Exact Fermi coordinates for a class of space-times

David Klein[1,a)] and Peter Collas[2,b)]

[1]*Department of Mathematics, California State University, Northridge, Northridge, California 91330-8313, USA*

[2]*Department of Physics and Astronomy, California State University, Northridge, Northridge, California 91330-8268, USA*

We find exact Fermi coordinates for timelike geodesic observers for a class of space-times that includes anti-de Sitter space-time, de Sitter space-time, the constant density interior Schwarzschild space-time with positive, zero, and negative cosmological constant, and the Einstein static universe. Maximal charts for Fermi coordinates are discussed. © *2010 American Institute of Physics.*
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I. INTRODUCTION

The effects of a gravitational field are most naturally analyzed by using a system of locally inertial coordinates. For an observer following a timelike path, Fermi–Walker coordinates provide such a system. A Fermi–Walker coordinate frame is nonrotating in the sense of Newtonian mechanics and is realized physically as a system of gyroscopes.[1–4] Applications of these coordinate systems are extensive and include the study of tidal dynamics, gravitational waves, relativistic statistical mechanics, and quantum gravity.[5–12] In the case that the path of the observer is geodesic, Fermi–Walker coordinates are commonly referred to as Fermi or Fermi normal coordinates. The metric in that case is Minkowskian to first order near the path, with second order corrections involving only the curvature tensor.[13]

Under general conditions, a timelike path has a neighborhood on which a Fermi–Walker coordinate system can be defined[14] (p. 200). In addition, general formulas in the form of Taylor expansions for coordinate transformations to and from Fermi–Walker coordinates, valid in some neighborhood of a given timelike path in general space-times, were given in Ref. 15. However, to the best of our knowledge, rigorous results for the radius of a tubular neighborhood of a timelike path for the domain of Fermi coordinates are not available. In addition to potential applications, it is therefore revealing to find examples where exact coordinate transformations to and from Fermi coordinates can be calculated in order to determine the maximum extent of coordinate charts for those coordinate systems.

In this paper, we find exact transformations to and from Fermi coordinates for a class of space-times. Our starting point is a generic metric given by Eq. (2) below. In Sec. II, Theorems 1 and 2 give explicit charts with Fermi coordinates for metrics of the form of Eq. (2). We use sectional curvature of appropriate two-dimensional submanifolds to define Jacobi fields that measure the separation of (Fermi) coordinate, spacelike geodesics. Our examples, described in Sec. III, include the metrics for anti-de Sitter space-time (AdS_4), de Sitter space-time (dS_4), the interior constant density Schwarzschild space-time with positive, negative, or zero cosmological constant, and the Einstein static universe. We also discuss the breakdown of Fermi coordinates at the horizon in dS_4. Concluding remarks are given in Sec. IV.

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a)Electronic mail: david.klein@csun.edu.

b)Electronic mail: peter.collas@csun.edu.

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II. FERMI COORDINATES AND CURVATURE FOR A CLASS OF METRICS

In a space-time M, let $\sigma(\tau)$ be a timelike geodesic parametrized by proper time τ with unit tangent vector $e_0(\tau)$. A Fermi normal coordinate system along σ is determined by an orthonormal tetrad of vectors, $e_0(\tau)$, $e_1(\tau)$, $e_2(\tau)$, $e_3(\tau)$ parallel along σ. Fermi coordinates x^0, x^1, x^2, x^3 relative to this tetrad are defined by

$$x^0(\exp_{\sigma(\tau)}(\lambda^j e_j(\tau))) = \tau,$$

$$x^k(\exp_{\sigma(\tau)}(\lambda^j e_j(\tau))) = \lambda^k, \tag{1}$$

where here and below, Greek indices run over 0,1,2,3 and Latin over 1,2,3. The exponential map, $\exp_p(\vec{v})$, denotes the evaluation at affine parameter 1 of the geodesic starting at the point p in the space-time, with initial derivative $\vec{v}$, and it is assumed that the λ^j are sufficiently small so that the exponential maps in Eq. (1) are defined.

Consider a line element of the form

$$ds^2 = -(1 - f(x,y,z))dt^2 + dx^2 + dy^2 + dz^2 + [(1 - kr^2)^{-1} - 1]dr^2, \tag{2}$$

where $r^2 = x^2 + y^2 + z^2$, k is a constant, and $f(x,y,z)$ is a smooth function, which together with its first partial derivatives vanishes at $x = y = z = 0$. When $f(x,y,z) \equiv 0 = k$, Eq. (2) is the Minkowski metric. Although not essential, we assume for convenience that $1 - f(x,y,z)$ does not vanish when $1 - kr^2 > 0$, and that this last expression determines the range of spatial coordinates (x,y,z) for the chart on which the metric is described by Eq. (2).

