Singularly Perturbed Integral Equations

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Abstract We study singularly perturbed Fredholm equations of the second kind. We give sufficient conditions for existence and uniqueness of solutions and describe the asymptotic behavior of the solutions. We examine the relationship between the solutions of the perturbed and unperturbed equations, exhibiting the degeneration of the boundary layer to delta functions. The results are applied to several examples including the Volterra equations.

Keywords singular perturbation 34E15, asymptotic expansion 34E05

Dedicated to the memory of Charles Lange

Introduction

This work is concerned with singularly perturbed integral equations on the interval $I = (a, b)$. Let $\epsilon$ be a small positive parameter. The perturbed equation is a Fredholm equation of the second kind

\begin{equation}
\epsilon u_\epsilon(x) + \int_a^b K(x, y)u_\epsilon(y) \, dy = f(x).
\end{equation}

By setting $\epsilon = 0$, we obtain the unperturbed equation

\begin{equation}
\int_a^b K(x, y)u_0(y) \, dy = f(x)
\end{equation}

which is a Fredholm equation of the first kind. We consider kernels $K(x, y)$ defined on $[a, b] \times [a, b]$ which have the form

\[ K(x, y) = \begin{cases} 
K_1(x, y) & y < x \\
K_2(x, y) & x < y 
\end{cases} \]

We assume that $K_1(x, y)$ and $K_2(x, y)$ are smooth on the $[a, b] \times [a, b]$. We consider several different types of kernels. In particular we examine kernels $K(x, y)$ which either have a jump discontinuity along the diagonal

\begin{equation}
K_1(x, x^-) - K_2(x, x^+) = a(x),
\end{equation}

or a jump in the $n^{th}$ derivative

\begin{equation}
\frac{\partial^n}{\partial y^n} K_1(x, y)|_{y=x} - \frac{\partial^n}{\partial y^n} K_2(x, y)|_{x=y} = a(x)
\end{equation}

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where $a(x) \in C^\infty[a, b]$ and $a(x) \neq 0$ for $x \in [a, b]$. This is an ellipticity condition which will be described in more detail later, see (EC).

Suppose the integral equation (1) has a kernel which satisfies the ellipticity condition (EC). Suppose the reduced equation (2) has a unique solution, then the perturbed equation has a unique solution in a Sobolev space which is particularly defined for this problem.

The second theorem gives the principal term in the asymptotic development of the solution and error estimates. The solution to (1) is a product of transition function and the solution to the unperturbed equation which has delta masses at the endpoints but is otherwise smooth. The transition function tends to 1 in the interior of the interval $[0,1]$ and has boundary layer terms (decaying exponentials possibly with oscillation) which may become singular at the endpoints, $x = 0$ or $x = 1$ as $\epsilon \to 0$.

These equations can be seen from the viewpoint of singularly perturbed elliptic equations which degenerate to elliptic equations. The regular degeneration of elliptic operators of higher order to elliptic differential operators was studied extensively by Višik and Ljusternik [VL]. An example of this situation is $\epsilon^2 \Delta^2 - \Delta$.

Demidov [D], Eskin [E1], Frank [F], Frank and Wendt [FW1,2,3], and Grubb [G] have studied elliptic pseudodifferential equations with a small parameter. Lange and Smith, studied equations (1) and (2) with kernels of type (3) and (4) using formal methodologies in [LS1] and [LS2]. In [A01], [A02], and [A03], Angell and Olmstead also used formal methods to examine equations (1) and (2). They also considered nonlinear equations which are not accessible by these techniques. In addition, they consider integral equations which have reduced solutions that have a kernel or cokernel and degenerate elliptic kernels. These issues are addressed in [S].

The purpose of this paper is to form a bridge between the work of Lange and Smith [LS1] and [LS2] and Eskin [E1]. [LS1] and [LS2] use an additive multivariable technique to obtain solutions to singularly perturbed Fredholm equations. On the other hand, Eskin uses techniques from pseudo-differential calculus to obtain proofs of the existence and uniqueness of solutions and describe the asymptotic behavior of the solutions to a wide class of singularly perturbed elliptic pseudo-differential equations defined on manifolds. This work treats only one dimensional equations on an interval and provides a rigorous, largely self-contained presentation of the main results in a simple, straightforward manner accessible to the applied mathematician.

There are several new points introduced in this paper. In particular, the index of factorization is not uniquely defined in the one-dimensional case. A new criteria is found to define this index, see Section 3, Example 3. Also variable order norms are used to describe the solution. Although such norms have appeared before, see [EV2], they have not been used in this context. In addition, the case of Volterra operators is not present in [E1].

The contents of the paper is as follows: section 1 provides the reformulation of the problem, section 2 contains the main theorems about the existence and uniqueness of solutions to (1) and (2) for each type of kernel and the asymptotic behavior of the solutions is described, and section 3 gives four examples including Volterra integral equations.

**Notation**

We will use the convention that the Fourier transform of a function $\phi$ in the Schwarz class $S$, functions with rapid decay at infinity, is defined by

$$\hat{\phi}(\xi) = \int_{-\infty}^{\infty} \phi(x) e^{ix\xi} dx$$

and the inverse Fourier transform of $\phi$ is

$$\phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}(\xi) e^{-ix\xi} dx.$$
The following distributions and their Fourier transforms will be frequently employed. $\Theta(x)$ is the Heaviside function which equals 1 for $x > 0$ and 0 otherwise. The Dirac delta function is denoted by $\delta(x)$. Let $x_+^\lambda = x^\lambda$ if $x > 0$ and 0 otherwise and $x_-^\lambda = |x|^\lambda$ if $x < 0$ and 0 otherwise. Let $z = \xi + i\tau$. The function $z^\lambda$ defined on $C / R^-$ has boundary values $(\xi + i0)^\lambda$ on the real axis from the upper half plane and $(\xi - i0)^\lambda$ on the real axis from the lower half plane. If $\lambda \in C$ with $\text{Re} \lambda > -1$, then

\begin{equation}
F(x_+^\lambda) = e^{\pm i\pi \frac{\lambda + 1}{2}} \Gamma(\lambda + 1)(\xi \pm i0)^{-\lambda - 1}
\end{equation}

where $\Gamma$ is the Gamma function. For details see for example [GS].

