Advanced Placement Calculus AB Evaluation

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Documents reviewed:

- *Calculus: Calculus AB Calculus BC Course Description,* The College Entrance Examination Board, 2005
- *Teacher's Guide: AP Calculus,* The College Entrance Examination Board and Educational Testing Service, 1997
- *2003 AP Calculus AB and AP Calculus BC, Released Exams,* The College Entrance Examination Board, 2005
- *1998 AP Calculus AB and AP Calculus BC, Released Exams,* The College Entrance Examination Board and Educational Testing Service, 1999
- *AP Calculus AB Free Response Questions, AP Calculus AB Free Response Items, Form B, AP Calculus AB Scoring Guidelines, AP Calculus AB Scoring Guidelines, Form B* for the years 2004, 2005, 2006
- Four sample syllabi of Calculus AB classroom teachers provided by The College Board

Background

There are two AP Calculus courses, Calculus AB and Calculus BC. The College Board recommends that both be taught as a college-level courses. Calculus AB is intended to correspond to 2/3 of a year long college calculus sequence, and Calculus BC is intended to substitute for a full year of college calculus. There are separate exams for each of these courses, but the grade for the BC exam includes a subscore based on the portion of the exam devoted to Calculus AB topics, approximately 60% of the test. By design the overlapping topics are not covered in any greater depth than on the AB exam. According to the College Board, the reliability of the Calculus AB subscore is nearly equal to the reliabilities of the Calculus AB and BC exams.

The focus of this report is on Calculus AB, and the grades are for Calculus AB only. However, some discussion of Calculus BC is also included because of the overlap of topics, and to set a broader context for the first course.

The AP Calculus exams are graded on a five point scale:

AP Grade	Qualification
5	Extremely well qualified
4	Well qualified
3	Qualified
2	Possibly Qualified
1	No recommendation

The duration of each AB and BC Calculus examination is 3 hours and 15 minutes. Section I of each exam consists of multiple choice questions, and Section II consists of free response questions. The two sections receive equal weight in the grading, and each of the two sections is further divided into a Part A and a Part B.

Part A of Section I has 28 multiple choice questions to be completed in 55 minutes, and does not allow students to use calculators. Part B of Section I requires a graphing calculator and consists of 17 questions to be completed in 50 minutes.

Each of Parts A and B of Section II lasts 45 minutes and each consist of 3 free response problems. Calculators are allowed only for Part A. During the time allotted for Part B, students may continue to work on Part A questions, but without a calculator. Not all of the questions in the parts of the test that allow calculators necessarily require their use, but some do.

Each college and university sets its own AP credit and placement policies, but many institutions offer at least a semester of credit for high grades on the AB exam, and a year of credit for high scores on the BC exam.

The Teacher's Guide explains the philosophy, themes, and goals of the AP Calculus courses:

> "Calculus AB and Calculus BC are primarily concerned with developing the students' understanding of concepts of calculus, and providing experience with its methods and applications. The courses emphasize a multirepresentational approach to calculus, with concepts, results, and problems being expressed geometrically, numerically, analytically, and verbally."

Working with functions geometrically, numerically, analytically, and verbally, and understanding the interconnections is the first listed goal of AP Calculus. It is referred to as "the rule of four," which the *Teacher's Guide* describes as a "rallying cry for the calculus reform movement," in contrast to "the earlier paradigm of doing almost everything analytically."

Technology plays a central role in AP Calculus. One of the listed goals is "the incorporation of technology into the course." The *Guide* recommends that "students should be comfortable using machines to solve problems, experiment, interpret results, and verify conclusions," and further explains that, "The most natural way to achieve this goal is to let the students use their own technology at all times, except perhaps for certain targeted 'no-calculator' assessments (which should be rare, and never at the exploratory phase of student learning)."

The Teacher's Guide presents AP Calculus as an extension of the K-12 mathematics reform movement led by the National Council of Teachers of Mathematics or NCTM, as explained in this passage:

> "Teachers familiar with the NCTM *Standards* and/or with various education reform documents will recognize many of these goals as being part of a broader blueprint for educational change. Adopting them for our students has necessitated (for many of us) a change in the *way* we teach, and for the AP Calculus Committee the *Standards* have suggested some significant changes in *what* we will teach in the immediate future."