Since all first partial derivatives of the metric elements determined by Eq. (2) vanish on the timelike path $\sigma(t) = (t,0,0,0)$, it immediately follows that the connection coefficients also vanish on $\sigma(t)$, and that $\sigma(t)$ is a geodesic. Moreover, $t = \tau$ is proper time, and the following orthonormal tetrad is parallel along $\sigma(t)$,

$$\frac{\partial}{\partial t} = e_0(\tau) = (1,0,0,0),$$

$$\frac{\partial}{\partial x} = e_1(\tau) = (0,1,0,0),$$

$$\frac{\partial}{\partial y} = e_2(\tau) = (0,0,1,0),$$

$$\frac{\partial}{\partial z} = e_3(\tau) = (0,0,0,1). \tag{3}$$

We construct Fermi coordinates for $\sigma(t) = (t,0,0,0)$, beginning with the inverse transformation, from Fermi coordinates $\{x^0, x^1, x^2, x^3\}$ to Cartesian coordinates $\{t,x,y,z\}$, given by the following theorem.

In what follows, it is convenient to define $a \equiv \sqrt{|k|} > 0$.

Theorem 1:

(a)   *When $k > 0$, the transformation from Fermi coordinates along $\sigma(t)$ to the coordinates $\{t,x,y,z\}$ is given by*

$$t = x^0, \tag{4}$$

$$x = x^1\left(\frac{\sin(\rho a)}{\rho a}\right), \tag{5}$$

$$y = x^2\left(\frac{\sin(\rho a)}{\rho a}\right), \tag{6}$$

$$z = x^3\left(\frac{\sin(\rho a)}{\rho a}\right). \tag{7}$$

(b)   When $k < 0$, the transformation from Fermi coordinates along $\sigma(t)$ to the coordinates $\{t, x, y, z\}$ is given by

$$t = x^0, \tag{8}$$

$$x = x^1\left(\frac{\sinh(\rho a)}{\rho a}\right), \tag{9}$$

$$y = x^2\left(\frac{\sinh(\rho a)}{\rho a}\right), \tag{10}$$

$$z = x^3\left(\frac{\sinh(\rho a)}{\rho a}\right), \tag{11}$$

where $\rho^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$.

Proof: It follows from Eq. (1) that a necessary and sufficient condition for $\{x^0, x^1, x^2, x^3\}$ to be Fermi coordinates relative to a tetrad $e_0(\tau)$, $e_1(\tau)$, $e_2(\tau)$, $e_3(\tau)$ along a geodesic σ is that in these coordinates,

$$\exp_{\sigma(\tau)}(sa^j e_j(\tau)) = (\tau, sa^1, sa^2, sa^3), \tag{12}$$

where s measures proper distance and $\sqrt{(a^1)^2 + (a^2)^2 + (a^3)^2} = 1$. Thus, it suffices in our case to prove that $X_t(s) \equiv (t, sa^1, sa^2, sa^3)$ is geodesic in the coordinates $\{x^0, x^1, x^2, x^3\}$ given by Eqs. (4)–(7) for $k > 0$ and (8)–(11) for $k < 0$. This is readily verified by using these equations to transform the metric of Eq. (2), yielding the results of Corollary 1 below, from which the connection coefficients are determined. It then follows by direct calculation that

$$\Gamma^\nu_{ij}(t, x^1, x^2, x^3) x^i x^j = 0, \tag{13}$$

which is equivalent to

$$\frac{d^2 X^\nu}{ds^2} + \Gamma^\nu_{\alpha\beta}\frac{dX^\alpha}{ds}\frac{dX^\beta}{ds} = \Gamma^\nu_{ij}(t, sa^1, sa^2, sa^3) a^i a^j = 0. \tag{14}$$

Thus, $X_t(s) \equiv (t, sa^1, sa^2, sa^3)$ is geodesic for all choices of (a^1, a^2, a^3).   ∎

The following two corollaries follow from Theorem 1 and direct calculation.

