Dirac particles in a gravitational shock wave

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Abstract

Using the Newman-Penrose formalism we calculate the positive energy momentum eigenstates of the Dirac equation for a plane polarized gravitational wave pulse. We then consider Dirac particles whose spins are polarized in each orthonormal coordinate direction prior to the arrival of the gravitational wave pulse, and determine how the spins are changed by the pulse after its departure.

KEY WORDS: gravitational wave, Dirac equation, spin polarization, exact solutions, Fermi coordinates

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1   Introduction

Ever since Chandrasekhar's solution in Kerr spacetime [1], the Dirac equation has been used extensively to study the influence of curved spacetimes on the quantum mechanical behavior of particles, for example [2, 3, 4, 5, 6, 7, 8]. An excellent reference on quantum field theory in curved spacetime is Parker and Toms text [9].

Detailed analyses of spinning objects in general relativity for the classical case (non quantum) were carried out by Dixon [10, 11, 12]. Later Hehl and Ni made use of the Foldy-Wouthyusen representation to examine the behavior of a Dirac particle in a non-inertial frame in Minkowski spacetime [13]. Since then several authors have used the Foldy-Wouthyusen approach to relate the quantum behavior of Dirac particles to the behavior of classical spinning particles in various general relativistic backgrounds, for example [14, 15, 16] and references therein.

Of particular interest is the effect of gravitational waves on quantum particles with spin. Classical spinning particles in gravitational wave backgrounds, in the weak-field approximation, were considered in [17, 18] and recently in [19], but to our knowledge there has not been an investigation of a Dirac particle in an *exact* gravitational wave. This may be due in part to the non-existence of a conserved spin operator in curved spacetime [20].

In the present paper we overcome this difficulty for a particular gravitational wave spacetime. We consider the spacetime of a gravitational pulse or "sandwich" wave. For background on spacetimes of this type, we refer the reader to [21, 22, 23, 24, 25, 26]. The spacetime $(\mathcal{M}, \mathrm{g})$ that we consider is a union of three regions, as shown in Figure 1. The first region, or front part, F, is flat. The wave pulse, which travels in the positive z-direction, has not yet reached the spacetime points within it. The next region is the pulse or wave part, W. It has non vanishing Weyl tensor but is Ricci flat. The remaining region, or back part, B, is again flat and consists of all spacetime points that the wave pulse has already passed.

A single coordinate chart $\{t, x, y, z\}$ covers the spacetime $\mathcal{M}$ (and hence all three regions), and the metric tensor g expressed in those coordinates has the form given by the line element of Eq.(2.1). The metric g satisfies the vacuum field equations (see Eq. (2.5)) within all three regions, F, W, B, and it is of class C^1 on $\mathcal{M}$ (and is smooth within each of the three regions)[3]

We use the Newman-Penrose formalism to solve the Dirac equation on $\mathcal{M}$ for the positive energy momentum eigenstates and find explicit formulas for their restrictions to each of the three regions. At this stage of our analysis, the tetrad

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[3]There is a quasiregular singularity in the region B of $\mathcal{M}$, but this does not affect our analysis; see Remark 4.
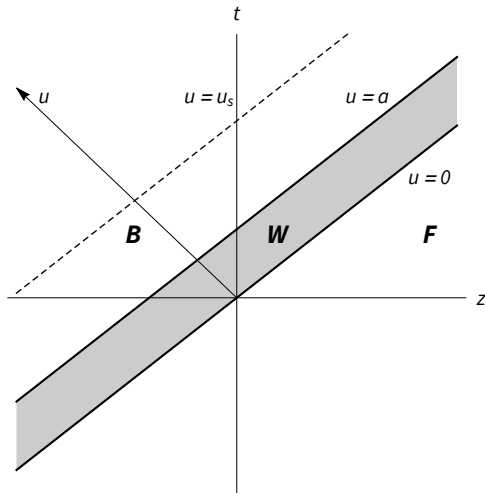
Figure 1: The 'sandwich' wave geometry. The shaded strip, W represents the gravitational wave pulse. To the right of W is region F, the front of the wave consisting of spacetime points not yet affected by the wave pulse, and to the left is region B, the back of the wave. The t-axis is the Fermi observer timelike geodesic.

field for the spinor solutions to the Dirac equation is determined in a natural way by the global coordinate chart, $\{t, x, y, z\}$. In region F (in front of the wave), the metric expressed in $\{t, x, y, z\}$ coordinates is Minkowskian and the tetrad field is the canonical one for Minkowski spacetime. Since there exists a well-defined spin operator in Minkowski spacetime, we can establish an initial spin polarization to the spinor solution of the Dirac equation restricted to region F, and we do so in the reference frame of the Dirac particle (i.e., with zero 3-momentum).

In order to analyze the effect of the shock wave on the initially polarized spinors, it is necessary to compare the spin of the Dirac particle in region B, to the original polarization in region F in a coherent way, but here a difficulty arises. Although the post wave region B is flat, the metric g expressed in the global $\{t, x, y, z\}$ coordinates is not Minkowskian, and there are infinitely many coordinate transformations that express g in Minkowskian form, each with an associated canonical tetrad field in region B (they are all related by Lorentz transformations). Regions B and F are the disjoint connected components of the flat portion of $\mathcal{M}$, and so the Minkowski coordinate patch in region F cannot be uniquely extended to B.

To find the appropriate Minkowski coordinate system and associated tetrad field in region B, so that pre- and post-spin calculations can be meaningfully compared, we first find Fermi coordinates along a timelike geodesic $\gamma(\tau)$ (where τ

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is proper time) that begins in region F, goes through the wave pulse W, and ends in region B. The path $\gamma(\tau)$, which we describe as the "laboratory frame," is chosen to remain at a fixed spatial location with respect to the Minkowski chart in F (see Figure 1). The Fermi coordinates are then uniquely determined by the parallel transport of the Minkowski tetrad in F along $\gamma(\tau)$.

The Fermi coordinates defined in this way are Minkowskian in both regions F and B. We derive the diffeomorphism from the initial (global) coordinates $\{t, x, y, z\}$ restricted to B, to the new Fermi coordinates $\{T, X, Y, Z\}$ in B (which are Minkowskian). We then change to the new Minkowski coordinates in B and carry out a tetrad rotation from the original tetrad field $h_A{}^\alpha$ to the canonical Minkowski tetrad field $\hat{e}_A{}^\alpha$ using a local Lorentz transformation Λ in B.

The local Lorentz transformation Λ determines a transformation matrix L according to which our spinor solution to the Dirac equation is transformed when the tetrad field is tranformed by Λ. We then have the relationship $\Psi(\hat{e}) = L \cdot \psi_{BFermi}(h)$, where $\Psi(\hat{e})$, is the new transformed spinor in region B in exact Fermi coordinates and with the canonical Minkowski (Fermi) tetrad $\hat{e}$, and $\psi_{BFermi}(h)$ is the original Dirac spinor restricted to region B relative to the original tetrad field h associated with the coordinates $\{t, x, y, z\}$, but whose components are functions of $\{T, X, Y, Z\}$. A calculation then shows that Ψ is not an eigenvector of the Minkowski spin operator in region B. It then follows, from a theorem we introduce and prove (Theorem 1), that our spinor solution to the Dirac equation in any possible Minkowski tetrad field is not an eigenvector to the stationary spin operator for that tetrad field in region B. Therefore the Dirac particle cannot be stationary and polarized in any of the infinitely many possible Minkowski charts in region B. The gravity wave pulse in this way disturbs the spins of the Dirac particle.

Our paper is organized as follows. In Sec. 2 we introduce the metric and derive the Dirac equations for the spinor components using the Newman-Penrose formalism. In Sec. 3 we separate the variables and find the general solutions. In Sec. 4 we provide some background information about the behavior of classical dust particles in the specific spacetime that we study. We then find the solutions for each of the three regions. Since our metric is of class C^1 it follows that the resulting solutions are also C^1. In Sec. 5 we parallel transport the Minkowski tetrad from region F to region B along the Fermi observer's worldline, and remove the coordinate singularity in region B by finding exact Fermi coordinates and transformation formulas. We explicitly show that these coordinates are Minkowskian. In Sec. 6, we give the local Lorentz transformation Λ that rotates of the tetrad field into a canonical Minkowski tetrad field, and transform the spinor solution to the Dirac equation accordingly. In Sec. 7 we state and prove our Theorem 1 on Lorentz transformations of the spin operators and compare the polarized spinors of the solution in region F to the spinors in region B and show that the gravitational wave pulse depolarizes the Dirac particle in any Minkowski tetrad field. Sec. 8 summarizes and discusses our results. The

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four Appendices follow the main exposition and provide details of the various calculations.

2   The Dirac equation in the N-P formalism

In all that follows we adopt Planck units so that, $c = G = \hbar = 1$. We consider a transverse plane polarized gravitational wave propagating in the z direction. The line element for such a wave may be written as shown below [22], [25], [26],

$$ds^2 = dt^2 - dz^2 - f^2(u)dx^2 - g^2(u)dy^2 \,, \tag{2.1}$$

where

$$u = \frac{1}{\sqrt{2}}(t - z) \,. \tag{2.2}$$

Introducing the variable

$$v = \frac{1}{\sqrt{2}}(t + z) \,, \tag{2.3}$$

we transform the metric to

$$ds^2 = 2dudv - f^2(u)dx^2 - g^2(u)dy^2 \,. \tag{2.4}$$

The functions $f(u)$ and $g(u)$ must satisfy the only nontrivial vacuum field equation, namely,

$$R_{uu} = -\left(\frac{f''(u)}{f(u)} + \frac{g''(u)}{g(u)}\right) = 0 \,. \tag{2.5}$$

Remark 1. *One may use a coordinate transformation to recast the Bondi-Pirani plane wave,* Eq.(2.1), *of ref.* [24]*, into the form of* Eq. (17.11) *of Griffiths-Podolský* [26] *(with $\omega = 0$) and hence to the form* Eq. (2.4).