For convenience we will work on the interval $I = (0,1)$ unless stated otherwise.

**Section 1: Kernels with jump discontinuities in the $n^{th}$ derivative along the diagonal**

**Reformulation of the problem**

We show that an important class of singularly perturbed integral equations which have been treated formally in [LS] by an additive multivariable technique can be treated rigorously using techniques from pseudo-differential calculus.

By Taylor’s theorem we can expand the kernel $K(x,y)$ around the diagonal $x=y$:

\[ K(x,y) = \begin{cases} 
K_1(x,y) = \sum_{i=0}^{N} f_i(x)(x-y)^i + R_1N(x,y) & y < x \\
K_2(x,y) = \sum_{i=0}^{N} g_i(x)(x-y)^i + R_2N(x,y) & x < y 
\end{cases} \]

where each $f_i(x)$ and $g_i(x)$ is $C^\infty([0,1])$ and $R_{ijN} \in C^{n+1}([0,1] \times [0,1])$ for $j = 1,2$. If $K(x,y)$ has a non-vanishing jump discontinuity its $n^{th}$ derivative along the diagonal as in (3) and (4), then set

\begin{equation}
q_n(x,y) = K(x,y) - \frac{a(x)}{n!} (x-y)^n.
\end{equation}

Here $\frac{a(x)}{n!} = f_n(x) - g_n(x)$ and $q_n(x,y) \in C^n([0,1] \times [0,1])$. If $n = 0$, we have $q_0(x,y) = K(x,y) - a(x)\Theta(x-y)$, i.e. we are in the case that $K_1(x,x^-) - K_2(x,x^+) = a(x) = f_0(x) - g_0(x)$.

We reformulate the problem as follows: take a nonzero extension smooth extension of $K$ to $R \times R$ such that $a, f_i, g_i$ all become constant for large $x$ and $R_{ijn}$ has compact support in $R \times R$ for $j = 1,2$. For $u(x) \in S(R)$ we define the unperturbed operator

\begin{equation}
A_0(x,D)u(x) = \int_{-\infty}^{\infty} K(x,y)u(y)dy \\
= \int_{-\infty}^{\infty} A_0(x,\xi)\hat{u}(\xi)d\xi
\end{equation}

where $A_0(x,\xi)$ is the symbol of the operator $A_0(x,D)$.

Now we compute the symbol of $A_0(x,D)$:

\[ A_0(x,D)u(x) = \frac{a(x)}{n!} \int_{-\infty}^{\infty} (x-y)^n u(y)dy + \int_{-\infty}^{\infty} q_n(x,y)u(y)dy \\
= \frac{a(x)}{n!} (x_+^n * u(x)) + Q_n(x,D)u(x).
\]
By (5)
\[ A_0(x, \xi) = \frac{a(x)}{n!} F(x^n) + Q_n(x, \xi) \]
\[ = a(x) e^{i\pi \frac{n+1}{2}} (\xi + i0)^{-(n+1)} + Q_n(x, \xi), \]
where \( Q(x, \xi) \) is the symbol of \( Q(x, D) \) and satisfies the estimate
\[ |Q_n(x, \xi)| \leq c (1 + |\xi|)^{-(n+2)} \quad \text{for} \quad \xi > 1. \]

Let
\[ A(x, D, \epsilon) u(x) = \epsilon u(x) + \int_{-\infty}^{\infty} K(x, y) u(y) dy \]
and let \( p \) denote the restriction operator to the interval \( I = (0, 1) \). Now we can rewrite the integral equation (1) as a pseudo-differential equation:
\[ pA(x, D, \epsilon) u(x) = f(x) \]
and the unperturbed equation (2) as
\[ pA_0(x, D) u(x) = f(x). \]

The principal symbol of \( A(x, D, \epsilon) \) is given by
\[ A_p(x, D, \epsilon) = \epsilon + A_{0p}(x, \xi). \]

If \( K \) has a jump discontinuity in the \( n \)th derivative then, for convenience we make the change of variables \( \epsilon = \epsilon^{n+1} \). Since \((\xi + i0)^n = \xi^n\), we can write \( A_p \) as
\[ A_p(x, \xi, \epsilon) = ((\epsilon \xi)^{n+1} + a(x) e^{i\pi \frac{n+1}{2}})(\xi + i0)^{-(n+1)}. \]

We note that since we are working on a finite interval \( I \), it is possible to avoid distribution symbols. Set
\[ B(x, D) u(x) = \int_{-\infty}^{\infty} e^{-(x-y)} K(x, y) u(y) dy. \]
By expanding \( e^{-(x-y)} \) around \( x = y \), one sees that the kernel of \( B(x, D) \) differs from the kernel of \( A_0(x, D) \) by smooth terms. Therefore the symbol of \( B(x, D) \) differs from the symbol of \( A_0(x, D) \) by lower order terms. Such terms are not important in the analysis of the problem present here. The principal symbol of \( B \) is a function, namely \( B_p(x, \xi) = a(x) e^{i\pi \frac{n+1}{2}}(\xi + i)^{-(n+1)}. \)

**Some Function Spaces**

Let \( s \in \mathbb{R} \). The space \( H_s(R) \) consists of those distributions \( u \) whose Fourier transform \( \hat{u}(\xi) \) is locally integrable and such that
\[ ||u||_s^2 = \int_{-\infty}^{\infty} (1 + |\xi|)^{2s} |\hat{u}(\xi)|^2 d\xi < \infty. \]