Corollary 1: The metric in Fermi coordinates for the observer $\sigma(t)$,

(a)   when $k > 0$ is given by

$$g_{00} = -\left[1 - f\left(x^1\left[\frac{\sin(\rho a)}{\rho a}\right], x^2\left[\frac{\sin(\rho a)}{\rho a}\right], x^3\left[\frac{\sin(\rho a)}{\rho a}\right]\right)\right], \tag{15}$$

$$g_{0i} = 0, \tag{16}$$

$$g_{ij} = \frac{x^i x^j}{\rho^2} + \frac{\sin^2(a\rho)}{a^2\rho^2}\left(\delta_{ij} - \frac{x^i x^j}{\rho^2}\right). \tag{17}$$

(b)   when $k < 0$, is given by,

$$g_{00} = -\left[1 - f\left(x^1\left[\frac{\sinh(\rho a)}{\rho a}\right], x^2\left[\frac{\sinh(\rho a)}{\rho a}\right], x^3\left[\frac{\sinh(\rho a)}{\rho a}\right]\right)\right], \tag{18}$$

$$g_{0i} = 0, \tag{19}$$

$$g_{ij} = \frac{x^i x^j}{\rho^2} + \frac{\sinh^2(a\rho)}{a^2\rho^2}\left(\delta_{ij} - \frac{x^i x^j}{\rho^2}\right). \tag{20}$$

Corollary 2: Under the change in spatial coordinates, $x^1 = \rho \sin\theta\cos\phi$, $x^2 = \rho\sin\theta\sin\phi$, $x^3 = \rho\cos\theta$, the Fermi metric given by Corollary 1,

(a)   for $k > 0$ becomes

$$ds^2 = g_{00}dt^2 + d\rho^2 + \frac{\sin^2(a\rho)}{a^2}(d\theta^2 + \sin^2\theta d\phi^2), \tag{21}$$

(b)   for $k < 0$ becomes

$$ds^2 = g_{00}dt^2 + d\rho^2 + \frac{\sinh^2(a\rho)}{a^2}(d\theta^2 + \sin^2\theta d\phi^2), \tag{22}$$

where g_{00} is given by Eq. (15) for part (a), and (18) for part (b).

Theorem 2:

(a)   When $k > 0$, the transformation from the coordinates $\{t, x, y, z\}$ to Fermi coordinates along $\sigma(t)$ is given by

$$x^0 = t, \tag{23}$$

$$x^1 = x\left(\frac{\sin^{-1}(ra)}{ra}\right), \tag{24}$$

$$x^2 = y\left(\frac{\sin^{-1}(ra)}{ra}\right), \tag{25}$$

$$x^3 = z\left(\frac{\sin^{-1}(ra)}{ra}\right). \tag{26}$$

(b)   When $k < 0$, the transformation from the coordinates $\{t, x, y, z\}$ to Fermi coordinates along $\sigma(t)$ is given by

$$x^0 = t, \tag{27}$$

$$x^1 = x\left(\frac{\sinh^{-1}(ra)}{ra}\right), \tag{28}$$

$$x^2 = y\left(\frac{\sinh^{-1}(ra)}{ra}\right), \tag{29}$$

$$x^3 = z\left(\frac{\sinh^{-1}(ra)}{ra}\right),\tag{30}$$

where, as above, $r^2 = x^2 + y^2 + z^2$.

Proof: To prove part (a), observe that squaring and adding Eqs. (5)–(7), gives

$$r^2 = \frac{\sin^2(\rho a)}{a^2}.\tag{31}$$

Solving for ρ in the above equation, and then for x, y, and z in Eqs. (5)–(7) yields Eqs. (24)–(26). The proof of part (b) using

$$r^2 = \frac{\sinh^2(\rho a)}{a^2}\tag{32}$$

is similar.   ∎

Remark 1: The independence from the function $f(x,y,z)$ of the coordinate transformations appearing in Theorems 1 and 2 is a consequence of Eq. (13) and the assumption that $f(x,y,z)$ and its first partial derivatives vanish on σ.

Remark 2: Under the assumptions made in the paragraph below Eq. (2), it follows from Eqs. (31) and (32) that for $k > 0$, the domain of the spatial Fermi coordinates may be chosen to include any open set in which $0 \leq \rho < \pi/2a$, and for $k < 0$, $0 \leq \rho < \infty$.

The following corollary will be used to identify a Jacobi field for the congruence of spatial geodesics orthogonal to the Fermi observer's world line.

