

NOTE
AN INEQUALITY FOR KRUSKAL-MACAULAY FUNCTIONS

BERNARDO M. ÁBREGO, SILVIA FERNÁNDEZ-MERCHANT, AND BERNARDO LLANO

ABSTRACT. Given integers $k \geq 1$ and $n \geq 0$, there is a unique way of writing n as $n = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \dots + \binom{n_1}{1}$ so that $0 \leq n_1 < \dots < n_{k-1} < n_k$. Using this representation, the *Kruskal-Macaulay function* of n is defined as $m_k(n) = \binom{n_k-1}{k-1} + \binom{n_{k-1}-1}{k-2} + \dots + \binom{n_1-1}{0}$. We show that if $a \geq 0$ and $a < m_{k+1}(n)$, then $m_k(a) + m_{k+1}(n-a) \geq m_{k+1}(n)$. As a corollary, we obtain a short proof of Macaulay's Theorem.

1. INTRODUCTION

Given integers $k \geq 1$ and $n \geq 0$, there is a unique way of writing n as

$$n = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \dots + \binom{n_2}{2} + \binom{n_1}{1} \tag{1.1}$$

so that $0 \leq n_1 < n_2 < \dots < n_{k-1} < n_k$. Using this representation, called the *k-binomial representation* of n , the *Kruskal-Macaulay function* of n is defined as,

$$m_k(n) = \binom{n_k-1}{k-1} + \binom{n_{k-1}-1}{k-2} + \dots + \binom{n_2-1}{1} + \binom{n_1-1}{0}.$$

(See [2], [5], [8], [10] for details.) The main goal of this note is to prove the following inequality for Kruskal-Macaulay functions and show some of its consequences.

Theorem 1. *Let k, a , and n be integers such that $k \geq 1$ and $n \geq a \geq 0$. If $a < m_{k+1}(n)$, then*

$$m_k(a) + m_{k+1}(n-a) \geq m_{k+1}(n). \tag{1.2}$$

Kruskal-Macaulay functions are relevant, among other things, for their applications to the study of antichains in multisets (see for example [8], [2]), posets, rings and polyhedral combinatorics (see [4] and the survey [3]). In particular, they play an important role in proving results, extensions and generalizations of classical problems concerning the Kruskal-Katona ([9],[7]) and Erdős-Ko-Rado [6] theorems. For instance, as a corollary of Theorem 1, we obtain a short proof of Macaulay's Theorem [10] (see Section 2). More recently, the authors [1] applied Theorem 1 to the problem of finding the maximum number of translated copies of a pattern that can occur among n points in a d -dimensional space, a typical problem related to the study of repeated patterns in Combinatorial Geometry.

Theorem 1 is tight in the sense that (1.2) does not necessarily hold if $a \geq m_{k+1}(n)$. For instance, whenever $k \geq 2$, $n_3 = 4$, $n_2 = 2$, $n_1 = 1$, and $a = m_{k+1}(n)$, we have that $m_k(a) + m_{k+1}(n-a) = m_{k+1}(n) - 1 < m_{k+1}(n)$.

We conclude this introduction by pointing out that the interesting problem of characterizing the equality case in Theorem 1 is still open.

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2. CONSEQUENCES OF THE THEOREM

Consider the set of nonincreasing sequences of length k

$$M_k = \{(x_1, x_2, \dots, x_k) \in \mathbb{N}^k : x_1 \geq x_2 \geq \dots \geq x_k \geq 1\}$$

with its lexicographic order. That is, for x and y in M_k , we write $x \prec y$ if for some index i , $x_i < y_i$ and $x_j = y_j$ whenever $j < i$. If $A \subseteq M_k$, then the *shadow* of A , denoted by ∂A , consists of all subsequences of length $k - 1$ of elements of A ($\partial(\emptyset) = \emptyset$). That is,

$$\partial A = \{x \in M_{k-1} : \exists y \in A \text{ such that } x \text{ is a subsequence of } y\}.$$

Using these definition, observe that

$$\text{if } A, B \subseteq M_k, \text{ then } \partial(A \cup B) = \partial A \cup \partial B. \tag{2.1}$$

There is an important relationship between shadows of sets and Kruskal-Macaulay functions. Namely, if we denote by $F_k(N)$ the set consisting of the first N members of M_k in the lexicographic order, then

$$|\partial F_k(N)| = m_k(N). \tag{2.2}$$

We now prove Macaulay's Theorem as a corollary of Theorem 1.