We use the Newman-Penrose (N-P) formalism to write and solve the Dirac equation in the spacetime of Eq. (2.4). A set of orthogonal tetrad vectors for the metric of Eq. (2.4) is given below for which we adopt the ordering convention $e_A = (e_A{}^x, e_A{}^y, e_A{}^v, e_A{}^u)$.

$$e_1 = \left(\frac{1}{f(u)}, \, 0, \, 0, \, 0\right) = \frac{1}{f(u)}\partial_x \,, \tag{2.6}$$

$$e_2 = \left(0, \, \frac{1}{g(u)}, \, 0, \, 0\right) = \frac{1}{g(u)}\partial_y \,, \tag{2.7}$$

$$e_3 = \left(0, \, 0, \, \frac{1}{\sqrt{2}}, \, -\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\partial_v - \frac{1}{\sqrt{2}}\partial_u \,, \tag{2.8}$$

$$e_4 = \left(0, \, 0, \, \frac{1}{\sqrt{2}}, \, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\partial_v + \frac{1}{\sqrt{2}}\partial_u \,. \tag{2.9}$$

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From these vectors we construct the complex null tetrad $\{l, n, m, \overline{m}\}$ as follows [27]:

$$\lambda_1 = l = \frac{1}{\sqrt{2}}(e_4 + e_3) = \frac{1}{\sqrt{2}}\left(0, 0, \sqrt{2}, 0\right), \tag{2.10}$$

$$\lambda_2 = n = \frac{1}{\sqrt{2}}(e_4 - e_3) = \frac{1}{\sqrt{2}}\left(0, 0, 0, \sqrt{2}\right), \tag{2.11}$$

$$\lambda_3 = m = \frac{1}{\sqrt{2}}(e_1 + ie_2) = \frac{1}{\sqrt{2}}\left(\frac{1}{f(u)}, i\frac{1}{g(u)}, 0, 0\right), \tag{2.12}$$

$$\lambda_4 = \overline{m} = \frac{1}{\sqrt{2}}(e_1 - ie_2) = \frac{1}{\sqrt{2}}\left(\frac{1}{f(u)}, -i\frac{1}{g(u)}, 0, 0\right). \tag{2.13}$$

We write the Dirac equation in the N-P formalism using the standard notation for the spin coefficients, [1], [28] as follows:

$$(\Delta + \mu^* - \gamma^*)G_1 - (\delta^* + \beta^* - \tau^*)G_2 = i\mu_* F_1 \,,$$

$$(D + \varepsilon^* - \rho^*)G_2 - (\delta + \pi^* - \alpha^*)G_1 = i\mu_* F_2 \,,$$

$$(D + \varepsilon - \rho)F_1 + (\delta^* + \pi - \alpha)F_2 = i\mu_* G_1 \,, \tag{2.14}$$

$$(\Delta + \mu - \gamma)F_2 + (\delta + \beta - \tau)F_1 = i\mu_* G_2 \,,$$

where $\mu_* \sqrt{2} = m = $ the mass of the particle (not to be confused with the null vector m in Eq. (2.12)), and the spinor $\psi(x, y, v, u)$ has the form:

$$\psi = \begin{pmatrix} F_1 \\ F_2 \\ G_1 \\ G_2 \end{pmatrix}. \tag{2.15}$$

We use the standard notation to designate the null tetrad vectors as directional derivatives

$$\lambda_1 = l = D, \quad \lambda_2 = n = \Delta, \quad \lambda_3 = m = \delta, \quad \lambda_4 = \overline{m} = \delta^*. \tag{2.16}$$

From these we have the four scalar operators,

$$D = l^\alpha \nabla_\alpha, \quad \Delta = n^\alpha \nabla_\alpha, \quad \delta = m^\alpha \nabla_\alpha, \quad \delta^* = \overline{m}^\alpha \nabla_\alpha, \tag{2.17}$$

and for any scalar function h we write

$$Dh = l^\alpha h_{,\alpha}, \quad \Delta h = n^\alpha h_{,\alpha}, \quad \delta h = m^\alpha h_{,\alpha}, \quad \delta^* h = \overline{m}^\alpha h_{,\alpha}. \tag{2.18}$$

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In our case the Dirac equation Eq. (2.14) reduces to,

$$(\Delta + \mu)G_1 - \delta^* G_2 = i\mu_* F_1\,,$$

$$DG_2 - \delta G_1 = i\mu_* F_2\,,$$

$$DF_1 + \delta^* F_2 = i\mu_* G_1\,, \tag{2.19}$$

$$(\Delta + \mu)F_2 + \delta F_1 = i\mu_* G_2\,.$$

3 Separation of variables and the solutions

In working out the details of Eqs. (2.19), first we cancel out all the $1/\sqrt{2}$ factors appearing on both sides of the equations (recall that $\mu_* \sqrt{2} = m$). Since the metric in Eq. (2.4) does not depend on x, y, and v, the corresponding canonical momenta are constants, both in classical and quantum mechanics. We take advantage of this fact and write the 4-component spinor ψ as

$$\psi(x, y, v, u) = \begin{pmatrix} F_1 \\ F_2 \\ G_1 \\ G_2 \end{pmatrix} = e^{-i(p_x x + p_y y + p_v v)} \begin{pmatrix} \mathbb{F}_1(u) \\ \mathbb{F}_2(u) \\ \mathbb{G}_1(u) \\ \mathbb{G}_2(u) \end{pmatrix}, \tag{3.1}$$

where, p_A, $A = x, y, v$, are the eigenvalues of the operators $i\partial_A$. Thus ψ is an eigenfunction of the canonical momenta p_A (see Sec. 4.1).

Remark 2. *If we write the metric in the coordinates of Eq. (2.1), then the conserved p_v is given by*

$$p_v = \frac{p_t + p_z}{\sqrt{2}}\,, \tag{3.2}$$

where the non-conserved p_t and p_z, can be obtained from the Lagrangian corresponding to the metric (2.1).

The resulting equations for the $\mathbb{F}_i(u)$ and the $\mathbb{G}_i(u)$ are

$$\left(\frac{ip_x}{f(u)} + \frac{p_y}{g(u)} \right) \mathbb{G}_2 + \sqrt{2}\, \mu\, \mathbb{G}_1 + \sqrt{2}\, \mathbb{G}_1' = im\mathbb{F}_1 \,, \tag{3.3}$$

$$\left(\frac{ip_x}{f(u)} - \frac{p_y}{g(u)} \right) \mathbb{G}_1 - i\sqrt{2}\, p_v\, \mathbb{G}_2 = im\mathbb{F}_2 \,, \tag{3.4}$$

$$-\left(\frac{ip_x}{f(u)} + \frac{p_y}{g(u)} \right) \mathbb{F}_2 - i\sqrt{2}\, p_v\, \mathbb{F}_1 = im\mathbb{G}_1 \,, \tag{3.5}$$

$$\left(-\frac{ip_x}{f(u)} + \frac{p_y}{g(u)} \right) \mathbb{F}_1 + \sqrt{2}\, \mu\, \mathbb{F}_2 + \sqrt{2}\, \mathbb{F}_2' = im\mathbb{G}_2 \,, \tag{3.6}$$

where the spin coefficient μ is given by

$$\mu = \frac{1}{2} \left(\frac{f'(u)}{f(u)} + \frac{g'(u)}{g(u)} \right) \,. \tag{3.7}$$

We now solve for $\mathbb{F}_1(u)$ and $\mathbb{G}_2(u)$ using the algebraic Eqs. (3.4) and (3.5), and substitute the results in the differential Eqs. (3.3) and (3.6) obtaining two decoupled and *identical* differential equations for $\mathbb{G}_1(u)$ and $\mathbb{F}_2(u)$,

$$\left[i \left(m^2 f^2 g^2 + p_x^2\, g^2 + p_y^2\, f^2 \right) + p_v fg \left(gf' + fg' \right) \right] \mathbb{G}_1(u) + \\ 2p_v\, f^2 g^2\, \mathbb{G}_1'(u) = 0 \,, \tag{3.8}$$

$$\left[i \left(m^2 f^2 g^2 + p_x^2\, g^2 + p_y^2\, f^2 \right) + p_v fg \left(gf' + fg' \right) \right] \mathbb{F}_2(u) + \\ 2p_v\, f^2 g^2\, \mathbb{F}_2'(u) = 0 \,. \tag{3.9}$$

Given explicit expressions for the functions $f(u)$ and $g(u)$, the general solutions to the Eqs. (3.8) and (3.9), can be obtained by performing the integration

$$\frac{\mathbb{G}_1(u)}{b_1} = \frac{\mathbb{F}_2(u)}{a_2} = \\ \exp\left[\int_{u_0}^u \frac{-i\left(p_x{}^2 g^2 + p_y{}^2 f^2 + m^2 f^2 g^2 \right) - p_v fg\left(fg' + gf' \right)}{2p_v f^2 g^2} dw \right] \,, \tag{3.10}$$

where in the integrand $f = f(w)$, $g = g(w)$. The constants a_2 and b_1, are chosen in region F and they determine the polarization of the particle in its rest frame as we show in Sec. 7. They depend on m, p_x, p_y, p_v, p_u and as we will see below, they must remain the same through all the three regions of our shock wave spacetime, in order for the different parts of the wavefunction to match properly at the shock wave boundaries, i.e., for continuous differentiability of the spinor ψ to hold.

4 The sandwich wave

In general one may choose an arbitrary function $g(u)$ and solve Eq. (2.5) for $f(u)$. In this section, however, we shall solve Eq. (3.10) for a *sandwich* gravitational wave [22, 23, 24, 25, 26]. As described in the introduction and illustrated in Figure 1, the spacetime $\mathcal{M}$ is a union of three regions, i.e.,

$$\mathcal{M} = F \cup W \cup B, \tag{4.1}$$

where B represents the *back* or post-wave region, W represents the *wave* pulse, and F represents the *front* of the wave, that is, the points in advance of the wave. Each of these regions is characterized by the variable $u = (t - z)/\sqrt{2}$, as indicated below. The functions $f(u)$ and $g(u)$ which appear in the metric Eq. (2.4) are now defined by,

$$f(u) = \begin{cases} 1, & u \in (-\infty, 0] \equiv F, \\ \cos(ku), & u \in [0, a] \equiv W, \\ \cos(ka) + k(a - u)\sin(ka), & u \in [a, \infty) \equiv B, \end{cases} \tag{4.2}$$

$$g(u) = \begin{cases} 1, & u \in (-\infty, 0] \equiv F, \\ \cosh(ku), & u \in [0, a] \equiv W, \\ \cosh(ka) + k(u - a)\sinh(ka), & u \in [a, \infty) \equiv B, \end{cases} \tag{4.3}$$

where $a > 0$ and k are constants. The functions $f(u)$ and $g(u)$ satisfy the vacuum Eq. (2.5) in all three regions. The resulting composite metric g is of class C^1. Using Eq. (2.2) it is easy to see that the plane wave 'slab,' W, consisting of the spacetime points satisfying $u \in [0, a]$, is moving in the positive z-direction as t increases. Since f and g are linear functions of u on F and B, we have $f''(u) = g''(u) = 0$, so all the curvature components, $R_{\alpha\beta\gamma\delta}$, vanish in these regions. In light of this we define the limit $k \to 0$, as the flat spacetime limit, wherein the shock wave disappears.

We shall choose a and k so that $ka \in (-\pi/2, \pi/2)$ so that $f(u) > 0$ in region W. However, $f(u)$ necessarily vanishes at one point, $u = u_s$, in region B. We deal with this singularity in Section 5.

Before turning to the solution to the Dirac equation for this spacetime, we briefly describe the effect of the wave pulse on classical test particles in analogy to the classical "sticky bead" problem. Similar descriptions for spacetimes of this type are given in [22, 25] and systematic analyses of timelike geodesics in wave pulse spacetimes were carried out by Penrose and other researchers, cf., [26].

