Math 462, Test 1, Dr. Klein

Solutions

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1. (a) (5 points) Define what it means for a collection B of vectors in a vector space V to be a basis of V.

   This question was asked on Quiz 2. See text or class notes for definition.

   (b) (10 points) Let $B = \{\vec{v}_1, ..., \vec{v}_n\}$ be a basis of a vector space V. Prove that for any $\vec{u} \in V$ there is exactly one choice of scalars $\lambda_1, ..., \lambda_n$, for which $\vec{u} = \lambda_1\vec{v}_1 + ... + \lambda_n\vec{v}_n$.

   Solution: Since Span $B = V$, there exist scalars, $\lambda_1, \ldots, \lambda_n$ such that $\vec{u} = \lambda_1\vec{v}_1 + \cdots + \lambda_n\vec{v}_n$. Suppose that $\vec{u} = \mu_1\vec{v}_1 + \cdots + \mu_n\vec{v}_n$ for scalars $\mu_1, \ldots, \mu_n$. Then

   $$\lambda_1\vec{v}_1 + \cdots + \lambda_n\vec{v}_n = \mu_1\vec{v}_1 + \cdots + \mu_n\vec{v}_n.$$

   Therefore,

   $$(\lambda_1 - \mu_1)\vec{v}_1 + \cdots + (\lambda_n - \mu_n)\vec{v}_n = \vec{0}.$$

   Since B is linearly independent, $\lambda_1 - \mu_1 = 0, \ldots, \lambda_n - \mu_n = 0$, and $\mu_i = \lambda_i$ for $i = 1, \ldots, n$.

   (c) (5 points) Let A be an $n \times n$ matrix over a field F. Use part b to show that if the columns of A constitute a basis of F^n then the linear system of equations $A\vec{x} = \vec{y}$ has exactly one solution for each vector $\vec{y}$ in F^n.

   Solution: Let the columns of A be $A^1, \ldots, A^n$ and let,

   $$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

   Then $A\vec{x} = \vec{y}$ implies that $x_1A^1 + \cdots + x_nA^n = \vec{y}$. Since the vectors $A^1, \ldots, A^n$ constitute a basis of F^n, it follows from part b that there exist unique scalars $x_1, \ldots, x_n$ such that $A\vec{x} = \vec{y}$.

2. (10 points) Let $T : V \to W$ be a one-to-one, onto linear transformation between vector spaces V and W, so that the inverse function $T^{-1} : W \to V$ exits. Prove that T^{-1} is linear.

Solution: Let $\vec{w}_1, \vec{w}_2 \in W$. Since T is a bijection, there exist unique vectors $\vec{v}_1, \vec{v}_2 \in V$ such that $T\vec{v}_1 = \vec{w}_1$ and $T\vec{v}_2 = \vec{w}_2$. Then,

$$T^{-1}(\vec{w}_1 + \vec{w}_2) = T^{-1}(T\vec{v}_1 + T\vec{v}_2) = T^{-1}(T(\vec{v}_1 + \vec{v}_2)) = \vec{v}_1 + \vec{v}_2 = T^{-1}\vec{w}_1 + T^{-1}\vec{w}_2.$$

Let λ be a scalar and let $\vec{w} \in W$ with $T\vec{v} = \vec{w}$. Then,

$$T^{-1}(\lambda\vec{w}) = T^{-1}(\lambda T\vec{v}) = T^{-1}(T(\lambda\vec{v})) = \lambda\vec{v} = \lambda T^{-1}\vec{w}.$$

Therefore T^{-1} is linear.

3. (15 points) Let $T : V \to W$ be a surjective linear transformation and let $\vec{v}_1, \ldots, \vec{v}_n$ be a spanning set of vectors in V. Prove that the vectors $T(\vec{v}_1), \ldots, T(\vec{v}_n)$ span W.

This was assigned as problem 15 from chap 4. **Solution:**

Let $\vec{w} \in W$. Since T is onto, there is a vector $\vec{v} \in V$ such that $T\vec{v} = \vec{w}$. Since $\vec{v}_1, \ldots, \vec{v}_n$ span V, there are scalars $\lambda_1, \ldots, \lambda_n$ such that $\vec{v} = \lambda_1\vec{v}_1 + \cdots + \lambda_n\vec{v}_n$. Then,

$$\vec{w} = T\vec{v} = T(\lambda_1\vec{v}_1 + \cdots + \lambda_n\vec{v}_n) = \lambda_1 T\vec{v}_1 + \cdots + \lambda_n T\vec{v}_n.$$

Therefore, the vectors $T(\vec{v}_1), \ldots, T(\vec{v}_n)$ span W.

