

## Problem of the Week 5, Fall 2005

**Solution by George Craciun (edited).** Let  $x_1, x_2, x_3, x_4$  the roots of the equation  $x^4 - (3m + 2)x^2 + m^2 = 0$ . Then, in order for these roots to be in arithmetic progression we set

$$x_1 = a - 3r, x_2 = a - r, x_3 = a + r, x_4 = a + 3r \quad (1)$$

where  $a, r$  are real numbers. Now, writing Viète formulas for the initial equations we have:

$$x_1 + x_2 + x_3 + x_4 = 0 \quad (2)$$

$$x_1(x_2 + x_3 + x_4) + x_2(x_3 + x_4) + x_3x_4 = -(3m + 2) \quad (3)$$

$$x_1x_2(x_3 + x_4) + x_2x_3x_4 + x_1x_3x_4 = 0 \quad (4)$$

$$x_1x_2x_3x_4 = m^2 \quad (5)$$

and additionally,  $x_1 + x_4 = x_2 + x_3$  since the roots are in arithmetic progression. From (1) we get that  $a = 0$ . Then from (2) we get

$$\begin{aligned} (-3r)(3r) + (-r)(4r) + 3r^2 &= -(3m + 2) \\ -9r^2 - 4r^2 + 3r^2 &= -(3m + 2) \\ -10r^2 &= -(3m + 2) \end{aligned} \quad (6)$$

From (5) we have

$$9r^4 = m^2. \quad (7)$$

Now, plugging (6) into (7) we get

$$9 \left( \frac{3m + 2}{10} \right)^2 = m^2$$

which is equivalent to

$$\begin{aligned} 19m^2 - 108m - 36 &= 0 \\ (19m + 6)(m - 6) &= 0 \end{aligned}$$

The solutions to this equation are  $m = 6$  and  $m = -6/19$ , both values satisfying that the resulting roots of the original equation are in arithmetic progression.