Math 651B & Phys 640
Assignment 4

Problem I. Consider the 3-dim spacetime metric determined by the line element,

$$\begin{aligned} ds^2 &= -2dtdx + \cos^2(kt)dy^2\,, \\ &= g_{tt}dt^2 + g_{tx}dtdx + g_{ty}dtdy \\ &+ g_{xt}dxdt + g_{xx}dx^2 + g_{xy}dxdy \\ &+ g_{yt}dydt + g_{yx}dydx + g_{yy}dy^2\,, \end{aligned} \tag{1}$$

In the above expression "$-2dtdx$" is a common abuse of notation and should be interpreted as "$-dt \otimes dx - dx \otimes dt$." Define the vectors,

$$\vec{A} = (A^t, A^x, A^y) = (1, 0, \sin(kt))\,, \quad \vec{B} = (B^t, B^x, B^y) = (0, 1, \sin(kt))$$

(a) Find the one-form $\tilde{A} = (A_t, A_x, A_y)$,
(b) Evaluate the scalar product, $g(\vec{A}, \vec{B})$,
(c) Evaluate $\tilde{A}(\vec{B})$.

Problem II. Find the nonvanishing connection coefficients for the 2-dim spacetime metric below

$$ds^2 = -dt^2 + a(t)^2 dx^2\,.$$

Problem III. Given the connection coefficients of a 2-dim. spacetime (you will not need all of them),

$$\Gamma^t{}_{tr} = \Gamma^t{}_{rt} = \frac{m}{2r(r-m)}\,, \quad \Gamma^r{}_{rr} = -\frac{(2r-m)}{2r(r-m)}\,, \quad \Gamma^r{}_{tt} = \frac{2m(r-m)}{r}\,,$$

and the curve $\gamma(\tau)$ below

$$\gamma(\tau) = (t(\tau)\,, r(\tau)) = \left(\tau + \frac{1}{2}\log\left(\frac{\tau-1}{\tau+1}\right)\,, m\tau^2\right)\,,$$

(a) find the tangent vector (the velocity vector) $\vec{U}(\tau) \equiv \vec{\gamma}'(\tau)$.
(b) Evaluate the t component of $\nabla_{\vec{U}}\vec{U}$, and express your answer as a function of τ.

Problem IV.: Time Dilation for a Traveler Accelerated by One Earth Gravity

Sheldon Cooper, from the series "Big Bang Theory," is in outer space near Earth in a spaceship with engines that can give it constant acceleration or deceleration of 1 earth gravity relative to its own instantaneous inertial frame. Because of this Sheldon can always walk around comfortably in his spaceship during his trips with his usual weight, just as he would on Earth (or sit in his favorite spot on his couch in the space ship).

Sheldon is planning a trip to a distant planet 32,000 light years away from Earth. For the trip he will accelerate with constant acceleration $1g$ for the first half of the trip and then decelerate at $1g$ for the second half of the trip coming, to rest near the destination planet. He will then return to earth in the same way, accelerating at $1g$ for half the return trip and then decelerating at $1g$ for the final leg of the journey.

The goal of this problem is to determine how much older Sheldon is at the end of this trip and what calendar year he arrives back on Earth. Follow the steps below to find the answer. Assume throughout that the Earth's frame is inertial (this isn't quite true, but it's good enough for this problem). Also, throughout this problem, time is measured in years and distance is measured in lightyears. In these units, the speed of light, $c = 1$.

a. The acceleration of gravity at the surface of the Earth is $g = 9.8$ m/sec^2. Verify that (by lucky numerical coincidence), $g \approx 1$ light year/ year^2. Use this value for g (actually this is slightly less than one earth gravity, but close enough for comfort on a spaceship).

b. The 4-velocity, U of a test particle may be expressed as $U = \gamma(1, \vec{v})$ as in lecture. With the notation of your textbook, and with an overdot signifying differentiation with respect to coordinate time t, verify that:

a) $\dot{\gamma} = \gamma^3 \beta \dot{\beta}$

b) $|| \, \vec{v} \, ||^2 = \beta^2$, where the norm is Euclidean norm

c) $\vec{v} \cdot \vec{a} = \beta \dot{\beta}$, where $\vec{a} = \dot{\vec{v}}$ is the usual 3-acceleration.

c. Let η denote the Minkowski metric and let A be the 4-acceleration of a test particle with 4-velocity U. Use problem 2 to prove that,

$$\eta(A, A) = -\gamma^4 || \, \vec{a} \, ||^2 - \gamma^6 \beta^2 (\dot{\beta})^2$$

Show therefore that in the instantaneous rest frame of the particle,

$$\eta(A, A) = -|| \, \vec{a} \, ||^2$$

Thus, the norm of the 4-acceleration is equal to the magnitude of the 3-acceleration in the instantaneous rest frame.

d. It is sufficient to consider only 2 spacetime dimensions for this problem, Earth time t and Sheldon's linear distance from earth. For Sheldon's trip, $|| \, \vec{a} \, || = g$ in his instantaneous rest frame. Deduce the following equations:

$$(U^0)^2 - (U^1)^2 = 1 \qquad U^0 A^0 - U^1 A^1 = 0 \qquad (A^0)^2 - (A^1)^2 = -g^2$$

Show therefore that,

$$\frac{dU^1}{d\tau} = gU^0 \quad \text{and} \quad \frac{dU^0}{d\tau} = gU^1,$$

where τ is Sheldon's proper time on the space ship.

e. Using problem 4, find a formula for Sheldon's worldcurve $(t(\tau), x(\tau))$, as measured in Earth's (inertial) frame during the first quarter of his total journey, as a function of his proper time. Hint: to check your result, review page 21 of the textbook.

f. Using problem 5, find a formula in terms of inverse hyperbolic functions for τ as a function of x and also t as a function of x. Find how many years it takes Sheldon to go half the distance, 16,000 light years, toward the distant planet. Find the answer both in terms of Sheldon's proper time τ and in terms of Earth time t for this part of the journey (1/4 of the total journey).

g. The two times, as measured by Sheldon and by Earth clocks, for the entire trip (to the planet and back) can be obtained by multiplying your two answers from problem 6 by 4. How much older is Sheldon by the time he comes back to Earth and in what calendar year will he arrive if he begins his journey today? Will he be able to visit his friend Leonard when he comes back?