Corollary 1. (Macaulay's Theorem, 1927 [10]) *Let $k \geq 0$. For every $A \subseteq M_{k+1}$,*

$$|\partial A| \geq |\partial F_{k+1}(|A|)|.$$

Proof. Let $A \subseteq M_{k+1}$. We proceed by induction on $k + |A|$. If $k = 0$ or $A = \emptyset$, the result is trivially true. Suppose $k \geq 1$ and $A \neq \emptyset$. Set $A_{11} = \{x \in M_k : x_k = 1 \text{ and } x * 1 \in A\}$, $A_{12} = \{x \in M_k : x_k \geq 2 \text{ and } x * 1 \in A\}$, and $A_2 = \{x \in A : x_{k+1} \geq 2\}$. Here $x * 1$ denotes the concatenation of x and 1, that is $x * 1$ is the k -tuple x with an entry 1 appended in the $(k + 1)^{\text{th}}$ position. Clearly, $A = (A_{11} * 1) \cup (A_{12} * 1) \cup A_2$ and the terms in the union are pairwise disjoint. Moreover, we can assume that $A_{11} \cup A_{12} \neq \emptyset$. Otherwise, since all entries of members of A are ≥ 2 , we can work with the set A' obtained by subtracting 1 to every entry in the sequences of A ($|A'| = |A|$ and $|\partial A'| = |\partial A|$.) Let $a = |A_{11}| + |A_{12}|$ and $b = |A_2|$. Note that $|A| = a + b$ and $a \geq 1$.

If $x = (x_1, x_2, \dots, x_k) \in A_{11}$, then $(x_1, x_2, \dots, x_{k-1}) \in \partial A_{11}$ and $(x_1, x_2, \dots, x_{k-1}, 1) = x \in \partial A_{11} * 1$. That is, $A_{11} \subseteq \partial A_{11} * 1$. We now calculate ∂A in terms of A_{11} , A_{12} , and A_2 using (2.1).

$$\begin{aligned} \partial A &= \partial A_2 \cup A_{12} \cup A_{11} \cup (\partial A_{11} * 1) \cup (\partial A_{12} * 1) \\ &= \partial A_2 \cup A_{12} \cup (\partial A_{11} * 1) \cup (\partial A_{12} * 1) = (\partial A_2 \cup A_{12}) \cup (\partial(A_{11} \cup A_{12}) * 1). \end{aligned}$$

If $x \in (\partial A_2 \cup A_{12})$, then $x_k \geq 2$. Thus $(\partial A_2 \cup A_{12}) \cap (\partial(A_{11} \cup A_{12}) * 1) = \emptyset$, and consequently

$$|\partial A| = |\partial A_2 \cup A_{12}| + |\partial(A_{11} \cup A_{12})|. \tag{2.3}$$

We consider two cases. If $a \geq m_{k+1}(|A|)$, then

$$|\partial A| = |\partial A_2 \cup A_{12}| + |\partial(A_{11} \cup A_{12})| \geq |A_{12}| + |A_{11}| = a \geq m_{k+1}(|A|).$$

Assume $a < m_{k+1}(|A|)$. Since $a \geq 1$ then $b < |A|$ and thus, by induction and (2.2), $|\partial A_2 \cup A_{12}| \geq |\partial A_2| \geq |\partial F_{k+1}(b)| = m_{k+1}(b)$ and $|\partial(A_{11} \cup A_{12})| \geq |\partial F_k(a)| = m_k(a)$. Therefore, by (2.3), Theorem 1, and (2.2); we have

$$|\partial A| \geq m_{k+1}(b) + m_k(a) \geq m_{k+1}(|A|) = |\partial F_{k+1}(|A|)|.$$

□

In terms of shadows of sets, and using our previous corollary, Theorem 1 can be generalized as follows.