Content

The AP Calculus curriculum has noteworthy strengths. One is the emphasis on the definite integral as a limit of Riemann sums to counter the tendency of students to think of integrals only as anti-derivatives. The explicit inclusion of the Mean Value Theorem along with geometric consequences (for both AB and BC) is also commendable, due to its theoretical importance in calculus. Also of value for students who will apply calculus to scientific and engineering problems is a focus on correct units to answers to word problems, and an emphasis within the curriculum on verbal descriptions of mathematical concepts using correct terminology. The value of this emphasis is two-fold: it helps students to understand the meanings of word problems and therefore is a first step to problem-solving, and it helps students communicate their solutions to others.

There are also deficiencies and controversial features in the AP Calculus program. Among them are the following.

1) Calculators vs. Analytic Methods

Of the categories of the "rule of four," analytic methods receive the least emphasis in the *Teacher's Guide.* The topic, "Computation of derivatives," which calls for the ability to compute derivatives of standard functions, along with knowledge of the chain rule, and the rules for finding derivatives of sums, differences, products, and quotients of functions, comes at the end of the list of topics. The *Guide* explains,

> "Perhaps the most significant thing about this topic [computation of derivatives] is that it is listed last, consistent with the philosophy that the emphasis of the course is not on manipulation."

To that end, the *Guide* explains, "Logarithmic differentiation is no longer on the list of topics." This is a mismatch with mainstream university calculus courses, where this is a standard topic. Practice with logarithmic differentiation helps to develop technical fluency in computations involving logarithms and exponentials, and it should be included in the curriculum.

The AP Calculus exams require the use of graphing calculators that can at minimum graph functions within an arbitrary viewing window, numerically calculate derivatives and definite integrals, and find roots of functions. The exams also allow calculators with Computer Algebra Systems (CAS) that can symbolically calculate limits, derivatives, and integrals. For the sake of equity, however, the exam questions are purposefully crafted in such a way so as to avoid giving advantage to examinees with these more powerful machines. For example, students are asked only to find definite integrals with numerical answers, and not indefinite integrals, in those parts of the AB and BC exams that allow calculators. In this way calculators with CAS provide no direct advantage over what the other allowed calculators can do. Here and elsewhere, technology determines mathematical content, a negative feature.

One of the topics in the AB and BC courses is the Trapezoidal Rule for numerical integration. This is a standard topic in first year calculus courses. However, Simpson's Rule, also a standard topic, is not included in the AP Calculus curriculum because, the Teacher's Guide explains, "it was viewed by most students as just another formula to memorize...(The Trapezoidal Rule is also a formula, but more students can see exactly where it comes from)." Ironically, in Appendix 3, the *Teacher's Guide* provides graphing calculator programs for Simpson's Rule that students are invited to enter into their calculators if they do not already come equipped with one. Students are permitted to use these programs during the AP Exams, thus adding to the "black box" role played by the calculator.

As described above, fluency in hand calculations receives relatively low emphasis in the AP Calculus curriculum, by design, and that choice is reflected by the exams. Only the simplest paper and pencil calculations involving algebra and calculus are required on the AP Calculus exams. This curricular choice is flawed. Technical fluency in hand calculations is essential for following – or producing – some mathematical proofs, and for the purpose of deriving scientific formulae in the mathematical sciences. The Calculus Committee of the College Board was aware of the controversial nature of this de-emphasis. The Teacher's Guide includes the following passage:

> "A final concern about calculators is the unfortunate fact that not all teachers at the college level approve of their use. It is therefore quite possible that an AP student will do well in your course, become comfortable with technology, and then enter a college mathematics course in which no calculators are allowed."

The Guide nevertheless gives overly optimistic assurances of the appropriateness of the AP Calculus curriculum.

