_i}(p), \quad \forall i, j \quad \text{and} \quad \forall x \in \Omega.$$

We say that a differential 1*-form* ω *is* exact *if there exists a smooth function* $f : \Omega \to \mathbf{R}$ *such that* $\omega = df$.

Proposition 2.3. *The following are equivalent:*
1) ω is exact in a connected open set Ω
2) $\int_c \omega$ depends only on the end points of c for all $c \subset \Omega$
3) $\int_c \omega = 0$, for all closed curves $c \subset \Omega$

This is a standard result on differential 1-forms, the reader can find the proof in several textbooks, for instance [1].

Recall that the Theorem of Schwarz states that if f is smooth then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Since

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n,$$

that is, $a_i = \partial f / \partial x_i$, we conclude that *exact* implies *closed.*

The converse of the last result is not true as we will show in the next example.

Example 2.4. *Let us consider the differential 1-form*

$$\omega = -\frac{y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy$$

defined before. We claim that ω is is closed but is not a exact 1-form. In fact, let γ be a closed curve such that

$$\begin{array}{cccc} \gamma : & [0, 2\pi] & \longrightarrow & \mathbf{R}^2 \\ & \theta & \mapsto & (cos\theta, sin\theta) \end{array}$$

Computing the line integral,

$$\begin{aligned} \int_\gamma \omega &= \int_\gamma -\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy \\ &= \int_0^{2\pi} -\frac{sin\theta}{sin\theta^2 + cos\theta^2}(-sin\theta)dt + \frac{cos\theta}{sin\theta^2 + cos\theta^2}(cos\theta)dt \\ &= \int_0^{2\pi} dt \\ &= 2\pi \end{aligned}$$

Since $\int_\gamma \omega \neq 0$, ω is not exact.

On oder hand, if we compute $d\omega$ we have

$$d\omega = dA \wedge dx + dB \wedge dy,$$

where,

$$A = -\frac{y}{x^2 + y^2} \qquad \text{and} \qquad \mathrm{B} = \frac{\mathrm{x}}{\mathrm{x}^2 + \mathrm{y}^2}.$$

$$\begin{aligned} d\omega &= \left(\frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy\right) \wedge dx + \left(\frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial y} dy\right) \wedge dy \\ &= \frac{\partial A}{\partial y} dy \wedge dx + \frac{\partial B}{\partial x} dx \wedge dy \\ &= -\frac{\partial A}{\partial y} dx \wedge dy + \frac{\partial B}{\partial x} dx \wedge dy \\ &= (-\tfrac{\partial A}{\partial y} + \tfrac{\partial B}{\partial x}) dx \wedge dy \\ &= 0 \end{aligned}$$

Thus ω is a closed 1-form but is not exact.

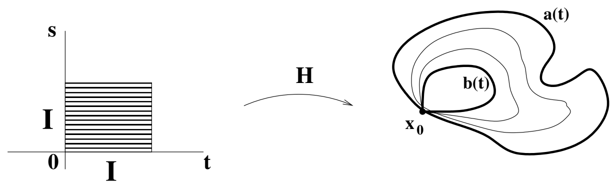
We note here that for domains of $\mathbf{R}^n$ with some special topological property, the converse is true, namely, closed implies exact. We explain next such a topological property.

Definition 2.5. *Let us consider the closed and bounded interval $I = [0, 1]$. A* path *in $\mathbf{R}^n$ is a continous map $a : I \to \mathbf{R}^n$. The path is called a* loop *if it is closed, that is, $a(0) = a(1)$ and if it does not intersect itself.*

We say that a loop a is *homotopic to a point p if it is homotopic to the constant path $c(t) = p$ for all $t \in [0, 1]$.*

Definition 2.6. *Two closed paths $a, b : I \to X$, with $a(0) = a(1) = x_0 \in X$, are said to be* homotopic *(denoted by $a \simeq b$) if there is a continuous map $H : I \times I \to X$ such that*

$$H(t, 0) = a(t), \;\; H(t, 1) = b(t), \;\; H(0, s) = H(1, s) = x_0 \quad \forall s, t \in I$$



Intuitively, a homotopy is a continuous deformation that starts in a stage 0 *(with* $a(t)$*) and ends in the stage* 1 *(with* $b(t)$*). During this deformation, the end points* $a(0) = b(0) = a(1) = b(1)$ *remain fixed.*

Proposition 2.7. *Let be* $a, b : I \to \Omega \subset \mathbf{R}^n$*. If* $a \simeq b$ *and* $f : \Omega \to \mathbf{R}^n$ *is a continuous function, then* $f(a) \simeq f(b)$*.*

Lemma 2.8. *The relation* $a \simeq b$ *is an equivalence relation.*

The proof of this result can be found in all textbooks on the subject. We call the reader's attention that the main point is to show the transitivity property of the relation, namely, if $a \simeq b$ *and* $c \simeq d$ *then* $ac \simeq bd$*, where* ac *is also a loop parametrized as follows*