It is easily verified that the equations, $x, y, z =$ constant, describe timelike geodesics with affine parameter t. Consider two dust particles at rest in region F before the wave pulse arrives, separated only by a small x-difference dx.

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The proper distance between them for all times is $ds = f(u)dx$, where dx is constant. Similarly two dust particles separated only by a small y-distance dy have proper distance between them $ds = g(u)dy$. Thus, as depicted in Figure 2, during the passage of the wave, the x-separation of the particles decreases and the y-separation increases since $f(u) = \cos u$ decreases and $g(u) = \cosh u$ increases, as u increases from zero.
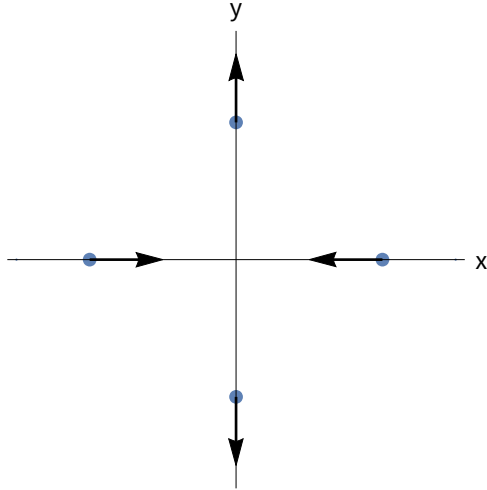
Figure 2: Motion of classical test particles in the wave pulse.

Returning to the Dirac equation, for convenience, we write the argument of the exponential in Eq. (3.10) as

$$\Phi(u_0, u) = \int_{u_0}^{u} \phi(w)dw \,, \tag{4.4}$$

where the lower limit of integration, u_0, is chosen so that no singularity of the integrand $\phi(w)$ lies between u_0 and u, but is otherwise arbitrary. In what follows it is convenient to choose $u_0 = 0$. It will follow from the fact that each term in the integrand of Eq. (3.10) is of class C^1, that the spinor component functions $\mathbb{F}_2(u)$ and $\mathbb{G}_1(u)$ will be of class C^2. However as one can tell from, e.g., Eq. (4.35), the components $\mathbb{F}_1(u)$ and $\mathbb{G}_2(u)$ are C^1 and therefore the spinors will be of class C^1.

4.1 The solution in region F

We begin with the solution in region F. Evaluating Eq. (3.10), with the appropriate choices from the definitions (4.2), (4.3), we obtain,

$$\Phi_F(0, u) = -i\frac{\left(m^2 + p_x{}^2 + p_y{}^2\right)}{2p_v} u = -ip_u u. \tag{4.5}$$

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Since in region F, p_u and p_t are conserved, we have the useful relations below,

$$m = \sqrt{2 p_v p_u - p_x^2 - p_y^2} = \sqrt{p_t^2 - \left(p_x^2 + p_y^2 + p_z^2\right)}, \qquad (4.6)$$

$$\begin{aligned} p_\mu x^\mu &= p_x x + p_y y + p_v v + p_u u, \\ &= p_x x + p_y y + p_z z + p_t t, \end{aligned} \qquad (4.7)$$

and

$$p_t + p_z = \sqrt{2}\, p_v, \qquad (4.8)$$
$$p_t - p_z = \sqrt{2}\, p_u. \qquad (4.9)$$

Thus

$$\mathbb{F}_{2F}(u) = a_2\, e^{\Phi_F(0,u)}, \qquad (4.10)$$

$$\mathbb{G}_{1F}(u) = b_1\, e^{\Phi_F(0,u)}. \qquad (4.11)$$

Then Eqs. (3.4) and (3.5) give us the $\mathbb{F}_{1F}(u)$ and $\mathbb{G}_{2F}(u)$ below

$$\mathbb{F}_{1F}(u) = \frac{e^{\Phi_F(0,u)}\left[-b_1 m - a_2\left(p_x - i p_y\right)\right]}{\sqrt{2}\, p_v}, \qquad (4.12)$$

$$\mathbb{G}_{2F}(u) = \frac{e^{\Phi_F(0,u)}\left[-a_2 m + b_1\left(p_x + i p_y\right)\right]}{\sqrt{2}\, p_v}. \qquad (4.13)$$

Remark 3. *We shall consider only positive energy solutions. In addition, we shall omit the overall normalization factors from all solutions, as they will not play a role in our analysis.*

We write the solution in region F in accordance with the notation of Eqs. (3.1) and (4.5). The positive energy solution in region F is the usual Dirac plane wave

in the chiral representation,

$$\psi_F = \begin{pmatrix} F_{1F} \\ F_{2F} \\ G_{1F} \\ G_{2F} \end{pmatrix} = e^{-i(p_x x + p_y y + p_v v)} \begin{pmatrix} \mathbb{F}_1(u) \\ \mathbb{F}_2(u) \\ \mathbb{G}_1(u) \\ \mathbb{G}_2(u) \end{pmatrix}, \tag{4.14}$$

$$= e^{-i(p_x x + p_y y + p_v v)} \begin{pmatrix} \dfrac{(-b_1 m - a_2 (p_x - i p_y))}{\sqrt{2}\, p_v} \\[2ex] a2 \\[2ex] b1 \\[2ex] \dfrac{(-a_2 m + b_1 (p_x + i p_y))}{\sqrt{2}\, p_v} \end{pmatrix} e^{-i p_u u}. \tag{4.15}$$

4.2   The solution in region W

For spacetime points in W, we have that $f(u) = \cos(ku)$, $g(u) = \cosh(ku)$. As before, we assume that

$$ka \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right), \tag{4.16}$$

in order to avoid the singularity arising from the vanishing of $f(u)$. Then, following the same steps as in Sec. 4.1, we obtain,

$$\Phi_W(0, u) = -iA(u) + \ln C(u), \tag{4.17}$$

where $A(u)$ and $C(u)$ are given by

$$A(u) = \frac{\left(m^2 ku + p_x^2 \tan(ku) + p_y^2 \tanh(ku) \right)}{2kp_v}, \tag{4.18}$$

$$C(u) = \frac{1}{\sqrt{f(u)\, g(u)}}, \tag{4.19}$$

with the appropriate $f(u)$ and $g(u)$ from Eqs. (4.2) and (4.3).

Then,

$$\mathbb{F}_{1W}(u) = e^{-iA(u)}C(u)\frac{[-b_1 m - a_2 p_x \sec{(ku)} + ia_2 p_y \operatorname{sech}{(ku)}]}{\sqrt{2}\,p_v}, \qquad (4.20)$$

$$\mathbb{F}_{2W}(u) = a_2\,e^{-iA(u)}C(u), \qquad (4.21)$$

$$\mathbb{G}_{1W}(u) = b_1\,e^{-iA(u)}C(u), \qquad (4.22)$$

$$\mathbb{G}_{2W}(u) = e^{-iA(u)}C(u)\frac{[-a_2 m + b_1 p_x \sec{(ku)} + ib_1 p_y \operatorname{sech}{(ku)}]}{\sqrt{2}\,p_v}, \qquad (4.23)$$

Thus

$$\psi_W = \begin{pmatrix} F_{1W} \\ F_{2W} \\ G_{1W} \\ G_{2W} \end{pmatrix} = e^{-i(p_x x + p_y y + p_v v)} \begin{pmatrix} \mathbb{F}_{1W}(u) \\ \mathbb{F}_{2W}(u) \\ \mathbb{G}_{1W}(u) \\ \mathbb{G}_{2W}(u) \end{pmatrix}. \qquad (4.24)$$

On the boundary $u = 0$, we have

$$\mathbb{F}_{1W}(0) = a_1 = \frac{[-b_1 m - a_2\,(p_x - ip_y)]}{\sqrt{2}\,p_v}, \qquad (4.25)$$

$$\mathbb{F}_{2W}(0) = a_2, \qquad (4.26)$$

$$\mathbb{G}_{1W}(0) = b_1, \qquad (4.27)$$

$$\mathbb{G}_{2W}(0) = b_2 = \frac{[-a_2 m + b_1\,(p_x + ip_y)]}{\sqrt{2}\,p_v}, \qquad (4.28)$$

as expected (see Eqs. (4.12), (4.13)). Furthermore one can check in a straight-forward manner that the solution on the boundary between F and W, $\partial(FW)$, is C^1.

4.3   The solution in region B

In region B we have the singularity due to the vanishing of $f(u)$ at $u = u_s$, where

$$u_s = a + \frac{\cot{(ka)}}{k}. \qquad (4.29)$$

Remark 4. *The singularity at $u = u_s$ is in fact a quasiregular singularity* [26], [29]. *This singularity is a manifestation of the fact that no complete spacelike hypersurface exists that is adequate for the global specification of Cauchy data. We also remark that the singularity moves along with the wave in the positive z-direction as time increases,* (see Figure 1).

The singularity at $u = u_s$, is present in the tetrad field and ultimately, as we shall see below, in the solution of the Dirac equation.

For region B, we follow the same basic procedure as in Sections 4.1 and 4.2. We can write, for all $a \leq u < u_s$,

$$\Phi_B(0, u) = \Phi_W(0, a) + \Phi_B(a, u), \tag{4.30}$$

where $\Phi_W(0, a)$, is obtained from Eq. (4.17). We have

$$\Phi_W(0, a) = -iA(a) + \ln C(a), \tag{4.31}$$

$$\Phi_B(a, u) = \frac{i(a-u)}{2p_v} \left(m^2 + \frac{p_x{}^2 \sec(ka)}{f(u)} + \frac{p_y{}^2 \operatorname{sech}(ka)}{g(u)} \right)$$
$$- \ln C(a) - \ln \left(\sqrt{f(u)g(u)} \right). \tag{4.32}$$

Thus

$$\Phi_B(0, u) = \frac{i(a-u)}{2p_v} \left(m^2 + \frac{p_x{}^2 \sec(ka)}{f(u)} + \frac{p_y{}^2 \operatorname{sech}(ka)}{g(u)} \right)$$
$$- iA(a) - \ln \left(\sqrt{f(u)g(u)} \right). \tag{4.33}$$