2) Definition and Computation of Limits

The mathematical definition of limit is not part of the AP Calculus syllabus and is not tested. The Teacher's Guide explains,

> "Epsilons and deltas are gone from BC, and should not be missed by anyone embracing the goals of the course. (Their most apparent effect on student understanding in the first calculus course in the past, even in college, has been for some students to understand that they had better drop the course. After that, what will the *second* week be like?) On the other hand, students will need an intuitive understanding of limits that can manifest itself graphically, numerically, analytically, and verbally. This is another area in which technology has really transformed the landscape, rendering the traditional algebraic approach to finding limits almost to the status of last resort."

The above paragraph argues that the definition of limit should not be taught because it is difficult. That is not a good reason. Not all first year college and university calculus courses include the definition of limit, and those that do vary in their depth of treatment. However, there are sound reasons for including it. Students who continue their studies as math majors benefit from an early exposure (revisited in depth in later courses) to a precise definition of this fundamental concept. Inclusion of precise definitions is the first step in recognizing that calculus *can* be developed rigorously with proofs, even though an introductory course cannot avoid a large dose of hand waving.

Of the four exemplary course syllabi provided by the College Board which I examined for this review, only one went beyond the AP Calculus syllabus by including the definition of the limit as an optional topic. But even in this isolated example, the syllabus indicated that "we no longer do limit proofs."

3) Inverse Function Theorem

The Inverse Function Theorem, once part of the AP Calculus syllabus, now appears in a weaker form: "Use implicit differentiation to find the derivative of an inverse function." The Teacher's Guide explains,

> "The derivative of an inverse function has always been in the course description, but note here that the course description specifies the 'use of implicit differentiation' to discover the derivative, as opposed to the bottom-line rule found in the Inverse Function Theorem. This is consistent with the philosophy that the course should focus on broad concepts and methods rather than on the memorization of formulas or theorems."

Using implicit differentiation to find the derivative of an inverse function is a valuable exercise, but the Inverse Function Theorem goes beyond that kind of calculation by guaranteeing the differentiability of the inverse function. Science courses make regular use of its conclusion, that dx/dy is the reciprocal of dy/dx, without deriving it each time via implicit differentiation, so it is a useful result for students to know. Furthermore, the Inverse Function Theorem for functions of one real variable has important generalizations to higher dimensions in later courses, and deserves a place in the curriculum for that reason alone.

To its credit, the Calculus AB exam does call upon students to find derivatives of inverse functions without first prompting the use of implicit differentiation. The following is a released exam question from the 2003 Calculus AB test, Section I, Part A:

> Let f be the function defined by $f(x) = x^3 + x$. If $g(x) = f^{-1}(x)$ and $g(2) = 1$, what is the value of $g'(2)$?

The question is followed by five choices for the correct answer. Many university calculus instructors would feel gratified for their own students to be able to answer such a question correctly, as would many high school Calculus AB teachers. For the 2003 Calculus AB exam, only 18% of the test takers correctly answered this question – a rate slightly worse than expected for random guessing. Moreover, fewer than half of the students who received the highest possible grade of 5, for the exam as a whole, correctly answered it. Certainly it is desirable and even necessary for an exam of this type to include difficult questions, but results for this topic might be improved if the Inverse Function Theorem were an explicit part of the syllabus.

4) Applications of Integrals and Anti-Derivatives

Some standard applications of Riemann sums and integrals, such as calculating work done by a force applied through a displacement, or volumes by cylindrical shells, are not part of the syllabus. Instead the goal is to teach modeling more generally so that students can apply their understanding of Riemann sums to a broad range of problems. The *Teacher's Guide* explains that "there might well be a problem on the AP Examination that starts by defining work as force times displacement, then proceeds to a point where my students could be expected to write the integral for themselves." However, in looking through released test questions and sample questions from the *Course Description*, I found no such examples, and there is a sameness of the problem types. Typical problems involve a necessarily artificial variable rate (for example, rates of traffic, sand, fuel or water flow) that the examinee is expected integrate with respect to time in order to find a net accumulation.