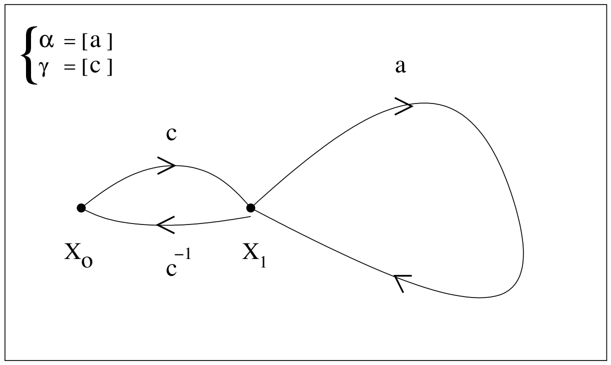
$$ac(t) = \begin{cases} a(2t) & 0 \le t \le 1/2 \\ c(2t-1) & 1/2 \le t \le 1 \end{cases}$$

Definition 2.9. *The* fundamental group *of a set* X *with respects to a basepoint* x_0*, denoted by* $\pi_1(X, x_0)$*, is the group formed by the set of equivalence classes of all loops, i.e., paths with initial and final points at the basepoint* x_0*, under the equivalence relation of homotopy.*

Since $\pi_1(X, x_o)$ *is a group it has an identity element, which is the homotopy class of the constant path at the basepoint* x_0*.*
If x_0 *and* x_1 *are on the same connected by path component of* X *then the group* $\pi_1(X, x_o)$ *is isomorphic to the group* $\pi_1(X, x_1)$*. In other words, if* $\alpha \in \pi_1(X, x_1)$ *there exists a correspondent element, that is* $\gamma\alpha\gamma^{-1}$*, in* $\pi_1(X, x_o)$*. The intuitive ideia comes from the picture below.*

3. The Theorem for $n = 2$

As mentioned in the introduction, for domains of $\mathbf{R}^2$ *we have the following theorem.*



Theorem 3.1. *If Ω is a subset of $\mathbf{R}^2$ such that any closed 1-form ω defined in Ω is exact, then Ω is simply connected.*

The proof of theorem 3.1 is a consequence of two classical theorems for closed curves that are quoted below.

Recall that a curve is called a simple curve *if it does not intersect itself.*

Theorem 3.2. *Jordan Curve Theorem: Let $C \subset \mathbb{R}^2$ be a closed simple curve. Then C divides the plane in two connected components: one bounded by C and the other unbounded.*

Theorem 3.3. (Schoenflies Theorem) *If C is a closed simple curve in $\mathbf{R}^2$ then C bounds a region that is homeomorphic to a disk $D = \{(x, y) \in \mathbf{R}^2 | x^2 + y^2 \leq r\}$.*

Proof of the 3.1 *- Suppose that the domain Ω is not simply connected. Then, by definition, there exists a point $p \in \Omega$ and a loop $\gamma \subset \Omega$ such that γ is not homotopic to p. Let J denote the interior of γ union with γ. It follows from Schoenflies theorem that J is homeomorphic to a disk in the plane. Therefore, γ is homotopic to any point p in J, since a disk of $\mathbf{R}^2$ is simply connected. It follows that there exists q in the interior of J such that $q \notin \Omega$. Without loss of generality, suppose that $q = (0, 0)$ and consider*

$$\omega = \frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy$$

The 1-form ω is then a differential form on Ω.

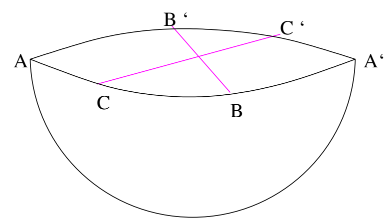
It is easy to see that

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2 + y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right),$$

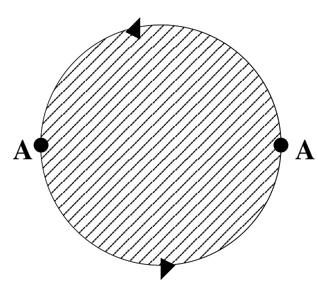
which implies that ω is closed. Since we are supposed that all closed 1-forms are exact, $\int_\gamma \omega = 0$. However we know that for γ as in the example 2.4, $\int_\gamma \omega \neq 0$. We then have a contradiction.

4. The Projective Plane

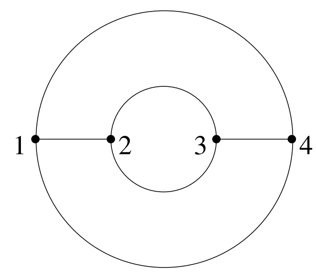
Our objective is to present a counterexample for a statement similar to the theorem 3.1 in higher dimensions. So let us start by defining the topological space $\mathbf{R}P^2$, that is called Real Projective Plane. *We construct the projective plane, by considering the $S^2 = \{(x, y) \in \mathbf{R}^2 | x^2 + y^2 = 1\}$ and indentifying each point with its antipodal point. $\mathbf{R}P^2$ is then the quotient space $S^2/(v \sim -v)$. Observe that it can be defined as the quotient space of a hemisphere identifying antipodal points on the boundary.*



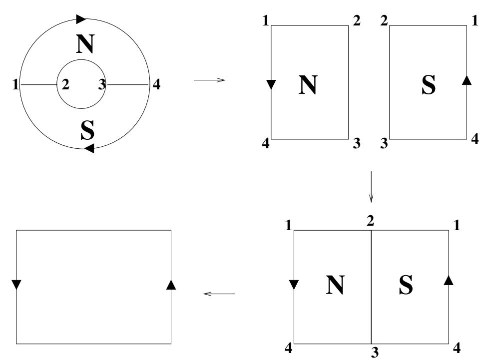
The projective plane can also be represented by the plane diagram below. Imagine the region above as an hemisphere after a "nice deformation".