In Eq. (4.32), and below, the $f(u)$ and $g(u)$, are given in Eqs. (4.2) and (4.3). For convenience we let

$$B(u) = \frac{i(a-u)}{2p_v} \left(m^2 + \frac{p_x{}^2 \sec(ka)}{f(u)} + \frac{p_y{}^2 \operatorname{sech}(ka)}{g(u)} \right) - iA(a), \tag{4.34}$$

with $A(a)$ given by Eq. (4.18) and $C(u)$ by Eq. (4.19) with the appropriate $f(u)$ and $g(u)$. Thus we have,

$$\mathbb{F}_{1B}(u) = C(u) \, e^{B(u)} \left(\frac{1}{\sqrt{2}\,p_v} \right) \left(-b_1 m - \frac{a_2 p_x}{f(u)} + \frac{i a_2 p_y}{g(u)} \right), \tag{4.35}$$

$$\mathbb{F}_{2B}(u) = a_2 C(u) \, e^{B(u)}, \tag{4.36}$$

$$\mathbb{G}_{1B}(u) = b_1 C(u) \, e^{B(u)}, \tag{4.37}$$

$$\mathbb{G}_{2B}(u) = C(u) \, e^{B(u)} \left(\frac{1}{\sqrt{2}\,p_v} \right) \left(-a_2 m + \frac{b_1 p_x}{f(u)} + \frac{i b_1 p_y}{g(u)} \right). \tag{4.38}$$

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Therefore,

$$\psi_B = \begin{pmatrix} F_{1B} \\ F_{2B} \\ G_{1B} \\ G_{2B} \end{pmatrix} = e^{-i(p_x x + p_y y + p_v v)} \begin{pmatrix} \mathbb{F}_{1B}(u) \\ \mathbb{F}_{2B}(u) \\ \mathbb{G}_{1B}(u) \\ \mathbb{G}_{2B}(u) \end{pmatrix}. \tag{4.39}$$

It is easy to verify using Eqs. (4.20)-(4.23) that on the boundary $u = a$, we have

$$\mathbb{F}_{1B}(a) = \mathbb{F}_{1W}(a), \tag{4.40}$$

$$\mathbb{F}_{2B}(a) = \mathbb{F}_{2W}(a), \tag{4.41}$$

$$\mathbb{G}_{1B}(a) = \mathbb{G}_{1W}(a), \tag{4.42}$$

$$\mathbb{G}_{2B}(a) = \mathbb{G}_{2W}(a). \tag{4.43}$$

In this case again one can check that the match on the $\partial(WB)$ is C^1. We note that the spinor components have branch points at $u = u_s$ and in general, $\mathbb{F}_{1B}(u) = \kappa \, \mathbb{G}_{2B}(u) = O(f(u)^{-3/2})$, as $u \to u_s$, where κ is some constant (see Eqs. (4.19), (4.35), and (4.38)).

5 The Fermi Observer

Recall that region B is flat.[4] Since the Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$ is identically zero in the interior of region B except at the singularity, $R_{\alpha\beta\gamma\delta}$ can be extended as the zero tensor to all of the interior of B. Therefore, there are Minkowski coordinate charts that cover the (open) interior of B. By transforming to Minkowski coordinates in B, the singularity in the metric in region B can thus be eliminated by a coordinate transformation.

In this section we construct a consistent Minkowski tetrad field for a Fermi observer (lab frame) whose spatially-fixed timelike path extends from region F through the wave pulse W to region B. All other Minkowski tetrad fields in region B are then related to this one via Lorentz transformations. To proceed, we consider the timelike path,

$$\gamma(\tau) = (t(\tau), x(\tau), y(\tau), z(\tau)) = (\tau, 0, 0, 0), \quad \tau \equiv T \geq 0. \tag{5.1}$$

It is easily checked that the path $\gamma(\tau)$ is geodesic, and it is also easily verified that the following orthonormal tetrad is parallel (i.e., parallel transported) along

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[4] Since the metric is only C^1 on the boundary separating regions B and W in the global coordinate system $\{t, x, y, z\}$, the boundary of B must be excluded here.

$\gamma(\tau)$:

$$\bar{e}_0 = (1, 0, 0, 0) = \partial_t \,, \tag{5.2}$$

$$\bar{e}_1 = \left(0, \frac{1}{f(u)}, 0, 0\right) = \frac{1}{f(u)}\partial_x \,, \tag{5.3}$$

$$\bar{e}_2 = \left(0, 0, \frac{1}{g(u)}, 0\right) = \frac{1}{g(u)\partial_y} \,, \tag{5.4}$$

$$\bar{e}_3 = (0, 0, 0, 1) = \partial_z \,, \tag{5.5}$$

where u is given by Eq. (2.2). Recalling that $f(u) = g(u) = 1$ in region F, we see that a representative Minkowski tetrad (at $\gamma(0)$) from region F is parallel transported to region B for all $\tau > \sqrt{2}a$. We define our Fermi observer to be $\gamma(\tau)$, and we refer to the Fermi coordinates in this context as the 'laboratory frame' (see Fig. 1). As we show in the remainder of this section, and in Appendix A, the Fermi coordinates restricted to an open set in region B are Minkowski coordinates, and it will also be shown that the coordinate transformation may be extended to the entire interior of region B. Associated with this coordinate system is a canonical Minkowski tetrad field in the interior of region B.

In order to display the Fermi coordinate transformations, it is convenient first to introduce the following notation,

$$f(u) = p + qu \,, \tag{5.6}$$

$$g(u) = r + su \,, \tag{5.7}$$

where,

$$p = \cos(ka) + ka\sin(ka) \,, \tag{5.8}$$

$$q = -k\sin(ka) \,, \tag{5.9}$$

$$r = \cosh(ka) - ka\sinh(ka) \,, \tag{5.10}$$

$$s = k\sinh(ka) \,. \tag{5.11}$$

In what follows, we designate the Fermi coordinates restricted to region B by (T, X, Y, Z). Details of the following calculation from (t, x, y, z) to exact Fermi coordinates (T, X, Y, Z) are given in Appendix A.

$$x = \frac{X}{\zeta(T, Z)} \,, \tag{5.12}$$

$$y = \frac{Y}{\xi(T, Z)} \,, \tag{5.13}$$

$$z = Z - \frac{qX^2}{2\sqrt{2}\,\zeta(T, Z)} - \frac{sY^2}{2\sqrt{2}\,\xi(T, Z)} \,, \tag{5.14}$$

$$t = T - \frac{qX^2}{2\sqrt{2}\,\zeta(T, Z)} - \frac{sY^2}{2\sqrt{2}\,\xi(T, Z)} \,, \tag{5.15}$$

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where in Eqs. (5.12)-(5.15),

$$\zeta(T, Z) = p + q\left(\frac{T-Z}{\sqrt{2}}\right), \tag{5.16}$$

$$\xi(T, Z) = r + s\left(\frac{T-Z}{\sqrt{2}}\right). \tag{5.17}$$

The inverse of the transformation of Eqs. (5.12)-(5.15) is given by,

$$X = x\, f(t, z), \tag{5.18}$$

$$Y = y\, g(t, z), \tag{5.19}$$

$$Z = z + \frac{1}{4}\left(\sqrt{2}\left(pqx^2 + rsy^2\right) + \left(q^2x^2 + s^2y^2\right)(t-z)\right), \tag{5.20}$$

$$T = t + \frac{1}{4}\left(\sqrt{2}\left(pqx^2 + rsy^2\right) + \left(q^2x^2 + s^2y^2\right)(t-z)\right), \tag{5.21}$$

where

$$f(t, z) = p + q\left(\frac{t-z}{\sqrt{2}}\right), \tag{5.22}$$

$$g(t, z) = r + s\left(\frac{t-z}{\sqrt{2}}\right). \tag{5.23}$$

We note that since the inverse transformation given by Eqs. (5.18)-(5.21) consists of quadratic and cubic polynomials in (t, x, y, z), they are also Taylor expansions for the Fermi coordinates. It is readily verified that these equations also follow from the general Taylor expansion formula for Fermi coordinates given by Eq. (25) in reference [30], which verifies the correctness of our calculations.

Remark 5. *The two time coordinates t and T are equal along the Fermi observer's timelike path $(t, 0, 0, 0)$, but differ away from the path according to Eq.(5.21). As a consequence, they generate distinct foliations of an open neighborhood of the lab frame path. A general feature of Fermi coordinates is that the a priori time coordinate and the Fermi time coordinate are not equal. As a further illustration, it is shown in [31] and references therein that cosmological coordinate time decreases to zero (the instant of the big bang), on all hypersurfaces of constant Fermi time in Robertson-Walker cosmologies.*

The identity, $t - z = T - Z$, follows from either set of transformation formulas, so region B may be characterized by $t - z = T - Z \geq \sqrt{2}\, a$. The transformation formulas in Eqs. (5.12)-(5.15) are valid at all points in region B except

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for the case $T - Z = \sqrt{2}\,u_s$, where the singularity u_s is given by Eq. (4.29). However, the inverse transformation formulas of Eqs. (5.18)-(5.21) are valid even for $t - z = \sqrt{2}\,u_s$, i.e., they are valid at the singular points in region B. This is because those transformation formulas are polynomials. Thus, the Fermi/Minkowski coordinates (T, X, Y, Z) are valid in all points of the interior of region B, as discussed in the first paragraph of this section.

Using the transformation (5.12)-(5.15), the metric

$$ds^2 = dt^2 - dz^2 - f^2(u)dx^2 - g^2(u)dy^2 \,, \tag{5.24}$$

with the $f(u)$ and $g(u)$ given in Eqs. (5.6)-(5.7) for region B, is transformed into the Minkowski metric,

$$ds^2 = dT^2 - dX^2 - dY^2 - dZ^2 \,. \tag{5.25}$$

Remark 6. *Our Fermi observer, or lab frame, is one of an infinite set of timelike geodesics of the form,*

$$x(\tau) = C_x, \quad y(\tau) = C_y, \quad z(\tau) = C_z, \quad t(\tau) = \tau, \tag{5.26}$$

where the constants C_x, C_y, C_z may be chosen arbitrarily. This is the set of timelike geodesic paths that are stationary relative to any particle whose 3-momentum is zero in the global coordinate system (t, x, y, z). Referring to Eqs. (4.6), (4.8), (4.9), we note that for a Dirac particle with zero 3-momentum, we must choose $p_x = p_y = 0$ and $p_v = 1/\sqrt{2}$. A calculation shows that the tetrad field given by Eqs. (5.2)-(5.5) is parallel along each of these geodesic paths. Moreover, each such timelike geodesic passes through the singularity $u = u_s$ where it is well defined and smooth. Fermi coordinates for each such path are Minkowskian in the interior of region B and defines a canonical Minkowski tetrad field. We chose $C_x = C_y = C_z = 0$ for $\gamma(\tau)$ in Eq. (5.1) only for convenience.

6 Spinor solution in the Fermi tetrad field

The goal of this section is to transform our spinor solution to the Dirac equation, ψ_B, in region B, to its representation, Ψ, in Fermi coordinates (T, X, Y, Z) with the associated (canonical) Minkowski tetrad field on region B, Eqs. (6.21)-(6.24). To do this, we first need to find the local Lorentz transformation Λ that represents the rotation between the tetrad axes for $\{t, x, y, z\}$ and the tetrad axes for (T, X, Y, Z) at each spacetime point in region B (but we carry this out by using 1-forms instead of vectors).