Corollary 3: Let M be a space-time with metric given by Eq. (21) or (22). Let N be a two-dimensional submanifold of M generated by the Fermi coordinates t and ρ with the angular coordinates held fixed so that the induced metric on N is given by

$$ds^2 = g_{00}dt^2 + d\rho^2.\tag{33}$$

Then the Gaussian curvature K of N is given by

$$K = \frac{-1}{\sqrt{-g_{00}}}\frac{\partial^2}{\partial \rho^2}\sqrt{-g_{00}}.\tag{34}$$

Proof: The result follows easily from Proposition 44 (p. 81) of Ref. 14 and direct calculation. ∎

Remark 3: In the case that g_{00} is a function of ρ only, it is easy to verify that N [with the induced metric, Eq. (33)] is totally geodesic in M, i.e., the shape tensor vanishes. Thus, the intrinsic geometry of N coincides with its extrinsic geometry as a submanifold of M. In particular, the sectional curvature of N in M is the Gaussian curvature K.

We assume now that g_{00} is a function of ρ only, i.e.,

$$g_{00} = g_{00}(\rho).\tag{35}$$

The vector field $\partial/\partial t$ is a variation vector field for the geodesic variation in spacelike geodesics of the form, $X_t(\rho) = (t, \rho)$, parametrized in N by t. Therefore, the Jacobi equation,

$$\nabla_{\partial/\partial\rho}\nabla_{\partial/\partial\rho}(\partial/\partial t) = R_{(\partial/\partial t)(\partial/\partial\rho)}(\partial/\partial\rho),\tag{36}$$

is satisfied, where ∇ is the Levi–Civita connection (on either N or M) and R is the Riemann curvature operator. In light of Remark 3, the right side of Eq. (36) may be expressed in terms of the Gaussian curvature K, yielding

$$\nabla_{\partial/\partial\rho}\nabla_{\partial/\partial\rho}(\partial/\partial t) = -K\,\partial/\partial t. \tag{37}$$

The unit vector $T=(1/\sqrt{-g_{00}})(\partial/\partial t)$ is orthogonal to $\partial/\partial\rho$ and thus parallel along the spacelike geodesic $X_t(\rho)=(t,\rho)$ (with t fixed). It follows that

$$\nabla_{\partial/\partial\rho}\nabla_{\partial/\partial\rho}(\partial/\partial t) = \nabla_{\partial/\partial\rho}\nabla_{\partial/\partial\rho}(\sqrt{-g_{00}}\,T) = \left(\frac{\partial^2}{\partial\rho^2}\sqrt{-g_{00}}\right)T. \tag{38}$$

Equation (37) then becomes

$$\left(\frac{\partial^2}{\partial\rho^2}\sqrt{-g_{00}} + K\sqrt{-g_{00}}\right)T = 0, \tag{39}$$

which is equivalent to Eq. (34). Thus, $y=(t_2-t_1)\sqrt{-g_{00}}$ is a measure of separation of the spacelike geodesics $X_{t_1}(\rho)=(t_1,\rho)$ and $X_{t_2}(\rho)=(t_2,\rho)$ at proper distance ρ and is a solution of the initial value problem,

$$\frac{\partial^2 y}{\partial\rho^2} + K(\rho)y = 0,$$

$$y'(0) = (t_2-t_1)\frac{-g'_{00}(0)}{\sqrt{-g_{00}(0)}} = 0,$$

$$y(0) = (t_2-t_1)\sqrt{-g_{00}(0)} = t_2-t_1, \tag{40}$$

where, in the initial data, we have used the assumptions on g_{00} that immediately follow Eq. (2), and for convenience, we assume that $t_2 > t_1$.

The following lemma shows that when the Gaussian curvature on N is nonpositive, there is a natural timelike separation of the spacelike geodesics that define the Fermi space coordinate, which never becomes null.

Lemma 1: Let $K(\rho) \leq 0$ be continuous and suppose that y is a solution to the initial value problem, Eqs. (40). Then y has no positive roots.

Proof: Suppose to the contrary that ρ_0 is the least positive root of y. Then $y'(\rho_0) \leq 0$. Since by assumption, $y'(0)=0$, $y'(\rho)$ must be a decreasing function on some open subinterval of $[0,\rho_0]$. On that subinterval, $y''(\rho) < 0$, which contradicts the assumption on K. ∎

III. EXAMPLES

Using the results of Sec. II, we find in this section exact expressions for the metrics in Fermi coordinates along particular timelike geodesics in AdS_4, dS_4, the interior constant density Schwarzschild space-time with positive, zero, and negative cosmological constant, and the Einstein static universe. We also discuss the range of Fermi coordinates together with the Gaussian curvatures of the associated submanifolds (i.e., N) described in Sec. II.