Is easy to see that the disk above is homeomorphic to an annulus union a disk.



First, we claim that the annulus is homeomorphic to a Möbius band. To see this, identify the points 2 *and* 3 *in the retangles* N *and* S*.*



From the above construction we conclude that the projective plane is homeomorphic to a Möbius band union a disk.

4.1. $\mathbf{R}P^2$ **is compact.** *Let* $p : S^2 \to \mathbf{R}P^2$ *be the map that to* $x \in S^2$ *assigns its equivalence class* $[x] = \{x, -x\}$. *The map* p *is called* quotient map.

Definition 4.1. *Let* $U \subset \mathbf{R}P^2$. *We say that* U *is open in* $\mathbf{R}P^2$ *if* $p^{-1}(U)$ *is open in* S^2.

With this definition of open set for $\mathbf{R}P^2$ *we prove:*

Proposition 4.2. *The real projective plane is compact.*

Proof. *Let* $\{U_\alpha\}$ *be an open cover of* $\mathbf{R}P^2$*. Let* $V_\alpha = p^{-1}(U_\alpha)$*.*
We have that $\{V_\alpha\}$ *is an open cover of* S^2*. Recall that a subset of* $\mathbf{R}^n$ *is compact if it is closed and bounded. Since* S^2 *is closed and bounded,* S^2 *is compact.*
It then follows from the definition of compactness that there exist a finite subcover $\{V_i\}_i^n$ *that covers* S^2*. Therefore there exist a finite subcover* $\{U_i = p(V_i)\}_i^n$ *that covers* $\mathbf{R}P^2$*.*
As we said before, we are looking for the property of simply connected on a domain. So we shall show now that S^2 *is simply connected but* $\mathbf{R}P^2$ *does not satisfy this property.*

4.2. $\mathbf{R}P^2$ is not simply connected.

Proposition 4.3. *The unit sphere* S^2 *is simply connected.*

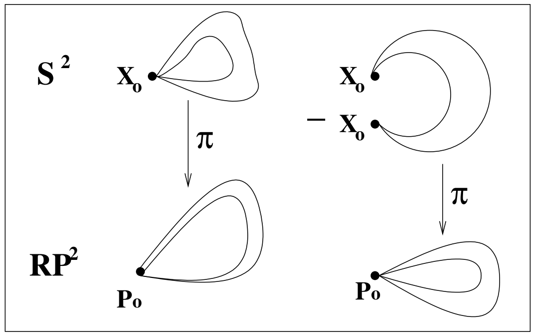
Proof. *Let* $c : I = [0,1] \to S^2$ *be a closed curve on* S^2 *with base point* x_0*. According to a lemma by Lima* [4] *there exists a loop, i.e, a closed simple curve, such that* $b \simeq c$*, the basepoint of* b *is* x_0 *and the image* $b(I)$ *is not the entire sphere* [1]*. Then we can take a point* $p \in S^2$ *such that* $p \notin b(I)$*. Let us consider the stereographic projection* $P : S^2 - \{p\} \to \mathbf{R}^2$*. Then* $\tilde{b} = P(b)$ *is a loop in the plane with basepoint* $P(x_0)$ *and hence* $\tilde{b}$ *can be continuously deformed to* $P(x_0)$*. Since the stereographic projection is a homeomorphism, we conclude that* $b \simeq x_0$*. Since* $b \simeq c$*, we have* $c \simeq x_0$*. Therefore, any loop on the sphere is homotopic to its basepoint on the sphere, which implies that the sphere is simply connected, or in other words, its fundamental group* π_1 *is trivial.*

Proposition 4.4. *The fundamental group of* $\mathbf{RP}^2$ *is* $\mathbf{Z}_2$ *and hence* $\mathbf{RP}^2$ *is not simply connected.*

Proof *Let* $\pi : S^2 \to \mathbf{RP}^2$ *be the quotient map, that is,* $\pi(x_0) = p_0 = \{x_0, -x_0\}$*. Let* c *and* d *be two closed loops in* $\mathbf{R}P^2$ *with basepoint* p_0 *and* $\tilde{c}$ *and* $\tilde{d}$ *be their liftings in* S^2 *with initial endpoint* x_0*. By a lifting we mean a curve* $\tilde{c}$ *such that* $\pi(\tilde{c}) = c$*.*

A result in Lima [4] *(page 77) implies that* $\tilde{c}(1) = \tilde{d}(1)$ *if and only if* $c \simeq d$*. This means that if* $\tilde{c}$ *is a loop at* x_0 *and* $d \simeq c$*, then* $\tilde{d}$ *is also a loop at* x_0*. Likewise, if* $\tilde{c}$ *is an open curve from* x_0 *to* $-x_0$*, then so is* $\tilde{d}$*, if* $d \simeq c$*. Therefore, there*

[1] Just as curiosity, there exist curves such that their images cover S^2, see Peano's curve in [4].



are two types of loops with base point p_0. The ones whose liftings close at x_0 and are all homotopic to each other and the ones whose liftings do not close, also all homotopic to each other. Therefore, there are only two equivalent classes implying that $\pi_1(\mathbf{RP}^2) = \mathbf{Z}_2$. □