Corollary 2. *Given sets $A \subseteq M_k$ and $B \subseteq M_{k+1}$ with $|A| < |\partial F_{k+1}(|A| + |B|)|$ we have*

$$|\partial A| + |\partial B| \geq |\partial F_{k+1}(|A| + |B|)|.$$

Proof. By the previous corollary and (2.2), $|\partial A| + |\partial B| \geq m_k(|A|) + m_{k+1}(|B|)$ and $|A| < m_{k+1}(|A| + |B|)$. Thus, by Theorem 1, $m_k(|A|) + m_{k+1}(|B|) \geq |\partial F_{k+1}(|A| + |B|)|$. \square

3. PROOF OF THE THEOREM

First an observation. If $n > k \geq 0$ then by Pascal's identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-2}{k-1} + \cdots + \binom{n-k}{1} + \binom{n-k-1}{0} \quad (3.1)$$

Let $a = \sum_{i=1}^k \binom{a_i}{i}$ be the k -binomial representation of a . We say that a is *long* if $a_1 \geq 1$, and *short* if $a_1 = 0$.

Lemma 1. *Let $a \geq 0$ be an integer. If a is short, then $m_k(a+1) = m_k(a) + 1$, otherwise $m_k(a+1) = m_k(a)$.*

Proof. The result is clear for $a = 0$. If $a \geq 1$ is short, then $a = \sum_{i=v}^k \binom{a_i}{i}$ for some $v \geq 2$ and $a_v \geq v$. Thus $a+1 = \sum_{i=v}^k \binom{a_i}{i} + \binom{v-1}{v-1}$ is the k -binomial representation of $a+1$. Then $m_k(a+1) = m_k(a) + \binom{v-2}{v-2} = m_k(a) + 1$.

Now suppose a is long. There is $v \geq 2$ such that $a_j = a_1 + j - 1$ for $j < v$, and either $v = k+1$ or $v \leq k$ and $a_v > a_1 + v - 1$. Then

$$a+1 = \binom{a_k}{k} + \cdots + \binom{a_v}{v} + \binom{a_1+v-2}{v-1} + \cdots + \binom{a_1+1}{2} + \binom{a_1}{1} + \binom{a_1-1}{0}$$

and by (3.1) the binomial representation of $a+1$ is

$$a+1 = \binom{a_k}{k} + \cdots + \binom{a_v}{v} + \binom{a_1+v-1}{v-1}.$$

Then, again by (3.1),

$$m_k(a+1) - m_k(a) = \binom{a_1+v-2}{v-2} - \left(\binom{a_1+v-3}{v-2} + \cdots + \binom{a_1}{1} + \binom{a_1-1}{0} \right) = 0.$$

\square

To prove the Theorem, we need to consider the *extended k -binomial representation* of a positive integer a , by requiring an a_0 coefficient. That is, we write

$$a = \binom{a'_a}{k} + \binom{a'_{a-1}}{k-1} + \cdots + \binom{a'_2}{2} + \binom{a'_1}{1} + \binom{a'_0}{0},$$

with $0 \leq a'_0 = a'_1 - 1 < a'_1 < \cdots < a'_k$. The condition $a'_0 = a'_1 - 1$ is necessary to make this representation unique when it exists. (Clearly $a = 0$ does not have an extended representation.)