4. (15 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x, y, z) = (x + 3z, 4y - x, y + z)$.

   (a) Find the matrix representation $M_B(T)$ of T with respect to the standard basis $B = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.

   Solution:
$$T\vec{e}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad T\vec{e}_2 = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} \quad T\vec{e}_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$
   Therefore,
$$M_B(T) = \begin{pmatrix} 1 & 0 & 3 \\ -1 & 4 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

   (b) Find the transition matrix P from the ordered basis $B' = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \right\}$ to B of $\mathbb{R}^3$ (given in part a).

   This was assigned as problem 20 in Chap 6. **Solution:**
$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 1\vec{e}_1 + 2\vec{e}_2 - 1\vec{e}_3 \quad \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix} = 6\vec{e}_1 + 0\vec{e}_2 + 1\vec{e}_3 \quad \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = -1\vec{e}_1 + 2\vec{e}_2 + 2\vec{e}_3$$
   Therefore,
$$P = \begin{pmatrix} 1 & 6 & -1 \\ 2 & 0 & 2 \\ -1 & 1 & 2 \end{pmatrix}$$

   (c) Express $M_{B'}(T)$, the matrix representation of T in the basis B', as a product of matrices and/or their inverses. You do not have to multiply the matrices or calculate inverses for full credit.

   Solution: $M_{B'}(T) = P^{-1}M_B(T)P$ (See Theorem 6.23 in Chap 6).

5. (10 points) Let $T : V \to W$ be a linear transformation between vector spaces V and W. Prove that if $\mathrm{Ker}(T) = \{\vec{0}\}$, then T is one-to-one.

   Solution: Suppose $T\vec{v} = T\vec{w}$ for some pair of vectors $\vec{v}, \vec{w} \in V$. Then $T\vec{v} - T\vec{w} = \vec{0}$. Therefore, $T(\vec{v} - \vec{w}) = \vec{0}$. Therefore, $\vec{v} - \vec{w} \in \mathrm{Ker}(T)$. Therefore, $\vec{v} - \vec{w} = \vec{0}$. Therefore, $\vec{v} = \vec{w}$. Therefore, T is one-to-one.

6. (15 points) Let V be a vector space and let $T : V \to V$ be a linear transformation such that $T \circ T = T$. Prove that $V = \mathrm{Ker}(T) \oplus \mathrm{Im}(T)$. Hint: What is the effect of T on $\vec{v} - T(\vec{v})$.

   This was assigned as prob 29 in Chap 3. It was solved in class. It also appeared on Quiz 1. Check your class notes for the solution.

7. (15 points) Let $V = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$ be the vector space over $\mathbb{R}$ of polynomials of degree two or less. Let the basis B of V be $\{1, x, x^2\}$, and let $B^* = \{1^*, x^*, x^{2*}\}$ be the dual basis of V^*. Let $f : V \to \mathbb{R}$ be the linear functional defined by $f(p) = \int_0^2 p(x)dx$. Express f as a linear combination of elements of B^*.

   This was a practice problem given out before the exam. **Solution:**

   Let $f = \lambda_1 1^* + \lambda_2 x^* + \lambda_3 x^{2*}$. To solve for the coefficients $\lambda_1, \lambda_2, \lambda_3$, evaluate both sides of the equation on each of the basis vectors $\{1, x, x^2\}$.

$$
\begin{aligned}
\int_0^2 1dx = f(1) &= \lambda_1 1^*(1) + \lambda_2 x^*(1) + \lambda_3 x^{2*}(1) \\
2 &= \lambda_1 + 0 + 0 \\
\lambda_1 &= 2 \\
\int_0^2 xdx = f(x) &= \lambda_1 1^*(x) + \lambda_2 x^*(x) + \lambda_3 x^{2*}(x) \\
2 &= 0 + \lambda_2 + 0 \\
\lambda_2 &= 2 \\
\int_0^2 x^2dx = f(x^2) &= \lambda_1 1^*(x^2) + \lambda_2 x^*(x^2) + \lambda_3 x^{2*}(x^2) \\
\frac{8}{3} &= 0 + 0 + \lambda_3 \\
\lambda_3 &= \frac{8}{3}
\end{aligned}
$$

   Therefore, $f = 2 \cdot 1^* + 2 \cdot x^* + \frac{8}{3} \cdot x^{2*}$.