Then, if $\tilde{c}$ is the closed loop in S^2 with $\tilde{c}(x_0) = \tilde{c}(0) = \tilde{c}(1)$ and $\tilde{d}$ is the open curve in S^2 with $\tilde{d}(x_0) = \tilde{d}(0)$ and $\tilde{d}(-x_0) = \tilde{d}(1)$. Then by Lima's proposition (see [4]*), the image of $\tilde{c}$ in $\mathbf{R}P^2$ denoted $\pi(\tilde{c}) = c$ is not homeomorphic to $\pi(\tilde{d}) = d$. Therefore, there are two loops in $\mathbf{R}P^2$ with the same base point $p_0 = \pi(x_0)$ such that $c \not\simeq d$. Therefore, $\mathbf{R}P^2$ is not simply connected.*

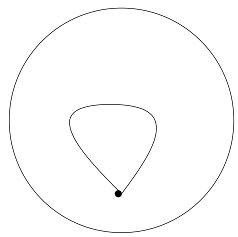
We finish this section by showing the following proposition.

Proposition 4.5. *Since ω be a closed 1-form in $\mathbf{R}P^2$. Then ω is exact.*

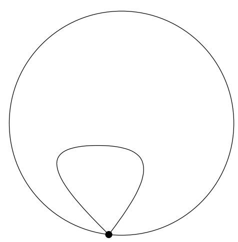
Proof *Let ω be any closed differential 1-form in $\mathbf{R}P^2$ and let c be any loop in $\mathbf{R}P^2$. Consider $\mathbf{R}P^2$ to be the disk, call it K, with the boundary containing the antipodal points which lie on the equator of the sphere (see subsection 1 where the construction of $\mathbf{R}P^2$ is defined). Then every loop in $\mathbf{R}P^2$ either intersects the boundary of K or does not.*

First consider the case in which loop c does not intersect the boundary of K. Then c lies entirely in the interior of disk K. As defined in subsection 1 the interior of K is one-to-one, closed, and homeomorphic to a regular disk in $\mathbf{R}^2$. So,

Schoenflie's theorem in $\mathbf{R}^2$ is applicable therefore the loop c is homotopic to a disk D. So, $\int_c \omega = \int_D \omega = 0$ since the disk is homeotopic to a point (see case n=2 for more detailed proof of this). Therefore every closed differential 1-form in $\mathbf{R}P^2$ is exact.



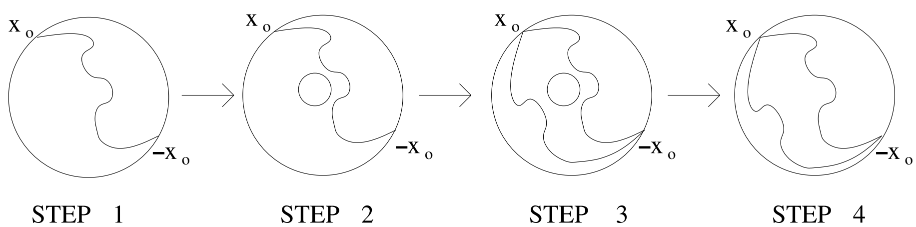
Now consider the case in which c is a loop in $\mathbf{R}P^2$ that intersects the boundary of K. Then either c has basepoint x_0 (i.e. c is a closed loop on the sphere) or c has endpoints x_0 and x_0 residing on the boundary of K (i.e. c is an open curve on the sphere) .



First consider the case in which c has basepoint x_0, then Schoenflie's theorem is applicable. So ω is exact.

Now consider the case in which c has endpoints x_0 and x_0 residing on the boundary of K.

Then remove an interior closed disk call it J (as described in subsection 1) such that no point of c is contained in J. Then, the disk K which is $\mathbf{R}P^2$ is homeomorphic to the mobius band plus disk J. Therefore c is contained entirely in the mobius band. Now consider 2c, which is a closed loop in disk K because in order to obtain 2c in $\mathbf{R}P^2$ orientation must be preserved so when one scribes c for the second time in disk K, c will go around the whole created by removing J (see the image below). Unlike c, 2c is a loop in the mobius band, which when disk J is reinserted, is a loop in the disk K. Therefore, schoenflies theorem is applicable, so



$\int_{2c}\omega = \int_D \omega = 0$ for some disk D since the disk is homeotopic to a point. Then by definition $\frac{1}{2}\int_{2c}\omega = \int_c \omega$. Therefore, $\int_c \omega = 0$. So, ω is exact. Therefore, every closed diferential 1-form in $\mathbf{R}P^2$ is exact. □

5. The counterexample for $n = 4$

5.1. $\mathbf{R}P^2$ embeds in R^4.

We first prove

Lemma 5.1. *There exists a map $f : \mathbf{R}^3 - \{(0,0,0)\} \longrightarrow \mathbf{R}^9$ with the following properties:*
(1) f is smooth;
(2) Rank $Df = 3$, that is, f has the maximal rank, where Df denotes the Jacobian matrix;
(3) For every point $q \in Imf$, there exists an open ball $B_r(q)$ in $\mathbf{R}^9$ such that if $V = B_r(q) \cap Imf$, thus $f^{-1} : V \longrightarrow \mathbf{R}^3 - \{(0,0,0)\}$ is continuous.