A function space depending on a parameter \( \epsilon \) that is appropriate for this problem. Let \( H^s_{\epsilon}(R) \) be the space of distributions such that for some \( \epsilon_0 \)
\[ ([u_{\epsilon}]_{s}^2)^2 = \sup_{0 < \epsilon < \epsilon_0} \int_{-\infty}^{\infty} (1 + |\xi|)^{2s} (1 + |\xi|)^{2r} |\hat{u}_{\epsilon}(\xi)|^2 d\xi < \infty. \]
Let $r(x)$ and $s(x)$ be smooth real valued functions and

$$
\Lambda(D)^r(x)u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 + |\xi|^2)^{r(x)/2} \hat{u}(\xi)e^{-ix\xi}d\xi
$$

and

$$
\Lambda(\epsilon D)^s(x)u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 + |\xi|^2)^{s(x)/2} \hat{u}(\xi)e^{-ix\xi}d\xi.
$$

Then a function space depending on a parameter $\epsilon$ with variable order of smoothness is obtained by letting $H^s_{r(x)}(R)$ denote the space of distributions with finite norm

$$
[u_{\epsilon}]^s_{r(x)} = \sup_{0<\epsilon<\epsilon_0} \|\Lambda^s(\epsilon D)\Lambda^r(D)u_{\epsilon}\|_{L^2}.
$$

Also $u \in H^s_{r(x)}(I)$ if $u$ is in $H^s_{r(x)}(R)$ restricted to $I$ and $u \in \tilde{H}^s_{r(x)}(I)$ if $u$ is in $H^s_{r(x)}(R)$ with compact support contained in $I$.

**Factorization of the symbols**

We will factor the principal symbol. The purpose of the factorization is to change the singular perturbation problem (1) in which the $\epsilon$ dependence is found in the leading order term into a regular perturbation problem in which the $\epsilon$ dependence appears in lower order terms. The first step is to let

$$
A_p(x, \xi, \epsilon) = A_1(x, \xi \epsilon)A_{op}(x, \xi)
$$

with

$$
A_1(x, \epsilon \xi) = \frac{(\epsilon \xi)^{n+1} + a(x)e^{i\xi \frac{n+1}{2}}}{a(x)e^{i\xi \frac{n}{2}}}.
$$

Let $\eta = \epsilon \xi$. We see that $A_1(x, \eta)$ is a polynomial in $\eta$ of degree $n+1$ and it is normalized so that $A_1(x, 0) = 1$. We say that the order of $A_1(x, \eta)$, denoted by $ordA_1$, is $n+1$.

The methods used in this paper require that $A_1(x, \epsilon D)$ is an elliptic operator; that is $A_1(x, \epsilon \xi) \neq 0$ for all $x$ and $\xi$ and $\epsilon > 0$. To ensure ellipticity we impose conditions on the sign of the function $a(x)$.

**(EC) Ellipticity Condition:** Suppose the kernel has a jump discontinuity in the $n^{th}$ derivative as in (3) or (4). Then we assume that $a(x) \neq 0$. Let $k$ be a non-negative integer. If $n = 4k + 1$, then assume that $\sup a(x) < 0$. If $n = 4k + 3$, then assume that $\inf a(x) > 0$.

The ellipticity condition implies that $A_1$ has no real roots. For kernels with jump discontinuities in the $n^{th}$ derivative, $A_1(x, \eta)$ is a polynomial of degree $n+1$ in $\eta$ it is easy to factor; just let $a_j(x)$, $j=1, ..., n+1$ be the roots of $A_1$ then

$$
A_1(x, \eta) = \prod_{j=1}^{n+1} \frac{(\eta - a_j(x))}{a_j(x)}
$$

Let $a_j^{\kappa^{\pm}}(x)$ denote the roots of $A_1$ which have negative imaginary parts, $j = 1, ..., \kappa^{+}$ and $a_j^{\kappa^{-}}(x)$ be the roots with $\Im a_j(x) > 0$, for $j = 1, ..., \kappa^{-}$.

The symbol $A_1(x, \eta)$ will be factored into two functions $A_1(x, \eta) = A_1^{+}(x, \eta)A_1^{-}(x, \eta)$. Set

$$
A_1^{+}(x, \eta) = \prod_{j=1}^{\kappa^{+}} \frac{(\eta - a_j^{\kappa^{+}}(x))}{a_j^{\kappa^{+}}(x)}
$$

and

$$
A_1^{-}(x, \eta) = \prod_{j=1}^{\kappa^{-}} \frac{(\eta - a_j^{\kappa^{-}}(x))}{a_j^{\kappa^{-}}(x)}
$$
Let \( z = \eta + i\tau \). A plus symbol like \( A^+_3(x,z) \) has the following properties: it is analytic for \( \tau > 0 \) and continuous for \( \tau \geq 0 \). On the other hand, a minus symbol like \( A^-_3(x,z) \) is analytic for \( \tau < 0 \) and continuous for \( \tau \leq 0 \).

Now we give a factorization of \( A_3(x,\epsilon D) \) which is valid on the interval \([0,1]\). Let \( 0 < \delta < 1/2 \) and \( \alpha(x) \in C^\infty \) where \( \alpha(x) = 1 \) for \(-\infty < x < \delta \) and \( \alpha(x) = 0 \) for \( 1 - \delta < x < \infty \). Let \( A^+_3(x,\epsilon D) \) be the operator whose symbol is

\[
(1.3) \quad A^+_3(x,\epsilon \xi) = (A^+_3(x,\epsilon \xi))^{\alpha(x)} (A^-_3(x,\epsilon \xi))^{1-\alpha(x)}
\]

and let \( A^-_3(x,\epsilon D) \) be the operator whose symbol is

\[
(1.4) \quad A^-_3(x,\epsilon \xi) = (A^-_3(x,\epsilon \xi))^{\alpha(x)} (A^+_3(x,\epsilon \xi))^{1-\alpha(x)}
\]

These operators are well-defined since neither \( A^+_3 \) or \( A^-_3 \) has roots on the real axis. Define

\[
(1.5) \quad \kappa(x) = \kappa^+ \alpha(x) + \kappa^- (1 - \alpha(x)).
\]