Example 1: AdS_4 and dS_4 metrics in Fermi coordinates.

In static coordinates of dS_4, or the analog for AdS_4, the metric is

$$ds^2 = -\left(1-\frac{\Lambda r^2}{3}\right)dt^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2) + \left(1-\frac{\Lambda r^2}{3}\right)^{-1}dr^2, \tag{41}$$

where the cosmological constant Λ is positive in the case of dS_4 and negative for AdS_4. In the case of dS_4, Eq. (41) is singular at the cosmological horizon where $r=\sqrt{3/\Lambda}$. The horizon divides space-time into four regions as may be seen from the Penrose diagram.[16] In one of these regions the timelike Killing vector $\partial/\partial t$ is future directed, $0 \leq r < \sqrt{3/\Lambda}$, and an observer at $r=0$ is surrounded by the cosmological horizon at $r=\sqrt{3/\Lambda}$. For the case of dS_4, we consider the Fermi observer at $r=0$ in this causal region.

By contrast, when $\Lambda < 0$ (for AdS_4), the range of r is unrestricted, i.e., $0 \leq r < \infty$. In either case, Eq. (41) may be rewritten as

$$ds^2 = -dt^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) + dr^2 + \frac{\Lambda r^2}{3}dt^2 + \left[\left(1 - \frac{\Lambda r^2}{3}\right)^{-1} - 1\right]dr^2. \qquad (42)$$

The first line of Eq. (42) is the Minkowski metric in spherical coordinates. Changing to Cartesian space coordinates x, y, z, and identifying $r^2 = x^2 + y^2 + z^2$, Eq. (42) becomes

$$ds^2 = -\left(1 - \frac{\Lambda r^2}{3}\right)dt^2 + dx^2 + dy^2 + dz^2 + \left[\left(1 - \frac{\Lambda r^2}{3}\right)^{-1} - 1\right]dr^2, \qquad (43)$$

which has the form of Eq. (2) with $f(x,y,z) = \Lambda r^2/3$, $k = \Lambda/3$.

Using Eq. (32), we find that the Fermi metric for the observer $\sigma(t) = (t,0,0,0)$ in AdS_4 is

$$ds^2 = -\cosh^2(a\rho)dt^2 + g_{ij}dx^i dx^j, \qquad (44)$$

where $a = \sqrt{|k|} = \sqrt{|\Lambda|/3}$ and the spatial metric coefficients g_{ij} are given by Eq. (20). Fermi coordinates $\{x^0, x^1, x^2, x^3\}$ are global on the covering space for AdS_4, and consistent with Remark 2, Eq. (44) is valid on the entire space-time. The associated polar metric given by Corollary 2, although heretofore not associated with Fermi coordinates, is independently well known and extant in the literature,

$$ds^2 = -\cosh^2(a\rho)dt^2 + d\rho^2 + \frac{\sinh^2(a\rho)}{a^2}(d\theta^2 + \sin^2 \theta d\phi^2). \qquad (45)$$

The Fermi metric for the observer $\sigma(t) = (t,0,0,0)$ in static coordinates in dS_4 is analogous. Using Eq. (31) for $\Lambda > 0$,

$$ds^2 = -\cos^2(a\rho)dt^2 + g_{ij}dx^i dx^j, \qquad (46)$$

where $a = \sqrt{k} = \sqrt{\Lambda/3}$ and the spatial metric coefficients g_{ij} are given by Eq. (17). Consistent with Remark 2, Fermi coordinates $\{x^0, x^1, x^2, x^3\}$ cover the region of dS_4 satisfying $\rho = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} < \pi/2a$, the same region covered by static coordinates, up to the cosmological horizon. The associated polar metric given by Corollary 2 is

$$ds^2 = -\cos^2(a\rho)dt^2 + d\rho^2 + \frac{\sin^2(a\rho)}{a^2}(d\theta^2 + \sin^2 \theta d\phi^2). \qquad (47)$$

Remark 4: We note that Eqs. (46) and (47) for dS_4 are not new. Chicone and Mashhoon, starting with a different coordinate system for the de Sitter universe, previously derived Eqs. (46) and (47) in Ref. 5, and observed that Eq. (47) appears in de Sitter's original 1917 investigations. Exact Fermi coordinates for Gödel space-time are also given in Ref. 5.