Ironically, the BC curriculum takes the opposite approach by requiring specific knowledge of the logistic differential equation, and using that particular equation in modeling problems on the Calculus BC exam. This is not a standard topic in many college and university first year calculus courses, but it appears with regularity on the Calculus BC exams that I examined.

The AP Calculus curriculum would achieve a better fit with college programs by reversing these choices. That is, by not specifying particular differential equations to be analyzed at the BC level, and by requiring more standard, and important, applications of the integral at the AB level.

5) Techniques of Integration

As with derivatives, hand calculation of antiderivatives receives low emphasis. For Calculus AB students, integration techniques are limited to knowing antiderivatives of basic functions and through simple substitution (including change of limits of integration). Integration by parts is a BC topic, and there exposure to partial fractions is limited to the case of non repeating linear factors, just enough to handle the logistic differential equation. The *Teacher's Guide* explains, "Again the impetus for this change stems ultimately from technology, but that is only because calculators are more dramatic than integral tables in replacing the need for this skill." This is insufficient. It is not clear from the materials I reviewed whether trigonometric substitutions are included in the curriculum. Trigonometric substitution allows for the evaluation of $\int_0^1 \sqrt{1-x^2}\,dx$ so that students can see how the number π arises in the formula for the area of a circle, a valuable lesson that goes beyond the insufficient explanation that a calculator says so. There are no integration problems requiring trigonometric substitution in the released examination questions reviewed for this report.

Conclusions

The Calculus AB curriculum is clearly written, with little ambiguity. The exams are well crafted and closely aligned to the curriculum. They appear to be effective in measuring the prescribed content. However, that content leaves much to be desired.

As indicated in the *Teacher's Guide*, the AP Calculus Committee looks to the NCTM for guidance. As with many NCTM aligned K-12 mathematics programs, the AP Calculus courses suffer from overuse of calculators and missing or abridged topics of importance. The roles should be reversed. University level mathematics programs should give guidance to K-12 mathematics education, not vice versa.

Technology is the centerpiece of the AP Calculus program. Two of the four parts of each AP Calculus examination require the use of graphing calculators. It is not then surprising that the mathematical content and the development of mathematical reasoning within the curriculum are strongly influenced by that technology. One negative consequence is a de-emphasis of analytical skills and important parts of mathematical reasoning.

A 1997 report from a task force formed by the Mathematical Association of America to advise the National Council of Teachers of Mathematics in its revision of the 1989 NCTM Standards described mathematical reasoning as follows:

> [T]he foundation of mathematics is reasoning. While science verifies through observation, mathematics verifies through logical reasoning. Thus the essence of mathematics lies in proofs, and the distinction among illustrations, conjectures and proofs should be emphasized. . .

The AP Calculus curriculum emphasizes "real world" applications through the use of technology, but not enough attention is given to the structural organization by which parts of calculus are connected to each other. The AP Calculus curriculum partially fulfills this obligation, e.g., through the use of the Mean Value Theorem to develop relationships between the graphs of a function and its first and second derivatives, and the definite integral as a limit of Riemann sums, as already noted. But definitions and proofs are largely swept aside.

There are obvious restrictions to what can be accomplished along these lines in an introductory calculus course, but some opportunities avail themselves. For example, a proof that differentiability at a point implies continuity at that point is accessible at this level, along with basic limit proofs, and derivation of the derivative formulas for trigonometric and other functions. Even the Mean Value Theorem could be more fully exploited within the curriculum by asking for its use in proving the Fundamental Theorem of Calculus, and the Mean Value Theorem for Integrals. Of course AP Calculus teachers are free to include all of this material in their own syllabi, but they are also free not to, and whether they do or not, their students will not see questions about such topics on the AP Calculus exams.

In spite of its shortcomings, the Calculus AB program has some positive features, as identified in this report, especially in the context of high school mathematics at the time of this writing. Nevertheless, university calculus courses generally provide a better foundation for students who will continue their studies in the mathematical sciences than what AP Calculus has to offer.

Grades

Clarity:	B
Content:	C
Rigor (Mathematical Reasoning):	C

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