So in particular, at \( x = 0 \), \( A^+_3(0,\eta) = A^+_3(0,\eta) \) and \( A^-_3(0,\eta) = A^-_3(0,\eta) \) and at \( x = 1 \), \( A^+_3(1,\eta) = A^+_3(1,\eta) \) and \( A^-_3(1,\eta) = A^-_3(1,\eta) \). Also note that \( (A^+_3)^{-1} \) and \( (A^-_3)^{-1} \) are well-defined. We point out that \( (A^+_3)^{-1}(x,z) \) is also a plus symbol. In particular we will show in the next lemma that if \( u \) has support in \([0,1]\) then \( (A^+_3)^{-1}(x,\epsilon D)u \) has support in \([0,1]\). On the other hand, \( (A^-_3)^{-1}(x,z) \) is a minus symbol, if \( v \) has support in \( R/\{0,1\} \) then \( (A^-_3)^{-1}(x,\epsilon D)v \) has support in \( R/\{0,1\} \). Let \( lv \) be an arbitrary extension of \( v \) onto \( R \), then \( p(A^-_3)^{-1}(x,\epsilon D)lv \) does not depend on the extension of \( v \).

**Lemma 1.** (1) \( A^+_3(x,D) \) isomorphically maps \( \tilde{H}^{s(x)}_r(I) \) onto \( \tilde{H}^{s(x)-\kappa(x)}_r(I) \). (2) \( (A^+_3)^{-1}(x,D) \) isomorphically maps \( \tilde{H}^{s(x)}_r(I) \) onto \( \tilde{H}^{s(x)+\kappa(x)}_r(I) \). (3) Let \( lf \) be an arbitrary extension of a function \( f \) defined on \( I \) then \( p(A^-_3)^{-1}(x,\epsilon D)lf \) isomorphically maps \( H^{\kappa(x)-n-1}_r(x,D) \) onto \( H^{\kappa(x)+n+1}_r(I) \).

The result is standard and more general theorems can be found in many places including [E2]. However, in the one-dimensional case where the operators are basically polynomials, more simple proofs can be provided. So for convenience of the reader we show (1). The rest will follow analogously.

Boundedsness of pseudo-differential operators is standard, see for example [E2, chapter 18.1 and 27.4]. The main result states that if \( A \) is a pseudo-differential operator of order \( \alpha \), then \( \|Au\|_{s-\alpha} \leq C\|u\|_s \). In our case, \( A^+_3 \) is explicitly known, see (1.3). The only special note is that we are using spaces of variable order and one keeps track of the \( \epsilon \xi \) dependence by the upper index in the space. Since \( A^+_3 \) has order \( \kappa(x) \) in \( \epsilon \xi \), \( [A^+_3 u]_{n(x)-\kappa(x)}(x) \leq C\|u\|_{s(x)} \).

Now we study the location of the support of a plus operator, i.e. we want to show that if \( u \in C^\infty(R) \) with support in \([0,1]\) then \( A^+_3(x,\epsilon D)u \) has support in \([0,1]\). Let \( \epsilon > 0 \). Take \( x \in R^- \), then

\[
A^+_3(x,\epsilon D)u = \frac{1}{2\pi} \int_{-\infty}^{\infty} A^+_3(x,\epsilon \xi) \hat{u}(\xi) e^{-i\epsilon \xi} d\xi.
\]

\[
e^{i\pi} A^+_3(x,\epsilon D)u = \frac{1}{2\pi} \int_{-\infty}^{\infty} A^+_3(x,\epsilon \xi) \hat{u}(\xi) e^{-i\epsilon \xi} d\xi.
\]
Let $s = \xi - i\tau$, then

$$|e^{-x\tau}A_3^+(x, \epsilon D)u| = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_3^+(x, \epsilon(s + i\tau))\hat{u}(s + i\tau)e^{-isx}ds.$$ 

Since $A_3^+(x, \epsilon D) = A_3^+(x, \epsilon D)$ for $x \in R^-$ and $u \in C^\infty(R)$ implies that $\hat{u}(\xi)$ has rapid decay,

$$|e^{-x\tau}A_3^+(x, \epsilon D)u| = C_N \int_{-\infty}^{\infty} \prod_{j=1}^{N} \frac{|\epsilon(s + i\tau) - a_j^+(\xi)|}{a_j^+(\xi)}(1 + |s| + |\tau|)^{-N}ds$$

is bounded for $N$ sufficiently large. Take a contour into the halfplane $\tau > 0$ and use the Cauchy theorem to see that $|e^{-x\tau}A_3^+u| \to 0$ as $\tau \to \infty$ for all $x \in R^-$. This implies that $A_3^+u = 0$ for $x \in R^-$. Similarly for $x \in (1, \infty)$, then $A_3^+(x, \epsilon D)u = A_1^+(x, \epsilon D)u$ and

$$|e^{-x\tau}A_3^+(x, \epsilon D)u| = C_N \int_{-\infty}^{\infty} \prod_{j=1}^{\kappa} \frac{|\epsilon(s + i\tau) - a_j^-(\xi)|}{a_j^-(\xi)}(1 + |s| + |\tau|)^{-N}ds < C.$$

Take a contour into the lower halfplane, $\tau < 0$ and use Cauchy’s theorem to see that $|e^{-x\tau}A_3^+u| \to 0$ as $\tau \to -\infty$ which implies that $A_3^+u = 0$ for $x \in (1, \infty)$.

$A_3^+$ has a right and left inverse. The leading term of the symbol of inverse is

$$((A_3^+(x, \epsilon \xi))^{-1} = (A_1^+(x, \epsilon \xi))^{-\alpha(x)}(A_1^-(x, \epsilon \xi))^{\alpha(x)-1}.$$ 

By the usual rules of composition of pseudo-differential operators, the other terms are both lower order and small. QED.