Proof *Let*

$$f(x_1,\ x_2,\ x_3) = (x_1^2,\ x_1x_2,\ x_1x_3,\ x_2x_1,\ x_2^2,\ x_2x_3,\ x_3x_1,\ x_3x_2,\ x_3^2)$$

Clearly, f is smooth. The Jacobian matrix,

$$Df = \begin{bmatrix} 2x_1 & 0 & 0 \\ x_2 & x_1 & 0 \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \\ 0 & 2x_2 & 0 \\ 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \\ 0 & x_3 & x_2 \\ 0 & 0 & 2x_3 \end{bmatrix}$$

has maximum rank 3, since $(x_1,\ x_2,\ x_3) \neq (0,0,0)$ *and so at least one* x_i *is not null.*

In order to prove (3), we suppose $x_1 \neq 0$ *and consider the projection*

$$\begin{array}{cccc} p: & \mathbf{R}^9 & \longrightarrow & \mathbf{R}^3 \\ & (x_1,\ x_2, \cdots,\ x_9) & \mapsto & (x_1,\ x_2,\ x_3) \end{array}$$

We know that p is continuous and observe that

$$\begin{array}{cccc} g = p \circ f: & \mathbf{R}^3 - \{(0,0,0)\} & \longrightarrow & \mathbf{R}^3 \\ & (x_1,\ x_2,\ x_3) & \mapsto & (x_1^2,\ x_1x_2,\ x_1x_3) \end{array}$$

has the Jacobian matrix

$$Dg = \begin{bmatrix} 2x_1 & 0 & 0 \\ x_2 & x_1 & 0 \\ x_3 & 0 & x_1 \end{bmatrix}$$

Since rank $Dg = 3$*, the Inverse Function theorem implies that for every point* $(x_1,\ x_2,\ x_3) \in \mathbf{R}^3 - \{(0,0,0)\}$ *there exists an open ball B containing* $(x_1,\ x_2,\ x_3)$ *such that* $g|_B$ *is a diffeomorphism onto its image* $g(B)$*. This means that* $g^{-1}: g(B) \to \mathbf{R}^3 - \{(0,0,0)\}$ *is continuous. Since* $g(B) = p(f(B))$*, if* $q \in f(B)$ *we can find a small ball* $B_r(q)$ *such that* $V = (B_r(q) \cap \ Imf) \subset f(B)$*.*

Now observe that $p(V) \subset g(B)$ *and* $f^{-1}: V \to \mathbf{R}^3 - \{(0,0,0)\}$ *is such that* $f^{-1}(q) = g^{-1}(p(q))$*, in other words, on* V*,* $f^{-1} = g^{-1} \circ p$*. Since both* g^{-1} *and p are continuous, so is* f^{-1}*.* □

Corollary 5.2. *There exists a map* $\tilde{f}: S^2 \to \mathbf{R}^9$ *with the following properties:*
(1) $\tilde{f}$ *is smooth.*
(2) $\tilde{f}$ *has maximal rank* 2*.*
(3) For every point $q \in Im\tilde{f}$*, there exists an open ball* $B_r(q)$ *in* $\mathbf{R}^9$ *such that if* $V = B_r(q) \cap \ Im\tilde{f}$*, thus* $\tilde{f}^{-1}: V \longrightarrow \mathbf{R}^3 - \{(0,0,0)\}$ *is continuous.*

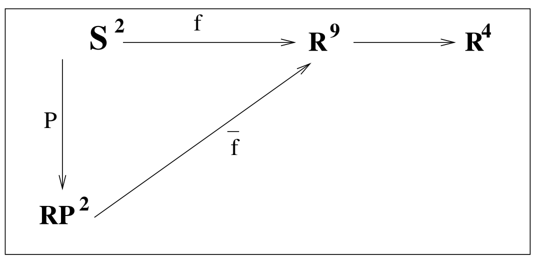
Proof *Such a map is the restriction of function f above to the sphere* S^2*. To verify smoothness and maximal rank property, for each point* $(x_1, x_2, x_3) \in S^2$ *compose* $\tilde{f}$ *with a parametrization around* (x_1, x_2, x_3) *(see Definition 1 on page 52 of do Carmo* [1]*).* □

Notice that the function $\tilde{f}$ *of the corollary is not injective, since* $(x_1,\ x_2,\ x_3)$ *and* $-(x_1,\ x_2,\ x_3)$ *have the same image. However, if we identify* $(x_1,\ x_2,\ x_3)$ *and* $(-x_1,\ -x_2,\ -x_3)$*, the map induced on the quocient space is* $1-1$.

Observe that $\dfrac{S^2}{\sim} = \mathbf{R}P^2$ *Where* $\sim$ *means* $(x_1,\ x_2,\ x_3) \sim (-x_1,\ -x_2,\ -x_3)$.