The local Lorentz transformation Λ determines a transformation matrix L that acts on spinors (for details see Appendix B). The two matrices are related by the consistency relations,

$$L^{-1}\gamma^A L = \Lambda^A{}_B \gamma^B \,, \tag{6.1}$$

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and
$$\gamma^0 L^\dagger \gamma^0 = L^{-1} \,, \tag{6.2}$$

where the γ matrices are given in the chiral representation in Appendix D. We note that Eq. (6.2) is explained and motivated in [9], (cf., Eq. (5.396) page 246). We then have the relationship $\Psi = L \cdot \psi_{B\mathrm{Fermi}}$.

We begin with the tetrad 1-forms in the region B for the metric (5.24),

$$\bar{e}^0 = (1, 0, 0, 0) \; = dt \,, \tag{6.3}$$
$$\bar{e}^1 = (0, f(t, z), 0, 0) \; = f(t, z)dx \,, \tag{6.4}$$
$$\bar{e}^2 = (0, 0, g(t, z), 0) \; = g(t, z)dy \,, \tag{6.5}$$
$$\bar{e}^3 = (0, 0, 0, 1) \; = dz \,. \tag{6.6}$$

The 1-forms Eqs. (6.3)-(6.6) are transformed, under the coordinate transformation (5.12)-(5.15), into the the tetrad 1-forms, $h^A = (h^A{}_T, h^A{}_X, h^A{}_Y, h^A{}_Z)$, below

$$h^0 = \left(1 + \frac{1}{2}\left(\mathbb{A}^2 + \mathbb{B}^2\right), \; -\mathbb{A}, \; -\mathbb{B}, \; -\frac{1}{2}\left(\mathbb{A}^2 + \mathbb{B}^2\right)\right) \,, \tag{6.7}$$

$$h^1 = (-\mathbb{A}, \; 1, \; 0, \; \mathbb{A}) \,, \tag{6.8}$$

$$h^2 = (-\mathbb{B}, \; 0, \; 1, \; \mathbb{B}) \,, \tag{6.9}$$

$$h^3 = \left(\frac{1}{2}\left(\mathbb{A}^2 + \mathbb{B}^2\right), \; -\mathbb{A}, \; -\mathbb{B}, \; 1 - \frac{1}{2}\left(\mathbb{A}^2 + \mathbb{B}^2\right)\right) \,, \tag{6.10}$$

where the components are the coefficients in respective order of dT, dX, dY, dZ, and

$$\mathbb{A} = \frac{qX}{\sqrt{2}\,\zeta(T, Z)} \,, \tag{6.11}$$

$$\mathbb{B} = \frac{sY}{\sqrt{2}\,\xi(T, Z)} \,. \tag{6.12}$$

Spinors transform like scalars (0-forms) under coordinate transformations, [9], thus we can easily obtain the transformed spinor, $\psi_{BFermi}(T, X, Y, Z)$, in Fermi coordinates. We let

$$\mathbb{D}(T, Z) = \frac{1}{\sqrt{\zeta(T, Z)\,\xi(T, Z)}} \,, \tag{6.13}$$

and

$$H = \left(\frac{1}{2\,pv}\right)\left(m^2 + \frac{{p_x}^2}{(p+qa)\zeta(T,Z)} + \frac{{p_y}^2}{(r+sa)\xi(T,Z)}\right)\left(\frac{T-Z}{\sqrt{2}} - a\right)$$

$$+ \frac{p_x X}{\zeta(T,Z)} + \frac{p_y Y}{\xi(T,Z)} + p_v\left(\frac{T+Z}{\sqrt{2}} - \frac{qX^2}{2\zeta(T,Z)} - \frac{sY^2}{2\xi(T,Z)}\right) + A\,, \quad (6.14)$$

where, of course, $H = H(T, X, Y, Z)$, and

$$A = \left(\frac{1}{2kp_v}\right)\left(m^2 ka - \frac{p_x^2\, q}{k(p+qa)} + \frac{p_y^2\, s}{k(r+sa)}\right). \quad (6.15)$$

The A above, in Eq. (6.15), is the $A(a)$ of Eq. (4.18) re-expressed in terms of the constants p, q, r, s, (see Eqs. (5.8)-(5.11)). Then, the wavefunction ψ_{BFermi}, that is, Eq. (4.39), but in Fermi (T, X, Y, Z) coordinates, is given by

$$\psi_{BFermi}(T, X, Y, Z) = \begin{pmatrix} F_1 \\ F_2 \\ G_1 \\ G_2 \end{pmatrix}, \quad (6.16)$$

where

$$F_1 = \frac{\mathbb{D}(T,Z)}{\sqrt{2}\,p_v}\, e^{-iH(T,X,Y,Z)}\left(-b_1 m - \frac{a_2 p_x}{\zeta(T,Z)} + \frac{ia_2 p_y}{\xi(T,Z)}\right), \quad (6.17)$$

$$F_2 = a_2 \mathbb{D}(T,Z)\, e^{-iH(T,X,Y,Z)}, \quad (6.18)$$

$$G_1 = b_1 \mathbb{D}(T,Z)\, e^{-iH(T,X,Y,Z)}, \quad (6.19)$$

$$G_2 = \frac{\mathbb{D}(T,Z)}{\sqrt{2}\,p_v}\, e^{-iH(T,X,Y,Z)}\left(-a_2 m + \frac{b_1 p_x}{\zeta(T,Z)} + \frac{ib_1 p_y}{\xi(T,Z)}\right), \quad (6.20)$$

We may now transform the tetrad 1-forms Eqs. (6.7)-(6.10), into the canonical 1-form fields,

$$\hat{e}^0 = (1, 0, 0, 0) = dT\,, \quad (6.21)$$
$$\hat{e}^1 = (0, 1, 0, 0) = dX\,, \quad (6.22)$$
$$\hat{e}^2 = (0, 0, 1, 0) = dY\,, \quad (6.23)$$
$$\hat{e}^3 = (0, 0, 0, 1) = dZ, \quad (6.24)$$

using the local Lorentz transformation below (see e.g., ref. [9], Eq. (5.234). Note that our notation differs from that of Parker and Toms)

$$\hat{e}^A{}_\alpha = \Lambda^A{}_B \, h^B{}_\alpha \,, \tag{6.25}$$

where Λ is a local Lorentz transformation which in our case is

$$\Lambda = \begin{pmatrix} 1 + \frac{1}{2}\left(\mathbb{A}^2 + \mathbb{B}^2\right) & \mathbb{A} & \mathbb{B} & -\frac{1}{2}\left(\mathbb{A}^2 + \mathbb{B}^2\right) \\[2mm] \mathbb{A} & 1 & 0 & -\mathbb{A} \\[2mm] \mathbb{B} & 0 & 1 & -\mathbb{B} \\[2mm] \frac{1}{2}\left(\mathbb{A}^2 + \mathbb{B}^2\right) & \mathbb{A} & \mathbb{B} & 1 - \frac{1}{2}\left(\mathbb{A}^2 + \mathbb{B}^2\right) \end{pmatrix} \,. \tag{6.26}$$

We let Ψ, be the spinor in region B, in Fermi coordinates but with the Fermi (canonical) tetrad, Eqs. (6.21)-(6.24). Thus Ψ is defined in a region of Minkowski spacetime (region B). The corresponding matrix L, relates the spinors, Ψ and ψ_{BFermi}, that is, $\Psi = L \cdot \psi_{BFermi}$. The calculation of L is given in Appendix B and the result is,

$$L = \begin{pmatrix} 1 & \mathbb{A} - i\mathbb{B} & 0 & 0 \\[2mm] 0 & 1 & 0 & 0 \\[2mm] 0 & 0 & 1 & 0 \\[2mm] 0 & 0 & -\mathbb{A} - i\mathbb{B} & 1 \end{pmatrix} \,. \tag{6.27}$$

It is easy to check that L and Λ satisfy the consistency relations

$$L^{-1}\gamma^A L = \Lambda^A{}_B \gamma^B \,, \tag{6.28}$$

and

$$\gamma^0 L^\dagger \gamma^0 = L^{-1} \,, \tag{6.29}$$

in the chiral representation (see Appendix D). We give, Ψ, below in a convenient shorthand notation using Eqs. (6.16) and (6.27),

$$\Psi = \begin{pmatrix} F_1 + (\mathbb{A} - i\mathbb{B})F_2 \\ F_2 \\ G_1 \\ G_2 - (\mathbb{A} + i\mathbb{B})G_1 \end{pmatrix} \,. \tag{6.30}$$

One may verify directly that both solutions, Eqs. (6.16) and (6.30), satisfy the Dirac equations with their respective tetrads (e.g., using the Fock-Ivanenko coefficient approach).

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7 Spin polarization tests

It is known that in a general semi-Riemannian spacetime we cannot define a conserved spin operator [20]. Nevertheless, in the situation discussed in the present paper, our Dirac particle is in flat (Minkowski) spacetime both before (region F) and after the passage of the shock wave (region B).[5] It therefore follows that we can define spin operators [32], Σ^A, corresponding to polarization in the positive X, Y, Z, directions, for the positive energy solutions of spin $\frac{1}{2}$ particles, in Minkowski spacetime (T, X, Y, Z), in the particle's rest frame. These are, apart from a factor of $\frac{1}{2}$,

$$\Sigma^A = \gamma^5 \gamma^0 \gamma^A, \qquad A = 1, 2, 3 \equiv X, Y, Z. \tag{7.1}$$

The Σ^A of Eq. (7.3) are correct provided our tetrad vectors are the canonical ones for Minkowski spacetime, namely, Eqs. (6.21)-(6.24). However, we must restrict our analysis in region B away from the singularity so that,

$$\sqrt{2}\, a \leq t - z = T - Z \neq \sqrt{2}\, u_s. \tag{7.2}$$

As mentioned above, the polarization of the wavefunction is determined by the choice of the constants a_2 and b_1. The details on such matters can be found in references [33] and [34], keeping in mind that our chiral representation of the γ matrices differs somewhat from theirs, (see Appendix D).

7.1 Transformation Properties of the Spin Operator

In this subsection we consider Minkowski spacetime (or an open submanifold of Minkowski spacetime such as region B) with Minkowski coordinates (T, X, Y, Z). We define spin operators, Σ^A as in [32], corresponding to polarization in the positive X, Y, Z, directions, for the positive energy solutions of spin $\frac{1}{2}$ particles in the particle's rest frame. These are, apart from a factor of $\frac{1}{2}$, given by,

$$\Sigma^A = \gamma^5 \gamma^0 \gamma^A, \qquad A = 1, 2, 3. \tag{7.3}$$

We show below how the operator Σ^A transforms under a (local) field of Lorentz transformations Λ acting on the Minkowski tetrad field associated with the Minkowski coordinates (T, X, Y, Z), and we find two equivalent expressions for the corresponding transformation of the operator Σ^A.