**Section 2: Main Theorems for kernels with jump discontinuities**

Let’s consider the unperturbed equation (2). The operator $A_0(x, D) (1.2)$ is elliptic since the principal symbol, $A_{op}(x, \xi) = a(x)e^{ix\frac{\alpha}{\kappa}(\xi + i0)^{-(n+1)}} \neq 0$ for all $x$ and $\xi \neq 0$. Notice also that the symbol of $A_0$ is $C^\infty$ and homogeneous for $\xi \neq 0$. Since the operator is $A_0$ is smoothing, i.e. has order $-n-1$, its inverse have positive order, i.e. will act to decrease regularity. If the right hand side of (2), $f$ is $C^\infty[0,1]$, the inverse of $pA_0$ can only introduce a singularity at the boundary, $x = 0$ or $x = 1$. Such a pseudo-differential operator is said to be smooth in a domain or to satisfy the transmission condition.

The degree of the singularity at the endpoints depends on the factorization of the symbol into plus and minus operators. However, the factorization of the symbol $A_{op}$ is not unique. We will pick a factorization for $A_{op}$ into $A^+$ and $A^-$ so that the solution of $pA_0u = f$ is in $H_{-\kappa(x)}$, i.e. if $\text{ord}A_1^+(x, \epsilon \xi) = \kappa(x)$ then pick $A^+$ such that $\text{ord}A^+(x, \xi) = -\kappa(x)$ and the ord$A^-(x, \xi) = -n-1+\kappa(x)$ . Locally, at $x = 0$, ord$A^+(0, \xi) = -\kappa^+$, and at $x = 1$, ord$A^+(1, \xi) = -\kappa^-$. Therefore it is reasonable to make the following assumption:

**Assumption:** We assume that there exists a unique solution $u_0 \in H_{-\kappa(x)}(I)$ for each $f \in H_{-\kappa(x)+n+1}(I)$. For $f \in C^\infty(I)$, $u_0$ has the following asymptotic behavior:

$$u_0(x) = \sum_{i=1}^{\kappa^+} c_i \delta_i^{-1}(x) + \sum_{i=1}^{\kappa^-} d_i \delta_i^{-1}(1-x) + v(x)$$

where $v(x) \in C^\infty(I)$ and $c_i$, $d_i$ are constants.
Theorem 1. Let $\epsilon > 0$ and $n$ be a non-negative integer. Suppose the integral equation (1) has a kernel which satisfies the ellipticity condition EC. Suppose the reduced equation (2) has a unique solution $u_0(x) \in H^{\kappa(x)}(I)$ for all $f \in H^{-\kappa(x)}(I)$. Then there exists an $\epsilon_0$ such that (1) has a unique solution $u_\epsilon(x) \in H^{\kappa(x)}(I)$ for all $f \in H^{\kappa(x)-n-1}(I)$ and $\epsilon < \epsilon_0$.

Proof of Theorem 1. Assume the EC and the above Assumption holds. We summarize the main steps in the proof (which can be found in [E2], chapter 27). We write (1) as

$$pAu_\epsilon = f \in H^{\kappa(x)-n-1}(I).$$

Let $u_\epsilon = (A_3^+)^{-1}v_\epsilon$, then

$$p(A_3^-)^{-1}A_3^+v_\epsilon = p(A_3^-)^{-1}lf \in H^{\kappa(x)}(I)$$

where $lf$ is an arbitrary extension of $f$ to $R$. By the usual rules of composition of pseudodifferential operators, this reduces to

$$p(A_0 + L)v_\epsilon = p(A_3^-)^{-1}lf$$

where $L$ has small operator norm, $\|Lv_\epsilon\|_{-\kappa(x)} < \alpha\|v_\epsilon\|$. The above expression is a regular perturbation of (2). This is the key idea of the proof – we turned a singular perturbation problem into a regular perturbation problem. By assumption (2) is solvable. Denote the inverse of $A_0$ by $R_0$, then

$$u_\epsilon = (A_3^+)^{-1}(I + R_0L)^{-1}R_0p(A_3^-)^{-1}lf. \tag{2.2}$$

Note that since ord$A_0p = -n-1$, then ord$R_0 = n+1$, so $R_0p(A_3^-)^{-1}lf \in H^{-\kappa(x)}$. Also, ord$(I + R_0pL)^{-1} = 0$. Since ord$(A_3^+)^{-1} = -\kappa(x)$, the unique solution to (1), $u_\epsilon(x) \in H^{\kappa(x)}(I)$.

Theorem 2. Let $f(x) \in C^\infty[0,1]$ and suppose the unperturbed equation (2) satisfies the Assumption. Then the principal asymptotic term for the solution to (1) is given by

$$u_\epsilon(x) = a_1(x, \frac{x}{\epsilon}, 1 - \frac{x}{\epsilon})v(x) + \sum_{j,l=1}^{\kappa} \alpha_{jl} e^{-i\omega_{jl}^+(0)\frac{x}{\epsilon}} + \sum_{j,l=1}^{\kappa} \gamma_{jl} e^{i\omega_{jl}^-(1)\frac{x}{\epsilon}} + w_\epsilon(x). \tag{2.3}$$

The constants $\kappa^+, \kappa^-, \alpha_{jl}$ and $\gamma_{jl}$ can be explicitly determined. The error term is small and satisfies the estimate

$$[w_\epsilon(x)]^{\kappa(x)} < C\epsilon.

The transition function $a_1(x, \frac{x}{\epsilon}, 1 - \frac{x}{\epsilon})$ has the behavior

$$a_1(x, y, z) = \sum_{l=1}^{\kappa^+} \rho_l (1 - e^{-i\omega_{l}^+(0)y})$$

as $y \to 0^+$ and $a_1(x, y, z) \to 1$ as $y, z \to 0$ and

$$a_1(x, y, z) = \sum_{l=1}^{\kappa^-} \gamma_l (1 - e^{i\omega_{l}^-(1)z}) \tag{2.5}$$

as $z \to 0^-$. The constants in the exponential terms are complex with $\text{Im}\omega_{l}^+(0) < 0$ and $\text{Im}\omega_{l}^-(1) > 0$, so that the exponentials in (7) and (8) are decaying oscillatory terms. $\rho_l$ and $\gamma_l$ are constants that can be explicitly determined.
Proof of Theorem 2

First let’s examine the behavior of the operator \((A_\eta^+)^{-1}\) in a small neighborhood around \(x = 0\). By Taylor’s theorem, expanding \((A_\eta^+)^{-1}\) around \(x = 0\), we get

\[
(A_\eta^+)^{-1}(x, \epsilon \xi) = (A_\eta^+)^{-1}(0, \epsilon \xi) + x S(x, \epsilon \xi)
\]

where \(\text{ord}_\eta S(x, \eta) \leq -\kappa^+\). So \(x S(x, \epsilon \xi) u_0 > -\kappa(x) \leq C\epsilon\). Then by (2.2),

\[
u_\epsilon = (A_\eta^+)^{-1}(0, \epsilon D) u_0 + w_\epsilon
\]

with \(< w_\epsilon >^{-\kappa(x)} \leq C\epsilon\).