Therefore we can state

Proposition 5.3. *There exists a map* $\overline{f} : \mathbf{R}P^2 \to \mathbf{R}^9$ *with the following properties:*
(1) $\overline{f}$ *is continuous;*
(2) $\overline{f}$ *is* $1-1$*;*
(3) $D\overline{f}$ *has maximal rank;*
(4) For every point $q \in \overline{f}(\mathbf{R}P^2)$*, there exist an open ball* $B_r(q)$ *in* $\mathbf{R}^9$ *such that if* $V = B_r(q) \cap \overline{f}(\mathbf{R}P^2)$*, thus* $f^{-1} : V \longrightarrow \mathbf{R}P^2$ *is continuous.*



A map with properties (1), (2), (3) and (4) is called an embedding.
We claim now that $\overline{f}(\mathbf{R}P^2)$ *actually lives in an affine subspace* V *(a translated subspace) of dimension 5 in* $\mathbf{R}^9$*. In fact, think of* $\overline{f}(x_1,\ x_2,\ x_3)$ *as a* 3×3 *symmetric matrix given by*

$$A = \begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2^2 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3^2 \end{bmatrix} \quad \text{such that } x_1^2 + x_2^2 + x_3^2 = 1$$

The subspace of symmetric matrices has dimension 6 *and since our matrices have trace* 1 *we have our claim.*

Moreover, taking the dot product of A with itself, we obtain

$$\begin{aligned} ||A||^2 &= A \cdot A \\ &= (x_1)^4 + (x_2)^4 + (x_3)^4 + 2(x_1)^2(x_2)^2 + 2(x_2)^2(x_3)^2 + 2(x_1)^2(x_3)^2 \\ &= ((x_1)^2 + (x_2)^2 + (x_3)^2)^2 \\ &= 1 \end{aligned}$$

which implies that $\bar{f}(\mathbf{RP}^2)$ is on the unit sphere of V. We then obtain the embedding of $\mathbf{RP}^2$ in $\mathbf{R}^4$ composing $\bar{f}$ with the stereographic projection.

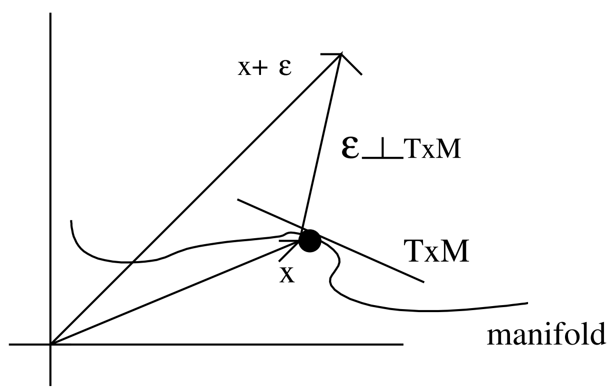
5.2. **The Tubular Neighborhood.**

It is clear that $\mathbf{R}P^2$ is not a domain of $\mathbf{R}^4$, since a domain is an open set and $\mathbf{R}P^2$, being compact, is closed in $\mathbf{R}^4$. However a domain can be constructed around $\mathbf{R}P^2$. Consider the following construction of a domain Ω in known as the tubular neighborhood.

Define $E = \{(x, \xi) | x \in \mathbf{R}P^2$ and $\xi \in T_x^{\perp}\mathbf{R}P^2\}$, where $T_x^{\perp}\mathbf{R}P^2$ denotes the plane orthogonal to the tangent plane at x.

Define the map

$$\begin{array}{rccc} f: & E & \to & \mathbf{R}^4 \\ & (x,\xi) & \mapsto & x+\xi \end{array}$$



Note that a point $x \in \mathbf{R}P^2$ is $(x, 0) \in E$. Also, the Jacobian Matrix of f at points $(x, 0)$ has rank 4, since it is the identity.

In fact, if v is tangent to $\mathbf{R}P^2$, there exists a curve $c(t)$ in $\mathbf{R}P^2$ such that $c'(0) = v$. The curve $c(t)$ in E is $(c(t), 0)$ and $f(c(t)) = c(t)$, i.e.,

$$Df \cdot v = c'(0) = v$$

Likewise, if ξ *is orthogonal to* $\mathbf{R}P^2$ *at point* x*, there exist a curve* $b(t)$ *in the orthogonal plane such that* $b'(0) = \xi$*. The curve* $b(t)$ *in* E *is* $(x, b(t))$ *and* $f(x, b(t)) = x + b(t)$*,*

$$Df \cdot \xi = b'(0) = \xi$$

Since $Df \cdot v = v$ *for all tangent vectors and* $Df \cdot \xi = \xi$ *for all orthogonal vectors, we get that* Df *is the identity matrix.*

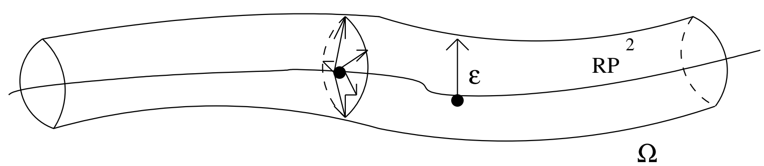
Therefore, the Inverse Function theorem implies that there exists an open ball $B_{\varepsilon_x}(x)$ *around* $(x, 0)$ *such that*

$$f|_{B_{\varepsilon_x}(x)} : B_{\varepsilon_x}(x) \to f(B_{\varepsilon_x}(x))$$

is a diffeomorphism. It follows that $V_x = f(B_{\varepsilon_x}(x))$ *is an open set in* $\mathbf{R}^4$*.*

Let $\Omega = \bigcup_{x \in \mathbf{R}P^2} V_x$*. The set* Ω *is open because it is the union of open sets.*

Remark *We actually do not need that all* V_x *have the same* ε *although we can have an* Ω *where* $\varepsilon = min\{\varepsilon_1, \cdots, \varepsilon_n\}$ *by taking a finite subcover of* $\mathbf{R}P^2$*, since it is compact.*