With the notation of Corollary 3, a short calculation shows that the Gaussian curvature K of the submanifold spanned by the Fermi coordinates t, ρ with the angular coordinates held fixed is given by

$$K = \frac{\Lambda}{3}, \qquad (48)$$

so that K is positive on the submanifold N of dS_4 and negative on the corresponding submanifold of AdS_4.

Equations (40) apply to these examples, but it is instructive to analyze directly the way in which the Fermi coordinate system breaks down at the horizon of dS_4, where $\rho = \pi/2a$. Consider two spacelike geodesics with the same fixed angular coordinates, orthogonal to the Fermi observ-

er's worldline. Without loss of generality we take the angular coordinates to be fixed at $\phi=0$ and $\theta=\pi/2$ and the Fermi time coordinates to be t_1 and t_2 with $t_1 < t_2$. The two spacelike geodesics are then given by

$$X_i(\rho) = (t_i, \rho, \pi/2, 0), \quad i = 1, 2. \tag{49}$$

When $\rho=0$, X_1 and X_2 lie on the timelike geodesic path of the Fermi observer. For $0 < \rho_0 < \pi/2a$, the two space-time points $X_1(\rho_0)$ and $X_2(\rho_0)$ are the same proper distance ρ_0 from the Fermi observer's path and are connected to each other by the timelike path,

$$\gamma_{\rho_0}(t) = (t, \rho_0, \pi/2, 0) \quad \text{for} \quad t_1 \le t \le t_2. \tag{50}$$

The observer following the path $\gamma_{\rho_0}(t)$ starts at $X_1(\rho_0)$, waits for the fixed Fermi coordinate time interval, $t_2 - t_1$ (without changing Fermi space coordinates), and then arrives at the space-time point $X_2(\rho_0)$.

However, the proper time along $\gamma_{\rho_0}(t)$ is less than the Fermi time interval by a factor of $\cos(a\rho_0)$, which decreases to zero as $\rho_0 \to \pi/2a$. Expressed another way, the tangent vector $\partial/\partial t$ of $\gamma_{\rho_0}(t)$ becomes null at the horizon, $\rho = \pi/2a$. Since the metric is Lorentzian, this alone is not enough to conclude that the two spacelike geodesics intersect at $\rho = \pi/2a$. This is because of the possibility that that $\gamma_{\rho_0}(t)$ becomes a lightlike path, but does not degenerate to a single space-time point. However, the point p of intersection can be identified via a different coordinate system, such as Kruskal coordinates, used for other purposes in Ref. 16. Thus, the Fermi coordinate patch cannot include points in the horizon or beyond.

Example 2: Fermi coordinates for the Einstein static universe.

The metric for the Einstein static universe may be written (cf. Ref. 17) as

$$ds^2 = -dt^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2, \tag{51}$$

where R is a constant that depends on energy density and the cosmological constant. Topologically, the space-time is $\mathbb{R} \times S_R^3$, where R is the radius of the 3-sphere S_R^3. The same calculation leading to Eq. (43) shows that this metric may be rewritten as

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + \left[\left(1 - \frac{r^2}{R^2}\right)^{-1} - 1\right] dr^2, \tag{52}$$

which has the form of Eq. (2) with $f(x,y,z) \equiv 0$ and $k = R^{-2}$ (and hence $a = R^{-1}$). Thus, the Fermi metric for the observer $\sigma(t) = (t, 0, 0, 0)$ is

$$ds^2 = -dt^2 + g_{ij} dx^i dx^j, \tag{53}$$

where the spatial metric coefficients g_{ij} are given by Eq. (17). The associated polar metric given by Corollary 2 is

$$ds^2 = -dt^2 + d\rho^2 + R^2 \sin^2\left(\frac{\rho}{R}\right)(d\theta^2 + \sin^2\theta d\phi^2), \tag{54}$$

a known form of the metric.[18] It follows trivially from Eq. (34) that the curvature $K = 0$. Consistent with Remark 2, if the range of r in Eq. (51) is $0 \le r < R$, then the corresponding range of the proper distance ρ is given by $0 \le \rho < \pi R/2$ in Eqs. (53) and (54). However, as expected for the case that $K \le 0$, Fermi coordinates may be extended beyond this range to cover the entire space-time, with the exception of the pole opposite to the origin or coordinates. Thus, we may take the range of ρ to be given by $0 \le \rho < \pi R$.