Let \(\kappa^+\) be the zeros of \((A_\eta^+)^{-1}\) with \(\text{Im} \kappa^+ < 0\). Set

\[
a_\epsilon^+(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (A_\eta^+)^{-1}(0, \xi)e^{-ix\xi} d\xi
\]

by partial fractions. Using the residue theorem, we have

\[
a_\epsilon^+(x) = i \sum_{l=1}^{\kappa^+} d_l e^{-ia\kappa^+} \Theta(x)
\]

Clearly, \(a_\epsilon^+(x) \in L^1(R)\) and

\[
\int_0^\infty a_\epsilon^+(x)e^{ix\xi}dx|_{x=0} = (A_\eta^+)^{-1}(0, 0) = 1.
\]

Thus \(\frac{1}{\epsilon} a_\epsilon^+(\frac{x}{\epsilon})\) is an approximate delta function.

Since we assume that the unperturbed equation (2) has a unique solution (2.1), we can now compute the principal asymptotic term of the perturbed solution near \(x = 0\). It is

\[
(A_\eta^+)^{-1}(0, \epsilon D) u_0(x) = (A_\eta^+)^{-1}(0, \epsilon D)[v(x) + \sum_{i=1}^{\kappa^+} c_i \delta^{i-1}(x)]
\]

\[
= \int_0^x \frac{1}{\epsilon} a_\epsilon^+(\frac{y}{\epsilon}) v(x - y) dy
\]

\[
+ \sum_{l,j=1}^{\kappa^+} d_{lj} (-1)^{l-1}(-ia\kappa^+)^{j-1} \frac{e^{-ia\kappa^+ \frac{x}{\epsilon}}}{\epsilon^j}.
\]

Let

\[
a_\epsilon^+(\frac{x}{\epsilon}) = \int_0^{\frac{x}{\epsilon}} \frac{1}{\epsilon} a_\epsilon^+(\frac{y}{\epsilon}) dy
\]

\[
= \int_0^{\frac{x}{\epsilon}} a_\epsilon^+(y) dy
\]

\[
= \sum_{l=1}^{\kappa^+} \gamma_l (1 - e^{-ia\kappa^+ \frac{x}{\epsilon}}).
\]
Note that \( a_1^+ (\frac{x}{\epsilon}) \to 1 \) as \( \frac{x}{\epsilon} \to \infty \). Now change variables \( s = \frac{y}{\epsilon} \) and expand \( v(x - \epsilon s) \) in powers of \( -\epsilon s \) to obtain

\[
(A_1^+)^{-1} (0, \epsilon D) v(x) = v(x) \int_0^x a_1^+ (s) ds + \epsilon v'(x) \int_0^x a_1^+ (s) ds = a_1^+ (\frac{x}{\epsilon}) v(x) + w_3.
\]

Using that \( v \in C^\infty [0, 1] \) and integrating by parts, we see that the error term \( w_3 \) is small, ie

\[
< w_3 >^\kappa(x) < C \epsilon.
\]

So putting together (x), (xx) and (xxx), we have

\[
(A_1^+)^{-1} (0, \epsilon D) u_0 (x) = a_1^+ (\frac{x}{\epsilon}) v(x) + \sum_{j,l=1}^{\kappa} \alpha_{jl} \frac{e^{-i \alpha_j \frac{x}{\epsilon}}}{\epsilon^j} + w_4 (x)
\]

with \( < w_4 >^\kappa(x) < C \epsilon \).

One can carry out the same sort of analysis at \( x = 1 \) to obtain the result (2.3).

**Section 3: Examples**

**Example 1** Let the kernel be \( K(x, y) = \Theta(x - y) \) where \( \Theta(x) \) is the Heaviside function, \( \Theta(x) = 1 \) for \( x > 0 \), and 0 for \( x < 0 \). The reduced equation is

\[
pA_0 (D) u_0 (x) = \int_0^x u_0 (y) dy = f(x).
\]

One can easily check that if \( f \in C^\infty [0, 1] \), the unique solution is

\[
u_0 (x) = f(0) \delta(x) + f'(x) \in H_{-1/2-\delta}[I]
\]

for all \( \delta > 0 \).

Now the perturbed equation

\[
pA(D, \epsilon) u_\epsilon (x) = \epsilon u_\epsilon (x) + \int_0^x u_0 (y) dy = f(x)
\]

can be solved by standard methods; i.e. divide by \( \epsilon \), differentiate, use \( e^{x/\epsilon} \) as an integrating factor, then one finds that the solution to the perturbed equation is

\[
u_\epsilon (x) = \frac{f(0)e^{-\frac{x}{\epsilon}}}{\epsilon} + \int_0^x \frac{e^{-\frac{x-y}{\epsilon}}}{\epsilon} f'(y) dy.
\]

The solution \( u_\epsilon (x) \in C^\infty [0, 1] \) for \( \epsilon > 0 \). Since \( \Theta(x) = \frac{\epsilon^x}{\epsilon} \to \delta(x) \) as \( \epsilon \to 0 \), \( u_\epsilon (x) \to \delta(x) f(0) + f'(x) \) as \( \epsilon \to 0 \) in \( H_{-1/2-\delta} \), which is the solution of the reduced equation. Thus \( u_\epsilon (x) \) differs from \( u_0 (x) \) principally at the boundary; there \( u_\epsilon (x) \) is smoother than \( u_0 (x) \). \( u_\epsilon (x) \) has a boundary term.