The Σ^A of Eq. (7.3) are spin operators in the particle's rest frame provided that our tetrad vectors are canonical for Minkowski spacetime. For the purposes of this subsection only, we express the canonical tetrad field as:

$$e_0 = (1, 0, 0, 0) = \partial_T, \tag{7.4}$$

$$e_1 = (0, 1, 0, 0) = \partial_X, \tag{7.5}$$

$$e_2 = (0, 0, 1, 0) = \partial_Y, \tag{7.6}$$

$$e_3 = (0, 0, 0, 1) = \partial_Z. \tag{7.7}$$

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[5]We recall that at this point that the singularity manifests itself only through the square root branch point in the function $\mathbb{D}$ given by Eq. (6.13).

The above tetrads satisfy the orthonormality relations,

$$e_A{}^\alpha e_B{}^\beta \eta_{\alpha\beta} = \eta_{AB}\,, \tag{7.8}$$

where η_{AB} is the Minkowski metric. If we now perform a local Lorentz transformation Λ on the tetrad vectors (7.4)-(7.7), we obtain a new set of tetrad vectors, $h_A{}^\alpha$, which are related to the ones above by the equation (written below in terms of the corresponding 1-forms)

$$h^A{}_\alpha = \Lambda^A{}_B\, e^B{}_\alpha\,, \tag{7.9}$$

where

$$\Lambda^A{}_C\, \Lambda^B{}_D\, \eta_{AB} = \eta_{CD}\,, \tag{7.10}$$

and, of course,

$$h^A{}_\alpha\, h^B{}_\beta\, \eta_{AB} = \eta_{\alpha\beta}\,. \tag{7.11}$$

The spacetime dependent γ matrices [9], [39] are then given by,

$$\bar\gamma^\alpha(x) := h_A{}^\alpha(x)\gamma^A\,. \tag{7.12}$$

Using Eq. (7.12) and $h^B{}_\alpha\, h_A{}^\alpha = \delta^B{}_A$, we obtain

$$\gamma^B = h^B{}_\alpha \bar\gamma^\alpha\,. \tag{7.13}$$

Then with the matrix L related to Λ by Eq. (6.28), we get, by rearranging terms,

$$L\,\gamma^C L^{-1} = \gamma^B\, \Lambda_B{}^C\,, \tag{7.14}$$

$$= h^B{}_\alpha\, \Lambda_B{}^C\, \bar\gamma^\alpha\,. \tag{7.15}$$

But from Eq. (7.9), $h^B{}_\alpha = \Lambda^B{}_D\, e^D{}_\alpha$, so that,

$$L\,\gamma^C L^{-1} = \Lambda^B{}_D\, e^D{}_\alpha\, \Lambda_B{}^C\, \bar\gamma^\alpha\,, \tag{7.16}$$

$$= \delta_D{}^C\, e^D{}_\alpha\, \bar\gamma^\alpha\,. \tag{7.17}$$

$$= e^C{}_\alpha\, \bar\gamma^\alpha := \bar\gamma^C\,. \tag{7.18}$$

It now follows immediately that for each $A = X, Y, Z$,

$$\overline\Sigma^A = \bar\gamma^5\bar\gamma^T\bar\gamma^A = (L\,\gamma^5 L^{-1})(L\gamma^T L^{-1})(L\,\gamma^A L^{-1}) = L\Sigma^A L^{-1}\,. \tag{7.19}$$

We summarize this results of this subsection in the form of the following theorem.

Theorem 1. *Let Ψ be a spinor defined in a region of Minkowski space with coordinates (T, X, Y, Z) with the canonical tetrad field given by Eqs. (7.4)-(7.7). Let Λ be a local Lorentz transformation and let $\bar\gamma^\alpha(x)$ be defined by Eqs. (7.12) and (7.9), and let $\overline\Sigma^A =: \bar\gamma^5\bar\gamma^T\bar\gamma^A$. Suppose that L and Λ satisfy (6.28) and (6.29). Then for $A = X, Y$ or Z,*

$$\Sigma^A\Psi = \Psi \quad \text{if and only if} \quad \overline\Sigma^A L\Psi = L\Psi\,. \tag{7.20}$$

7.2 Spin tests in region F

We begin this subsection by considering the situation in region F. As previously mentioned we ignore overall normalization constants. Of course the normalization will be the same in the different parts of the solution for each polarization case. Furthermore we shall also ignore the overall exponential factor. We denote by $u(p)$ the spinor part in Eq. (4.15), thus

$$u(p) = \begin{pmatrix} \dfrac{(-b_1 m - a_2 (p_x - ip_y))}{\sqrt{2}\, p_v} \\[2ex] a2 \\[1ex] b1 \\[2ex] \dfrac{(-a_2 m + b_1 (p_x + ip_y))}{\sqrt{2}\, p_v} \end{pmatrix}. \tag{7.21}$$

For polarization in the positive x-direction we let

$$a_2 = p_t + m - p_x - ip_y + p_z\,, \tag{7.22}$$
$$b_1 = -p_t - m - p_x + ip_y - p_z\,, \tag{7.23}$$

for polarization in the positive y-direction we let

$$a_2 = i(p_t + m + ip_x - p_y + p_z)\,, \tag{7.24}$$
$$b_1 = -p_t - m - ip_x - p_y - p_z. \tag{7.25}$$

Finally for polarization in the positive z-direction we let,

$$a_2 = -\frac{p_x + ip_y}{p_t + m}\,, \tag{7.26}$$

$$b_1 = -1 - \frac{p_z}{p_t + m}\,. \tag{7.27}$$

So in the "lab frame," where, $p_x = p_y = p_z = 0$, and $p_v = m/\sqrt{2}$, the spinor of Eq. (7.21) reduces to the three polarized polarized spinors below,

$$u_x(p) = \begin{pmatrix} 2m \\ 2m \\ -2m \\ -2m \end{pmatrix}, \quad u_y(p) = \begin{pmatrix} 2m \\ 2im \\ -2m \\ -2im \end{pmatrix}, \quad u_z(p) = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \tag{7.28}$$

It is easy to check that

$$\Sigma^1 \cdot u_x(p) - u_x(p) = 0. \tag{7.29}$$

$$\Sigma^2 \cdot u_y(p) - u_y(p) = 0. \tag{7.30}$$

$$\Sigma^3 \cdot u_z(p) - u_z(p) = 0. \tag{7.31}$$

Thus, the spinors in Eq. (7.28) are eigenvectors of the respective spin operators and are polarized in the positive x, y and z directions respectively.

7.3   Spin tests in region B

In this section we apply the tests of Eqs. (7.29)-(7.31) but use the Ψ given by Eq. (6.30). In the "lab frame" we denote by Ψ_{lab} the value of Ψ, when $p_x = p_y = p_z = 0$, and $p_v = m/\sqrt{2}$. The lab frame determines Fermi coordinates, which are also Minkowski coordinates, and the associated tetrad field (see Eqs. (7.4)-(7.7)). The spinor solution Ψ_{lab}, whose momentum is specified and therefore has arbitrary location is given by,

$$\Psi_{lab} = \frac{e^{-\frac{im}{2}(2T - X\mathbb{A} - Y\mathbb{B})}}{\sqrt{\zeta(T, Z)\xi(T, Z)}} \begin{pmatrix} -b_1 + a_2(\mathbb{A} - i\mathbb{B}) \\ a_2 \\ b_1 \\ -a_2 - b_1(\mathbb{A} + i\mathbb{B}) \end{pmatrix}, \tag{7.32}$$

and where $\zeta, \xi, \mathbb{A}, \mathbb{B}$ are given in Eqs. (5.16), (5.17), (6.11), (6.12), respectively.

To test the deviation from the initial x polarization eigenstate we choose the a_2 and b_1 given by Eqs. (7.22) and (7.23). Then we find that

$$\Psi_{lab}^{(x)} = \frac{2m\, e^{-\frac{im}{2}(2T - X\mathbb{A} - Y\mathbb{B})}}{\sqrt{\zeta(T, Z)\xi(T, Z)}} \begin{pmatrix} 1 + \mathbb{A} - i\mathbb{B} \\ 1 \\ -1 \\ -1 + \mathbb{A} + i\mathbb{B} \end{pmatrix}, \tag{7.33}$$

and that

$$\Sigma^1 \cdot \Psi_{lab}^{(x)} - \Psi_{lab}^{(x)} \equiv \Delta\Psi_{lab}^{(x)} \neq 0, \tag{7.34}$$

where

$$\Delta\Psi_{lab}^{(x)} = \frac{2m\, e^{-\frac{im}{2}(2T - X\mathbb{A} - Y\mathbb{B})}}{\sqrt{\zeta(T, Z)\xi(T, Z)}} \begin{pmatrix} -\mathbb{A} + i\mathbb{B} \\ \mathbb{A} - i\mathbb{B} \\ \mathbb{A} + i\mathbb{B} \\ -\mathbb{A} - i\mathbb{B} \end{pmatrix}. \tag{7.35}$$

Likewise for initial polarization in the y direction, we choose the a_2 and b_1 given by Eqs. (7.24) and (7.25) and obtain

$$\Psi_{lab}^{(y)} = \frac{2m\, e^{-\frac{im}{2}(2T - X\mathbb{A} - Y\mathbb{B})}}{\sqrt{\zeta(T, Z)\xi(T, Z)}} \begin{pmatrix} 1 + i\mathbb{A} + \mathbb{B} \\ i \\ -1 \\ -i + \mathbb{A} + i\mathbb{B} \end{pmatrix}, \tag{7.36}$$

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and that
$$\Sigma^2 \cdot \Psi_{lab}^{(y)} - \Psi_{lab}^{(y)} \equiv \Delta\Psi_{lab}^{(y)} \neq 0, \tag{7.37}$$

where
$$\Delta\Psi_{lab}^{(y)} = \frac{2m \, e^{-\frac{im}{2}(2T - X\mathbb{A} - Y\mathbb{B})}}{\sqrt{\zeta(T,Z)\xi(T,Z)}} \begin{pmatrix} -i\mathbb{A} - \mathbb{B} \\ -\mathbb{A} + i\mathbb{B} \\ -i\mathbb{A} + \mathbb{B} \\ -\mathbb{A} - i\mathbb{B} \end{pmatrix}. \tag{7.38}$$

Finally, to test the deviation from the z polarization eigenstate we choose the a_2 and b_1 given by Eqs. (7.26) and (7.27). We find that

$$\Psi_{lab}^{(z)} = \frac{e^{-\frac{im}{2}(2T - X\mathbb{A} - Y\mathbb{B})}}{\sqrt{\zeta(T,Z)\xi(T,Z)}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ \mathbb{A} + i\mathbb{B} \end{pmatrix}, \tag{7.39}$$

and that again
$$\Sigma^3 \cdot \Psi_{lab}^{(z)} - \Psi_{lab}^{(z)} \equiv \Delta\Psi_{lab}^{(z)} \neq 0, \tag{7.40}$$

where
$$\Delta\Psi_{lab}^{(z)} = -\frac{2 \, e^{-\frac{im}{2}(2T - X\mathbb{A} - Y\mathbb{B})}}{\sqrt{\zeta(T,Z)\xi(T,Z)}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathbb{A} + i\mathbb{B} \end{pmatrix} \neq 0. \tag{7.41}$$

Remark 7. *We note that the spin is unaltered only on the zero measure set of the Fermi observer's worldline, but the spatial position of the Dirac particle is completely unspecified so this is not a measurable phenomenon. If a different laboratory frame described in Remark 6 were used instead, the spinor along the original laboratory frame would be seen to be unpolarized.*

We see that $\Sigma^A\Psi_{lab}^{(A)} \neq \Psi_{lab}^A$ for each coordinate direction A. It now follows from Theorem 1 that no Minkowski tetrad field exists in region B with respect to which Ψ_{lab}^A, expressed in that tetrad field, is stationary and polarized in any coordinate direction. To see this, assume to the contrary that there does exist such a Minkowski tetrad field. Then that tetrad field must be related to the lab frame tetrad field through a Lorentz transformation Λ with associated spinor representation L. Using the notation of Theorem 1, it follows that

$$\overline{\Sigma}^A L\Psi_{lab}^A = L\Psi_{lab}^A, \tag{7.42}$$

for some direction A and at all spacetime points in region B. But this implies by Theorem 1 that $\Sigma^A\Psi_{lab}^A = \Psi_{lab}^A$, which is a contradiction.