In fact,

Lemma 2. *Let $a = \sum_{i=v}^k \binom{a_i}{i} \geq 1$ be the k -binomial representation of a , where the terms equal to zero have been omitted. The extended k -binomial representation of a exists (and it is unique), if and only if $a_v \geq v + 1$.*

Proof. If $a_v \geq v + 1$, then, by (3.1), $\binom{a_v}{v} = \binom{a_v-1}{v} + \binom{a_v-2}{v-1} + \dots + \binom{a_v-v-1}{0}$. Thus $a = \sum_{i=0}^v \binom{a_v-v-1+i}{i} + \sum_{i=v+1}^k \binom{a_i}{i}$ is an extended k -representation of a . Reciprocally, if $a = \sum_{i=0}^k \binom{a'_i}{i}$ is an extended k -representation, then $\binom{a'_0}{0} = \binom{a'_1-1}{0}$, and there is $v \geq 1$ such that $a'_j = a'_1 + j - 1$ for $1 \leq j \leq v$ with either $v = k$ or $a'_{v+1} > a'_1 + v$. Then, by (3.1),

$$a = \sum_{j=0}^v \binom{a'_1 + j - 1}{j} + \sum_{i=v+1}^k \binom{a'_i}{i} = \binom{a'_1 + v}{v} + \sum_{i=v+1}^k \binom{a'_i}{i},$$

is the k -representation of a . Thus $a_v = a'_1 + v \geq v + 1$. □

We can define $m'_k(a) = \sum_{i=1}^k \binom{a'_i-1}{i-1}$ for the extended k -representation of a (if it exists). It turns out that both definitions agree, i.e., $m_k(a) = m'_k(a)$. Indeed, if $a = \sum_{i=v}^k \binom{a_i}{i}$ with $a_v \geq v + 1$, then by (3.1) and the last proof, $m_k(a) - m'_k(a) = \binom{a_v-1}{v-1} - \left(\sum_{i=0}^v \binom{a_v-v-2+i}{i-1}\right) = 0$.

Let $n = \sum_{i=1}^{k+1} \binom{n_i}{i}$, $a = \sum_{i=1}^k \binom{a_i}{i}$, and $n - a = b = \sum_{i=1}^{k+1} \binom{b_i}{i}$, be binomial representations.

Lemma 3. *If $0 \leq a < m_{k+1}(n)$, then $a_k < n_{k+1} \leq b_{k+1} + 1$.*

Proof. We prove the contrapositives. If $a_k \geq n_{k+1}$, then $a \geq \binom{a_k}{k} \geq \binom{n_{k+1}}{k} = m_{k+1}\left(\binom{n_{k+1}+1}{k+1}\right) \geq m_{k+1}(n)$, since $\binom{n_{k+1}+1}{k+1} \geq n$ and m_{k+1} is a non-decreasing function by Lemma 1. Now, if $b_{k+1} + 1 \leq n_{k+1} - 1$, then $b < \binom{b_{k+1}+1}{k+1} \leq \binom{n_{k+1}-1}{k+1}$. Thus

$$a = n - b > n - \binom{n_{k+1} - 1}{k + 1} = \binom{n_{k+1} - 1}{k} + \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \dots + \binom{n_1}{1},$$

but

$$m_{k+1}(n) = \binom{n_{k+1} - 1}{k} + \binom{n_k - 1}{k-1} + \binom{n_{k-1} - 1}{k-2} + \dots + \binom{n_1 - 1}{0},$$

and clearly $\binom{n_i}{i} \geq \binom{n_i-1}{i-1}$. Thus $a \geq m_{k+1}(n)$. □

Proof of Theorem 1. Recall $b = n - a$. Clearly, (1.2) holds if $a = 0$, and the case $a = 1$ is a consequence of Lemma 1.

We first prove the case $a_k < b_{k+1}$. Assume that the pair (a, b) minimizes $m_k(a) + m_{k+1}(b)$ with a as small as possible. If $a \leq 1$, the result holds. Let $a = \sum_{i=v}^k \binom{a_i}{i} \geq 2$.