Let’s follow the method of solution described in this paper (which is obviously overkill to solve this problem but can be used in much more general situations):
Step 1: find the symbols of reduced operator $A_0$ and the perturbed $A;$

$$A_0(\xi) = F(\Theta(x)) = \frac{i}{\xi + i0},$$

$$A(\xi, \epsilon) = \epsilon + A_0(\xi).$$

We are in the most simple situation as there is no $x$ dependence in the symbols.

Step 2: factor $A(\xi, \epsilon)$ into two symbols,

$$A(\xi, \epsilon) = A_1(\epsilon \xi) A_0(\xi)$$

where

$$A_1(\epsilon \xi) = \frac{i \epsilon \xi + i}{i}.$$

Step 3: factor $A_1(\epsilon \xi)$ into two symbols, a plus symbol (analytic in the upper half plane) and minus symbol (analytic in the lower half plane)

$$A_1^+(\epsilon \xi) = A_1(\epsilon \xi), \quad A_1^-(\epsilon \xi) = 1.$$

The factorization indices are $\kappa^+ = 1$ and $\kappa^- = 0$. $A_1^+ = (A_1^+)^{\alpha(x)}$ where $\alpha(x)$ is $C^\infty$ which is 1 near $x = 0$ and 0 near $x = 1$.

Step 4: find the solution to the reduced equation

$$u_0(x) = f(0) \delta(x) + f'(x)$$

Step 5: find the solution of the perturbed equation using the asymptotics in Theorem 2.

$$u_\epsilon(x) = (A_1^+)^{-1}(\epsilon D)u_0(x).$$

Locally near $x = 0$,

$$a(x) = F^{-1}((A_1^+)^{-1}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i}{\xi + i} e^{i\xi x} d\xi = e^{-x}$$

Then the solution to the perturbed equation can be expressed as the convolution of the transition function and the solution to the reduced equation

$$u_\epsilon(x) = \frac{1}{\epsilon} a(\frac{x}{\epsilon}) * (f(0) \delta(x) + f'(x))$$

$$= \frac{f(0)e^{-\frac{x}{\epsilon}}}{\epsilon} + \int_0^x \frac{e^{-\frac{y-x}{\epsilon}}}{\epsilon} f'(y) dy.$$

Carrying the computation one step further

$$a_1^+(\frac{x}{\epsilon}) = \int_0^x a(t) dt = 1 - e^{-\frac{x}{\epsilon}}.$$

Then the solution of the perturbed equation is

$$u_\epsilon(x) = \frac{f(0)e^{-\frac{x}{\epsilon}}}{\epsilon} + (1 - e^{-\frac{x}{\epsilon}}) f'(x) + w_\epsilon$$
where \( < w_\epsilon > \frac{1}{\epsilon} < C \epsilon \) which agrees with the solution obtained by standard methods modulo an error or order \( \epsilon \). The error rises because of the factorization of the symbols which introduces an \( x \) dependence in the symbols (which locally are constant coefficient symbols).

**Example 2** Let \( K(x, y) = \Theta(y - x) \). Then the perturbed equation is

\[
pA(D, \epsilon)u_\epsilon(x) = \epsilon u_\epsilon(x) + \int_x^1 u_0(y)dy = f(x)
\]

and the unperturbed or reduced equation is

\[
pA_0(D)u_0(x) = \int_x^1 u_0(y)dy = f(x).
\]

When \( f \in C^\infty[0, 1] \), the unique solution is

\[
u_0(x) = f(1)\delta(1 - x) + f'(x).
\]

The symbols of \( A_0(D) \) and \( A(D, \epsilon) \) are given by

\[
A_0(\xi) = F(1 - \Theta(x)) = \frac{-i}{\xi - i0},
\]

\[
A(\xi, \epsilon) = \epsilon + A_0(\xi).
\]

So

\[
A(\xi, \epsilon) = A_1(\epsilon \xi)A_0(\xi)
\]

where

\[
A_1(\epsilon \xi) = \frac{\epsilon \xi - i}{-i}
\]

and

\[
A_1(\epsilon \xi) = A_1^-(\epsilon \xi), \quad A_1^+(\epsilon \xi) = 1.
\]

The two obvious differences between this example and Example 1 is that the solution to the reduced equation has a Dirac mass concentrated at \( x = 1 \) rather than at \( x = 0 \) and \( A_1 \) is a minus symbol. The change of variables \( x \to 1 - x \) induces the change of variables in the symbol \( A_1^- (\xi) \to A_1^- (-\xi) \) which is a plus symbol. So using the calculations in Example 1,

\[
u_\epsilon(x) = \frac{f(1)e^{-\frac{1-x}{\epsilon}}}{\epsilon} + (1 - e^{-\frac{1-x}{\epsilon}})f'(x) + w_\epsilon
\]

where \( < w_\epsilon > \frac{1}{\epsilon} < C \epsilon \).

**Example 3** Let \( K(x, y) = \frac{1}{2} \Theta(x - y) - \frac{1}{2} \Theta(y - x) \). Then the perturbed equation is

\[
pA(D, \epsilon)u_\epsilon(x) = \epsilon u_\epsilon(x) + \frac{1}{2} \int_0^x u_\epsilon(y)dy - \frac{1}{2} \int_x^1 u_\epsilon(y)dy = f(x)
\]

and the unperturbed or reduced equation is

\[
pA_0(D)u_0(x) = \frac{1}{2} \int_0^x u_0(y)dy - \frac{1}{2} \int_x^1 u_0(y)dy = f(x).
\]
The solution to the unperturbed equation depends on the Sobolev space. For example, if \( f \in C^\infty[0, 1] \),

\[
u_0(x) = c_1 \delta(x) + c_2 \delta(1 - x) + f'(x)
\]

where the constants can be

\[
(3.1) \quad c_1 = f(0) + f(1) \quad c_2 = 0
\]

\[
(3.2) \quad c_1 = f(0) \quad c_2 = -f(1)
\]

\[
(3.3) \quad c_1 = 0 \quad c_2 = -(f(0) + f(1))
\]

Note that each of the above solutions belong to different Sobolev space \( H_{\kappa(x)}(I) \), with \( \kappa(x) = \kappa^+ \alpha(x) + \kappa^- (1 - \alpha(x)) \). (3.1) is the unique solution if \( \kappa^+ = 1 \) and \( \kappa^- = 0 \). The solution given by (3.2) or (3.3) comes from being in the space, with \( \kappa^+ = 1 \) and \( \kappa^- = 1 \) with \( \kappa^+ = 0 \) and \( \kappa^- = 1 \).