Thus, the effect of the gravitational wave pulse is to alter the spins so that the Dirac particle loses its initial polarization. We note that we arrive at this same conclusion if any other timelike geodesic discussed in Remark 6 is used to define a Minkowski tetrad field on the interior of region B instead of the lab frame we used. The calculations are essentially the same.

8 Summary and concluding remarks

In this paper, we considered a spacetime $(\mathcal{M}, \mathrm{g})$ for the gravitational wave pulse (the 'sandwich' wave of [21]), which is a union of three regions, F (front), W (wave), and B (back) as shown in Figure 1. We solved the Dirac equation for a particle in this spacetime. We showed that starting with polarized Dirac particles (eigenstates of the operators Σ^A) in region F, regardless of the initial polarization state, the Dirac spinor cannot be an eigenfunction of the axial stationary spin operators in any possible Minkowski tetrad field in region B, after the passage of the gravitational wave pulse. Our choice for a lab frame geodesic was arbitrary, as described in Remark 6, but this conclusion that the spin is unpolarized holds with any such choice because of Theorem 1 and the remarks surrounding Eq (7.42). Our results are broadly consistent with those of [19], which analyzed the behavior of *classical* spinning particles polarized in advance of the arrival of a weak gravitational wave, and the effects of the wave on spin. In our case, all of our solutions are exact, including the solution to the Dirac equation, the Fermi coordinate transformations, transformations of the tetrad field and spinor, and there was no linearization of the metric.

Appendix A   The transformation to Fermi coordinates in region B

In this section we give the transformation to the exact Fermi coordinates, (T, X, Y, Z). For further details see references [30, 35, 36]. We take our Fermi observer to be the timelike geodesic,

$$\gamma(\tau) = (t(\tau), x(\tau), y(\tau), z(\tau)) = (\tau, 0, 0, 0), \quad \tau = T \geq 0. \tag{A.1}$$

We recall the metric Eq. (2.1)

$$ds^2 = dt^2 - dz^2 - f^2(u)dx^2 - g^2(u)dy^2, \tag{A.2}$$

where

$$u = \frac{t - z}{\sqrt{2}}. \tag{A.3}$$

Our first step is to write the geodesic equations using the Lagrangian for the metric expressed in the form of Eq. (2.4),

$$L = \dot{v}\dot{u} - \frac{f^2(u)}{2}\dot{x}^2 - \frac{g^2(u)}{2}\dot{y}^2, \tag{A.4}$$

where the overdot signifies derivative with respect to an affine parameter σ, and

$$f(\sigma) = p + qu(\sigma), \tag{A.5}$$

$$g(\sigma) = r + su(\sigma), \tag{A.6}$$

where p, q, r, s, are constants. Recall that these constants were appropriately chosen so that the sandwich wave metric is C^1.

Lagrange's equations are

$$p_x = \frac{\partial L}{\partial \dot{x}} = -f^2(u)\dot{x}\,, \tag{A.7}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = -g^2(u)\dot{y}\,, \tag{A.8}$$

$$p_v = \frac{\partial L}{\partial \dot{v}} = \dot{u}\,, \tag{A.9}$$

where p_x, p_y, p_v, are constants (conserved canonical momenta). Integrating Eq. (A.9), we obtain,

$$u(\sigma) = p_v \sigma + u_0. \tag{A.10}$$

From Eqs. (A.7), (A.8) we have

$$x(\sigma) = -\int \frac{p_x}{f^2(u)}\, d\sigma = \frac{p_x}{p_v q\,(p + qu(\sigma))} + x_0,, \tag{A.11}$$

$$y(\sigma) = -\int \frac{p_y}{f^2(u)}\, d\sigma = \frac{p_y}{p_v s\,(r + su(\sigma))} + y_0. \tag{A.12}$$

The easiest way to obtain $v(\sigma)$ is to substitute the solutions, (A.10)-(A.12) in Eq. (A.4), for the different kinds of geodesics, and integrate the resulting ODE below

$$2\dot{v}\dot{u} - f^2(\sigma)\dot{x}^2 - g^2(\sigma)\dot{y}^2 = \begin{cases} -1, & \text{spacelike,} \\ 0, & \text{null,} \\ +1, & \text{timelike}, \end{cases} \tag{A.13}$$

where, we have abused the notation slightly, so that $f(\sigma) = p + qu(\sigma)$, and so on. The results for the *spacelike* case follow:

$$x(\sigma) = \frac{p_x}{p_v\, q\, f(\sigma)} + x_0, \tag{A.14}$$

$$y(\sigma) = \frac{p_y}{p_v\, s\, g(\sigma)} + y_0, \tag{A.15}$$

$$v(\sigma) = \left(\frac{-1}{2p_v^2}\right)\left(u(\sigma) + \frac{p_x^2}{q\, f(\sigma)} + \frac{p_y^2}{s\, g(\sigma)}\right) + v_0, \tag{A.16}$$

$$u(\sigma) = p_v\, \sigma + u_0 \tag{A.17}$$

The geodesics in (x, y, z, t) coordinates follow from the relations,

$$z(\sigma) = \frac{v(\sigma) - u(\sigma)}{\sqrt{2}}, \tag{A.18}$$

$$t(\sigma) = \frac{v(\sigma) + u(\sigma)}{\sqrt{2}}, \tag{A.19}$$

A.1 Spacelike geodesics

To obtain the Fermi coordinate transformation equations, we first write the general expressions for the spacelike geodesics in region B. From the preceding calculations, we find,

$$x(\sigma) = \frac{p_x}{p_v \, q \, f(\sigma)} + x_0, \tag{A.20}$$

$$y(\sigma) = \frac{p_y}{p_v \, s \, g(\sigma)} + y_0, \tag{A.21}$$

$$z(\sigma) = \left(\frac{1}{\sqrt{2}}\right) \left(v_0 - u(\sigma) - \left(\frac{1}{2p_v^2}\right) \left(u(\sigma) + \frac{p_x^2}{q \, f(\sigma)} + \frac{p_y^2}{s \, g(\sigma)}\right)\right), \tag{A.22}$$

$$t(\sigma) = \left(\frac{1}{\sqrt{2}}\right) \left(v_0 + u(\sigma) - \left(\frac{1}{2p_v^2}\right) \left(u(\sigma) + \frac{p_x^2}{q \, f(\sigma)} + \frac{p_y^2}{s \, g(\sigma)}\right)\right), \tag{A.23}$$

where σ is the proper length, $x_0, y_0, v_0, u_0, p_x, p_y, p_v$, are constants, and the functions $f(\sigma), g(\sigma), u(\sigma)$, are

$$u(\sigma) = p_v \, \sigma + u_0, \tag{A.24}$$
$$f(\sigma) = p + qu(\sigma), \tag{A.25}$$
$$g(\sigma) = r + su(\sigma). \tag{A.26}$$

A.2 Fermi coordinate conditions

Our Fermi observer is the timelike geodesic

$$\gamma(\tau) = (\tau, 0, 0, 0). \tag{A.27}$$

Fermi coordinates in region B are uniquely determined by the parallel transport along $\gamma(\tau)$ of the tetrad of vectors given by Eqs. (5.2)-(5.5) to region B. This parallel transported tetrad frame constitutes the Fermi coordinate axes.

At any point Q on the Fermi observer worldline we may define a unit spacelike vector

$$\mathbb{V} = \left(\dot{t}(0), \dot{x}(0), \dot{y}(0), \dot{z}(0)\right). \tag{A.28}$$

The vector $\mathbb{V}$ is the tangent vector to a spacelike geodesic emanating from Q. The constants, x_0, y_0, v_0, u_0, in Eqs. (A.20)-(A.23) are chosen so that

$$(t(0), x(0), y(0), z(0)) = (\tau, 0, 0, 0) = \gamma(\tau). \tag{A.29}$$

We find that

$$\sqrt{2}\, u_0 = \tau \equiv T. \tag{A.30}$$

We now choose the constants p_x, p_y, p_v, so as to satisfy the conditions below, Eqs. (A.31)-(A.34),

$$f(0)\dot{x}(0) = \frac{X}{\lambda}, \tag{A.31}$$

$$g(0)\dot{y}(0) = \frac{Y}{\lambda}, \tag{A.32}$$

$$\dot{z}(0) = \frac{Z}{\lambda}, \tag{A.33}$$

$$\dot{t}(0) = 0, \tag{A.34}$$

where $\dot{x} = dx/d\sigma$, etc., and

$$\lambda = \sqrt{X^2 + Y^2 + Z^2}. \tag{A.35}$$

Conditions (A.31) and (A.32) are satisfied by letting

$$p_x = -\frac{(p + qu_0)X}{\lambda}, \tag{A.36}$$

$$p_y = -\frac{(r + su_0)Y}{\lambda}, \tag{A.37}$$

Conditions (A.33) and (A.34) become

$$\frac{-2p_v^2 - \mathbb{C}}{2\sqrt{2}\, p_v} = \frac{Z}{\lambda}, \tag{A.38}$$

$$\frac{2p_v^2 - \mathbb{C}}{2\sqrt{2}\, p_v} = 0, \tag{A.39}$$

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where

$$\mathbb{C} = \left(\frac{Z}{\lambda}\right)^2. \tag{A.40}$$

Condition (A.39) is satisfied with

$$p_v = \pm\sqrt{\frac{\mathbb{C}}{2}}, \tag{A.41}$$

however, it is evident from Eq. (A.38) that $p_v > 0$ gives the negative of the right-hand-side, so that the correct choice is

$$p_v = -\sqrt{\frac{\mathbb{C}}{2}}. \tag{A.42}$$

In (t, x, y, z) coordinates the Lagrangian L is given by

$$2L = \dot{t}^2 - \dot{z}^2 - f^2(u)\dot{x}^2 - g^2(u)\dot{y}^2 = -1. \tag{A.43}$$