Example 3: Fermi coordinates for the interior constant density Schwarzschild space-time with cosmological constant.

The metric for a constant density fluid may be written as

$$ds^2 = -A(r)d\bar{t}^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{55}$$

where M is the mass of the spherical fluid, Λ is the cosmological constant, R is the radial coordinate for the radius of the fluid, and

$$A(r) = \left[\frac{(3 - R_0^2\Lambda)}{2}\sqrt{1 - \frac{R^2}{R_0^2}} - \frac{(1 - R_0^2\Lambda)}{2}\sqrt{1 - \frac{r^2}{R_0^2}} \right]^2,$$

$$B(r) = \left(1 - \frac{r^2}{R_0^2}\right)^{-1}. \tag{56}$$

Here,

$$R_0^2 = \frac{3R^3}{6M + \Lambda R^3}. \tag{57}$$

We assume that $A(r)$, $B(r)$, and R_0 are all positive for $0 \le r \le R$ so that the metric is well defined. In order to find the metric form of Eq. (55) in Fermi coordinates, we first make a change of variable, $t = \sqrt{A(0)}\bar{t}$, with the space coordinates held fixed. Equation (55) then becomes

$$ds^2 = -(1 - f(x,y,z))dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{58}$$

where

$$f(x,y,z) = 1 - \frac{A(r)}{A(0)}. \tag{59}$$

The same calculation leading to Eq. (43) shows that this metric may be rewritten as

$$ds^2 = -(1 - f(x,y,z))dt^2 + dx^2 + dy^2 + dz^2 + [B(r) - 1]dr^2, \tag{60}$$

which has the form of Eq. (2) with $k = R_0^{-2} > 0$. Thus, the Fermi metric for the observer $\sigma(t) = (t,0,0,0)$ is

$$ds^2 = -\frac{A(r(\rho))}{A(0)}dt^2 + g_{ij}dx^i dx^j, \tag{61}$$

where the spatial metric coefficients g_{ij} are given by Eq. (17) with $a = 1/R_0$, and where $r(\rho)^2$ is given by Eq. (31). The interval of values for ρ corresponding to $0 \le r \le R$ is $0 \le \rho \le R_0 \sin^{-1}(R/R_0)$. The associated polar metric given by Corollary 2 is

$$ds^2 = -\frac{A(r(\rho))}{A(0)}dt^2 + d\rho^2 + \frac{\sin^2(a\rho)}{a^2}(d\theta^2 + \sin^2\theta d\phi^2). \tag{62}$$

The Gaussian curvature of the submanifold N generated by the Fermi coordinates t, ρ, given by Eq. (34) is

$$K = -\frac{1 - R_0^2\Lambda}{2R_0^2\sqrt{A(r(\rho))}}\cos(a\rho). \tag{63}$$

It is clear that $K \le 0$, and by Lemma 1, orthogonal spacelike geodesics with different Fermi time coordinates remain temporally separated for $\rho \le \pi/2a$. The restriction of ρ to smaller values, noted above, is a requirement of Buchdahl-type inequalities.[19,20]

IV. CONCLUDING REMARKS

Using the results of Sec. I, we have found Fermi coordinates in Cartesian and polar forms, for natural observers in AdS_4, dS_4, the Einstein static universe, and the interior Schwarzschild solution with cosmological constant. A Jacobi field measuring the separation of coordinate spacelike geodesics was described in terms of Gaussian curvature (or sectional curvature) of two-dimensional submanifolds defined in terms of Fermi time and distance.

A breakdown of Fermi coordinates occurs when two or more spacelike geodesics, orthogonal to the Fermi observer's worldline $\sigma(\tau)$, and originating from that worldline at two different proper times, intersect at some space-time point. This occurs for dS_4 at the horizon for the Fermi observer. In the other examples considered here, the charts for Fermi coordinates are global. In the case of the Einstein static universe, Fermi coordinates extend beyond the range of the coordinates used to define the metric given by Eq. (51). We note that it is not difficult to construct additional examples of space-times with exact transformation formulas to Fermi coordinates (using Theorems 1 and 2) by combining these examples so as to obtain Fermi coordinates for Schwarzschild-(anti) de Sitter space with interior constant density fluid. The Fermi observer in those cases remains for all proper times at the center of the fluid.

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