Here we use the factorization of \( A_1 \) to determine which solution of the unperturbed equation gives a unique solution for perturbed equation.

One can write the unperturbed equation as either

\[
pA_0(D)u_0(x) = \int_0^x u_0(y)dy - \frac{1}{2} \int_0^1 u_0(y)dy
\]

\[
= -\int_x^1 u_0(y)dy + \frac{1}{2} \int_0^x u_0(y)dy.
\]

In both cases the second integral on the right-hand side is a lower order term. So the principal symbol of \( A_0(D) \) is either \( A_{op} = \frac{i}{\epsilon - x} \) or \( A_{op} = \frac{i}{x - \epsilon} \). Regardless of which symbol we use for \( A_0 \), the principal symbol of the perturbed equation is

\[
A_p(\xi, \epsilon) = \epsilon + A_{op}(\xi) = A_1(\epsilon \xi)A_{op}(\xi)
\]

with

\[
A_1^+(\epsilon \xi) = \frac{\epsilon \xi + i}{i} \quad A_1^-(\epsilon \xi) = 1.
\]

The factorization indices are \( \kappa^+ = 1 \) and \( \kappa^- = 0 \). We take the factorization of \( A_{op} \) as \( A_{op}^+ A_{op}^- \) with \( A_{op}^+ = \frac{i}{\epsilon + x} \) and \( A_{op}^- = 1 \).

This indicates that we select space of the unperturbed solution to be with one with a delta mass at \( x = 0 \) and no delta masses at \( x = 1 \). In the space, we have a unique solution, i.e. \( u_0(x) = (f(0) + f(1)) \delta(x) + f'(x) \).

Again using Theorem 2 and the computations in Example 1,

\[
u_\epsilon(x) = \left( \frac{f(0) + f(1)}{\epsilon} \right) e^{-\frac{x}{\epsilon}} + \left( 1 - e^{-\frac{x}{\epsilon}} \right) f'(x) + w_\epsilon
\]

where \( < w_\epsilon >_{1,1} < C \epsilon \).

Example 4: We consider a singularly perturbed Volterra equation on \([0, T]\). Let \( p \) denote the restriction operator to \([0, T]\). Let \( a(x) \in C^\infty[0, T] \), \( a(x) \geq c > 0 \), \( Q(x, y) \) be a smooth function defined on \( 0 \leq y \leq x \leq T \) which vanishes along the diagonal and \( f \) a given smooth function on \([0, T]\). We are looking for the solution \( u_\epsilon(x) \in L^2(R) \) with support in \([0, T]\) to

\[
pA(x, \epsilon, D)u_\epsilon(x) = \epsilon u_\epsilon(x) + a(x) \int_0^x u_\epsilon(y)dy + \int_0^x Q(x, y)u_\epsilon(y)dy
\]

\[
= f(x).
\]
The associated unperturbed equation is

\[ pA_0(x, D)u_0(x) = a(x) \int_0^x u_0(y)dy + \int_0^x Q(x, y)u_0(y)dy = f(x). \]

For each \( f \in C^\infty[0, T] \) we assume that there exists a unique solution \( u_0(x) \in L^2(R) \) with compact support contained in \([0, T]\) of the form \( u_0(x) = v(x) + c\delta(x) \). Inserting \( u_0 \) into the unperturbed equation, we have

\[ c(a(x) + Q(x, 0)) + a(x) \int_0^x v(y)dy + \int_0^x Q(x, y)v(y)dy = f(x). \]

Evaluating at \( x = 0 \), we find

\[ c = \frac{f(0)}{a(0)} \]

since \( Q(0, 0) = 0 \). Then by dividing through by \( a(x) \)

\[ \int_0^x v(y)dy + \frac{1}{a(x)} \int_0^x Q(x, y)v(y)dy = \frac{f(x) - ca(x)}{a(x)}. \]

Differentiating the above expression, we have

\[ v(x) + \frac{d}{dx} \left( \frac{1}{a(x)} \int_0^x Q(x, y)v(y)dy \right) = \frac{d f(x)}{dx} \frac{1}{a(x)}. \]

This is a Volterra integral equation of the second kind which is uniquely solvable, (see, for example Yosida [Y]). Therefore, there exists a unique solution to (2), \( u_0(x) = f(0)\delta(x) + v_0(x) \) where \( v_0(x) \in C^\infty[0, T] \).

The principal symbol of \( A_0(x, D) \) is

\[ A_{0p}(x, \xi) = \frac{a(x)i\xi}{\xi + i\epsilon} \]

which is a plus symbol. Since

\[ A_1^+(x, \epsilon\xi) = \epsilon\xi + ia(x) \frac{ia(x)}{ia(x)} \]

and

\[ A_1^-(x, \epsilon\xi) = 1, \]

by Theorem 2 (and using the computations in Example 1), the solution to the perturbed equation is

\[ u_\epsilon(x) = (1 - e^{-a(0)\epsilon})v_0(x) + f(0)e^{-a(0)\epsilon} \frac{a(0)}{\epsilon} + w_\epsilon(x) \]

where \( \|w_\epsilon(x)\|_{L^2[0, T]} < c\epsilon \).

**Acknowledgment:** I would like to thank G. Eskin not only for suggesting the problem, his valuable comments, but also his patience. I’d like to thank Werner Horn and Blake Barley for some useful discussions.

**References**