So using Eq. (A.34) we have, as expected,

$$f^2(0)\dot{x}^2(0) + g^2(0)\dot{y}^2(0) + \dot{z}^2(0) = 1. \tag{A.44}$$

Finally using Eq. (A.30), we find that the transformation relating the coordinates (t, x, y, z) to the Fermi coordinates (T, X, Y, Z) is given by

$$x = x(\lambda) = \frac{X}{\zeta(T, Z)}, \tag{A.45}$$

$$y = y(\lambda) = \frac{Y}{\xi(T, Z)}, \tag{A.46}$$

$$z = z(\lambda) = Z - \frac{qX^2}{2\sqrt{2}\,\zeta(T, Z)} - \frac{sY^2}{2\sqrt{2}\,\xi(T, Z)}, \tag{A.47}$$

$$t = t(\lambda) = T - \frac{qX^2}{2\sqrt{2}\,\zeta(T, Z)} - \frac{sY^2}{2\sqrt{2}\,\xi(T, Z)}, \tag{A.48}$$

where in Eqs. (A.45)-(A.48),

$$\zeta(T, Z) = p + q\left(\frac{T - Z}{\sqrt{2}}\right), \tag{A.49}$$

$$\xi(T, Z) = r + s\left(\frac{T - Z}{\sqrt{2}}\right). \tag{A.50}$$

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Eqs. (A.45)-(A.48), are Eqs. (5.12)-(5.15) in Sec. 5, where we also give the inverse transformation, Eqs. (5.18)-(5.21). Using the transformation (A.45)-(A.48), the metric

$$ds^2 = dt^2 - dz^2 - f^2(u)dx^2 - g^2(u)dy^2 \,, \tag{A.51}$$

is transformed into the Minkowski metric,

$$ds^2 = dT^2 - dX^2 - dY^2 - dZ^2 \,. \tag{A.52}$$

Appendix B The matrix L

We use the equation below given in ref. [9], p. 225, with appropriate notational changes,

$$L(\Lambda(x)) = \exp\left(i\epsilon_{AB}\Sigma^{AB}\right), \tag{B.1}$$

where Λ is a local Lorentz transformation, ϵ_{AB} are the parameters characterizing the Lorentz transformation. Also from [9], Eq. (5.284), p. 228, we have that

$$\Sigma^{AB} = -\frac{i}{8}\left[\gamma^A, \gamma^B\right], \tag{B.2}$$

thus

$$L = \exp\left(\frac{\epsilon_{AB}}{8}\left[\gamma^A, \gamma^B\right]\right) \tag{B.3}$$

Our γ matrices will be in the chiral representation.

It is straightforward to show that, $\det(L) = 1$. We use the relation that for a matrix M we have that

$$\det\left(e^M\right) = e^{\operatorname{tr}(M)}. \tag{B.4}$$

Now we use Eq. (B.3) in its most general form. The exponent then is

$$\frac{1}{2}\left(\epsilon_{01}\gamma^0\gamma^1 + \epsilon_{02}\gamma^0\gamma^2 + \epsilon_{03}\gamma^0\gamma^3 + \epsilon_{12}\gamma^1\gamma^2 + \epsilon_{13}\gamma^1\gamma^3 + \epsilon_{23}\gamma^2\gamma^3\right). \tag{B.5}$$

By evaluating each product of pairs of γ matrices separately in the chiral representation, we find that they all vanish. Thus the trace of expression (B.5) vanishes. This is a representation independent result.

For the details of the relation between Λ and L, namely the calculation of the parameters ϵ_{AB}, a clear presentation is given in Chapter 1 of Hitoshi Yamamoto's lecture notes [37] where it may be seen that Λ may be expressed as

$$\Lambda = e^{\xi_i K_i + \theta_i L_i} \equiv e^M \,, \quad i = 1, 2, 3, \tag{B.6}$$

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where the matrices K_i and L_i are given in [37], furthermore we find that

$$M = \begin{pmatrix} 0 & \xi_1 & \xi_2 & \xi_3 \\ \xi_1 & 0 & -\theta_3 & \theta_2 \\ \xi_2 & \theta_3 & 0 & -\theta_1 \\ \xi_3 & -\theta_2 & \theta_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \epsilon_{01} & \epsilon_{02} & \epsilon_{03} \\ \epsilon_{01} & 0 & -\epsilon_{12} & -\epsilon_{13} \\ \epsilon_{02} & \epsilon_{12} & 0 & -\epsilon_{23} \\ \epsilon_{03} & \epsilon_{13} & \epsilon_{23} & 0 \end{pmatrix} , \qquad (B.7)$$

or

$$\epsilon_{01} = \xi_1, \ \epsilon_{02} = \xi_2, \ \epsilon_{03} = \xi_3, \ \epsilon_{12} = \theta_3, \ \epsilon_{13} = -\theta_2, \ \epsilon_{23} = \theta_1. \qquad (B.8)$$

Using Mathematica's MatrixLog[Λ] with our Λ, Eq. (6.14), we obtain our expression for M. Comparing coefficients with (B.3), we find the ϵ_{AB}. Finally Mathematica's MatrixExp[M] gives us L.

Appendix C Amplitude and polarization

The gravitational wave part in region W, Eqs. (4.2),(4.3), is characterized by the, in general complex, curvature component

$$\Psi_4(u) = |\Psi_4| e^{i\theta}. \qquad (C.1)$$

In the Newman-Penrose formalism Ψ_4 is related to the Weyl tensor components by the relation

$$\Psi_4 = -C_{\alpha\beta\gamma\delta}\, \overline{m}^\alpha n^\beta\, \overline{m}^\gamma n^\delta \,, \qquad (C.2)$$
$$= -C_{\alpha\beta\gamma\delta}\, \lambda_4{}^\alpha \lambda_2{}^\beta \lambda_4{}^\gamma \lambda_2{}^\delta \,. \qquad (C.3)$$
$$= -C_{4242} \,, \qquad (C.4)$$

We remark that since our spacetime is Ricci flat, the Weyl is equal to the Riemann tensor.

In Eq. (C.1) above, $|\Psi_4|$ is the amplitude of the wave and θ its polarization [26], p. 325. If θ is constant then the wave is said to be linearly polarized. In our case we have that

$$\Psi_4 = k^2 \,, \qquad (C.5)$$

thus our result for the amplitude of the wave agrees with Rindler's deduction, [22], p. 170.

Appendix D   The flat space solution

The solution in region F is the usual Dirac plane wave solution in the chiral representation.

Remark 8. *We choose the chiral representation γ matrices*

$$\gamma^0 = \begin{pmatrix} 0 & -I_2 \\ -I_2 & 0 \end{pmatrix}, \quad \gamma^K = \begin{pmatrix} 0 & \sigma^K \\ -\sigma^K & 0 \end{pmatrix}, \quad K = (1,2,3), \tag{D.1}$$

where the σ^K are the standard Pauli matrices, and $\gamma^5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3$. We have

$$\left(\gamma^0\right)^2 = I, \quad \left(\gamma^K\right)^2 = -I. \tag{D.2}$$

We write everything in x, y, v, u, and p_x, p_y, p_v, p_u, so that $p_\mu x^\mu = p_x x + p_y y + p_v v + p_u u$. Then, for example, the positive and negative energy solutions with spin along the z axis, in the Feynman-Stückelberg interpretation, i.e., $p_t > 0$ everywhere, are given below apart from an overall normalization factor, $N(p) = \sqrt{(p_t + m)/4m}$, where, we follow the Lorentz invariant normalization of [32] and [38], namely $\bar{u}(p,s)u(p,s') = \delta_{s,s'}$, etc.

$$\psi^{(+)(1)} = \begin{pmatrix} \dfrac{\sqrt{2}\left(m+\sqrt{2}\,p_u\right)}{\sqrt{2}\,m+p_v+p_u} \\[2ex] \dfrac{-\sqrt{2}(p_x+i\,p_y)}{\sqrt{2}\,m+p_v+p_u} \\[2ex] \dfrac{-\sqrt{2}\left(m+\sqrt{2}\,p_v\right)}{\sqrt{2}\,m+p_v+p_u} \\[2ex] \dfrac{-\sqrt{2}(p_x+i\,p_y)}{\sqrt{2}\,m+p_v+p_u} \end{pmatrix} e^{-ip_\mu x^\mu}, \quad \psi^{(+)(2)} = \begin{pmatrix} \dfrac{-\sqrt{2}(p_x-i\,p_y)}{\sqrt{2}\,m+p_v+p_u} \\[2ex] \dfrac{\sqrt{2}\left(m+\sqrt{2}\,p_v\right)}{\sqrt{2}\,m+p_v+p_u} \\[2ex] \dfrac{-\sqrt{2}(p_x-i\,p_y)}{\sqrt{2}\,m+p_v+p_u} \\[2ex] \dfrac{-\sqrt{2}\left(m+\sqrt{2}\,p_u\right)}{\sqrt{2}\,m+p_v+p_u} \end{pmatrix} e^{-ip_\mu x^\mu},$$

$$\tag{D.3}$$

$$\psi^{(-)(1)} = \begin{pmatrix} \dfrac{\sqrt{2}\left(m+\sqrt{2}p_u\right)}{\sqrt{2}\,m+p_v+p_u} \\[2ex] \dfrac{-\sqrt{2}(p_x+i\,p_y)}{\sqrt{2}\,m+p_v+p_u} \\[2ex] \dfrac{\sqrt{2}\left(m+\sqrt{2}\,p_v\right)}{\sqrt{2}\,m+p_v+p_u} \\[2ex] \dfrac{\sqrt{2}(p_x+i\,p_y)}{\sqrt{2}\,m+p_v+p_u} \end{pmatrix} e^{ip_\mu x^\mu}, \quad \psi^{(-)(2)} = \begin{pmatrix} \dfrac{-\sqrt{2}(p_x-i\,p_y)}{\sqrt{2}\,m+p_v+p_u} \\[2ex] \dfrac{\sqrt{2}\left(m+\sqrt{2}\,p_v\right)}{\sqrt{2}\,m+p_v+p_u} \\[2ex] \dfrac{\sqrt{2}(p_x-i\,p_y)}{\sqrt{2}\,m+p_v+p_u} \\[2ex] \dfrac{\sqrt{2}\left(m+\sqrt{2}p_u\right)}{\sqrt{2}\,m+p_v+p_u} \end{pmatrix} e^{ip_\mu x^\mu}.$$

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