Suppose first that $a_v \geq v + 1$. Then a has an extended representation, say $a = \sum_{i=0}^k \binom{a'_i}{i}$. Let

$$\alpha = \sum_{i=1}^k \binom{\min(a'_i, b_i)}{i} \text{ and } \beta = \binom{b_{k+1}}{k+1} + \sum_{i=1}^k \binom{\max(a'_i, b_i)}{i} + \binom{a'_0}{0}.$$

Note that $a + b = \alpha + \beta$ and $\alpha < a$. Also $0 \leq \min(a'_1, b_1) < \min(a'_2, b_2) < \dots < \min(a'_k, b_k)$ and $0 \leq a'_0 < \max(a'_1, b_1) < \dots < \max(a'_k, b_k) < b_{k+1}$ (since $a'_k \leq a_k < b_{k+1}$ by assumption). Therefore the definitions we gave for α and β are k -binomial representations (extended for β). This means that $m_k(\alpha) + m_{k+1}(\beta) = m_k(\alpha) + m'_{k+1}(\beta) = m_k(a) + m_{k+1}(b)$. This contradicts the minimality of a .

Assume now that $a_v = v$. This means that $a - 1 = a - \binom{a_v}{v} = \sum_{i=v+1}^k \binom{a_i}{i} \geq 1$ is the k -representation of $a - 1$, and thus $a - 1$ is short. Then by Lemma 1, $m_k(a - 1) + m_{k+1}(b + 1) = m_k(a) - 1 + m_{k+1}(b + 1) \leq m_k(a) + m_{k+1}(b)$, which again contradicts the minimality of a .

Now, assume $b_{k+1} \leq a_k$. Since $a < m_{k+1}(n)$ then, by Lemma 3, $a_k < n_{k+1} \leq b_{k+1} + 1$. That is, $a_k = b_{k+1} = n_{k+1} - 1$. We proceed by induction on k . If $k = 1$, then $a_1 = b_2 = n_2 - 1$.

Thus $\binom{n_2}{2} + \binom{n_1}{1} = n = a + b = \binom{n_2-1}{1} + \binom{n_2-1}{2} + \binom{b_1}{1}$, i.e., $b_1 = n_1$. Hence, $m_1(a) + m_2(b) = \binom{n_2-2}{0} + \binom{n_2-2}{1} + \binom{n_1-1}{0} = m_2(n)$.

Assume $k \geq 2$ and that the result holds for $k-1$. Let $n' = n - \binom{n_{k+1}}{k+1}$, $b' = b - \binom{n_{k+1}-1}{k+1}$, and $a' = a - \binom{n_{k+1}-1}{k}$. Clearly, $a' + b' = n'$, and $a' < m_k(n')$ since $a < m_{k+1}(n) = \binom{n_{k+1}-1}{k} + m_k(n')$. By induction on k the result holds for a', b', n' , and thus

$$\begin{aligned} m_{k+1}(b) + m_k(a) - m_{k+1}(n) &= \binom{n_{k+1}-2}{k} + m_k(b') + \binom{n_{k+1}-2}{k-1} + m_{k-1}(a') - \binom{n_{k+1}-1}{k} + m_k(n') \\ &= m_k(b') + m_{k-1}(a') - m_k(n') \geq 0. \end{aligned}$$

□

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(B. M. Ábrego and S. Fernández-Merchant) DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY, NORTHRIDGE AT 18111 NORDHOFF STREET, NORTHRIDGE, CA 91330

E-mail address: `bernardo.abrego@csun.edu` and `silvia.fernandez@csun.edu`

Current address: (B. M. Ábrego and S. Fernández-Merchant) CENTRO DE INVESTIGACIÓN EN MATEMÁTICAS, A.C., GUANAJUATO, GTO., MEXICO

(B. Llano) DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA METROPOLITANA, IZTAPALAPA, SAN RAFAEL ATLIXCO 186, COLONIA VICENTINA, 09340, MÉXICO, D.F.

E-mail address: `llano@xanum.uam.mx`