5.3. Ω **is Not Simply Connected.** *First observe that there is a map* $r : \Omega \to \mathbf{R}P^2$ *given by* $r(x + \xi) = x$*. It is clear that* r *is continuous and that* $r \circ i : \mathbf{R}P^2 \to \mathbf{R}P^2$ *is the identity, where* $i : RP^2 \to \Omega$ *denotes the inclusion map* $i(x) = x$*. Such a map is called a* Retraction. *It also has the following property:*

Lemma 5.4. *Let* c *be a loop in* Ω *with base point* p_0*. Then* c *is freely homotopic (see below) to loop* $\tilde{c} = r(c)$ *with base point* $x_0 = r(p_0)$*.*

Proof *Define* $H(t, s) = (1-s)r(c(t)) + sc(t)$*. Observe that* $H(t, 0) = r(c(t))$ *and* $H(t, 1) = c(t)$*. Observe also that for any fixed* s_0*,* $H(t, s_0) = (1-s_0)r(c(t)) + s_0c(t)$ *is a loop with base point* $(1 - s_0)x_0 + sp_0$*.* □

Roughly speaking, we say that two loops not having the same base point are freely homotopic *if one can be continuously deformed onto another but the loops given by the Homotopy* $H(t, s_0)$ *don't have the same base point.*

Proposition 5.5. *The tubular neighborhood Ω of $\mathbf{R}P^2$ is not simply connected.*

Proof *Each point in Ω can be represented as $x+\xi$ for some point $x \in \mathbf{R}P^2$ (as described in subsection 3), which creates a line segment conecting x to $x+\xi$. We know that there exists a retraction map $r : \Omega \to M$ such that $r : (x+\xi) \mapsto x$ along the line segment defined above for $x \in M$. Also, let us consider the identity map $i : \mathbf{R}P^2 \to \Omega$ such that $x \mapsto x + \overrightarrow{0}$. Then, $r \circ i$ is the identity map of $\mathbf{R}P^2$*

Let us define $r^{\#} : \Pi_1(\Omega) \to \Pi_1(M)$, meaning that $r^{\#}$ takes the equivalent class $[c]$ and finds the equivalence class of $[r \circ c]$, that is, $r^{\#}([c]) = [r \circ c]$. Likewise, we define $i^{\#} : \Pi_1(\mathbf{R}P^2) \to \Pi_1(\Omega)$ and $I_\Omega : \Pi_1(x) \to \Pi_1(x)$. According to the definition above, $(r \circ i)^{\#}$ must be the identity map, which in turn is given by $r^{\#} \circ i^{\#}$. Therefore, $i^{\#}$ has to be injective, otherwise composing it with $r^{\#}$ we would not obtain the identity map.

Now if $\Pi_1(\mathbf{R}P^2)$ is not trivial and $i^{\#}$ is injective, then Ω is not simply connected. $\square$

5.4. **Every closed 1-form defined on Ω is exact.** *For that we first recall that a 1-form ω on Ω is given by*

$$\omega = \sum_{i=1}^{4} a_i(x_1, \cdots, x_4) dx_i$$

Now we consider a closed form ω defined on Ω and restrict it to $\mathbf{R}P^2$. We will show that, as 1-form on $\mathbf{R}P^2$, it is also exact. Since $\mathbf{R}P^2$ is a regular surface, every point $p \in \mathbf{R}P^2$ has a neighborhood U of p such that $x_1,\ x_2,\ x_3,\ x_4$ in U are differentiable functions of x and y, for (x, y) in an open set in $\mathbf{R}^2$.

Therefore ω is locally written as

$$\begin{aligned} \omega &= \textstyle\sum_{i=1}^{4} a_i(x_1, \cdots, x_4) dx_i \\ &= \textstyle\sum_{i=1}^{4} a_i \dfrac{\partial x_i}{\partial x} dx + \displaystyle\sum_{i=1}^{4} a_i \frac{\partial x_i}{\partial y} dy \\ &= p(x,y)dx + q(x,y)dy, \end{aligned}$$

whre the second equlaity was implied by the chain rule,

$$dx_i = \frac{\partial x_i}{\partial x} dx + \frac{\partial x_i}{\partial y} dy.$$

In order to show that ω is a closed form of $\mathbf{R}P^2$, we need to show that $\dfrac{\partial p}{\partial y} = \dfrac{\partial q}{\partial x}$. In fact,

$$\begin{aligned}
\frac{\partial p}{\partial y} &= \sum_i \frac{\partial a_i}{\partial y}\frac{\partial x_i}{\partial x} + a_i \frac{\partial^2 x_i}{\partial y \partial x} \\
&= \sum_i \left(\sum_j \frac{\partial a_i}{\partial x_j}\frac{\partial x_j}{\partial y}\frac{\partial x_i}{\partial x} \right) + a_i \frac{\partial^2 x_i}{\partial y \partial x} \\
\frac{\partial q}{\partial x} &= \sum_i \frac{\partial a_i}{\partial x}\frac{\partial x_i}{\partial y} + a_i \frac{\partial^2 x_i}{\partial x \partial y} \\
&= \sum_i \left(\sum_j \frac{\partial a_i}{\partial x_j}\frac{\partial x_j}{\partial x}\frac{\partial x_i}{\partial y} \right) + a_i \frac{\partial^2 x_i}{\partial x \partial y}
\end{aligned}$$

Since ω is closed on Ω, we have that $\dfrac{\partial a_i}{\partial x_j} = \dfrac{\partial a_j}{\partial x_i}$ which substituded above implies $\dfrac{\partial p}{\partial y} = \dfrac{\partial q}{\partial x}$.

Next we show that

5.5. $\omega \in \Omega^*$ **is Exact.**

Proof: Let c_0 be a closed loop in Ω and let r denotes the retraction from Ω to $\mathbf{R}P^2$. As we proved before, such that there is a closed loop $c_1 = r(c_0)$ in $\mathbf{R}P^2$ is freely homotopic to c_0. Then by do Carmo [1]*, since c_0 and c_1 are two homotopic curves, we have that $\int_{c_0} \omega = \int_{c_1} \omega$ where ω is a closed 1-form in Ω. As before, for $\omega \in \Omega^*$ we consider its restriction on $\mathbf{R}P^2$, and we have that it is aclosed 1-form on $\mathbf{R}P^2$, which in turn is exact, by Proposition 4.5.*

Therefore

$$0 = \int_{c_1} \omega = \int_{c_0} \omega.$$

This implies, by definition, that ω is exact.

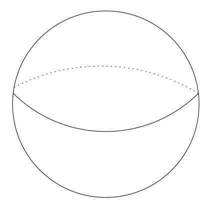
5.6. **Conclusion.**

The above construction shows that there exists a domain $\Omega \in \mathbf{R}^4$ such that Ω is not simply connected, yet every closed differential 1-form in Ω is exact, which is a complete counterexample to the question posed in the introduction.

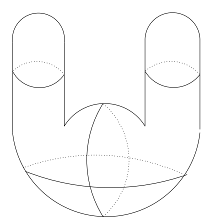
6. Case n=3 - Classical Counterexample: The Horned Sphere

Note: This case will not be explained with great detail since the detailed proof is beyond the scope of the paper's intended audience. For a more intricate explanation see Dale Rolfsen's book entitled Knots and Links.

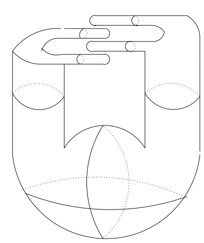
The domain Ω in R^3 is the complement of the Alexander horned sphere, which is construtucted in the following manner. Begin with a hollow sphere in $\mathbf{R}^3$.



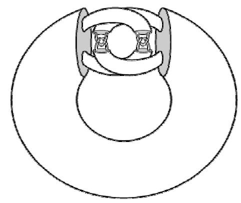
Then, push out two horns in the sphere.



Next, push out two horns from each horn.

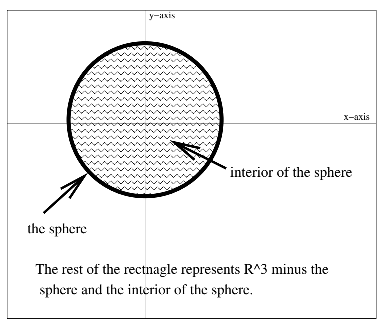


This method is continued in an infinite fashion until the Alexander horned sphere be obtained as shown below.



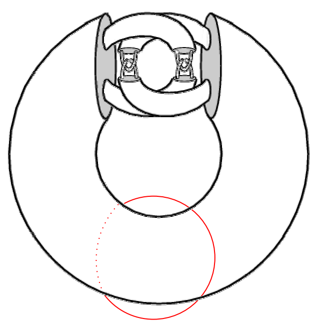
The Alexander horned sphere is homeomorphic to the hollow sphere, which is intuitively obvious by the construction defined above. Due to this homotopy, the Jordan Alexander Duality theorem applies, which states that H^1, the cohomology group, is the same for the complements of the defined spaces. The cohomology group is defined as $H^1 = \frac{closed\ forms}{exact\ forms}$.

For example, as shown earlier in the paper the sphere is simply connected. Thus, by a known result that can be found in several textbooks, for instance do Carmo [1]*, every closed 1-form defined on the sphere is exact. Therefore,* H^1 *equals the identity. The cohomology group for the complement of the hollow sphere also equals the identity. This is due to the geometry of the complement of the sphere. The complement of the sphere is composed of two pieces: the ball interior to the sphere and* R^3 *minus the sphere and its interior. A cross section of this is shown below.*



The interior of the sphere is a ball which is simply connected, so every closed loop is homotopic to a point. So the interior of the sphere is simply connected. R^3 *minus the sphere and its interior is simply connected Therefore, the complement of the sphere is simply connected, so* H^1 *equals the identity for the complement of the sphere. Then by the Jordan Alexander Duality theorem,* H^1 *of the complement of the horned sphere is also equal to the identity. So, for the defined space* Ω *in* R^3

defined as the complement of the Alexander horned sphere, every closed differential 1-form on Ω is exact. Yet Ω is not simply connected since a loop in Ω around the horned sphere (as shown in the image below) is not homotopic to a point, since the loop can not be deformed in a point without passing through the horned sphere, a fact that is not easily proved. (For a detailed proof of this conjecture see Dale Rolfsen's book entitled Knots and Links as referenced above.)



In conclusion, there is a domain Ω in R^3 defined as the complement of the horned sphere, such that every closed differential 1-form in Ω is exact, but Ω is not simply connected. Furthermore, this shows that if there exists a domain such that every closed 1-form is exact, that domain is not guaranteed to be simply connected.

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