MATH 350 FINAL Spring 2005

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section (4) your instructor's name, and (5) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and **BOX IN YOUR FINAL ANSWERS**. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Textbooks, classnotes, crib sheets, or calculators are not permitted.

1. (30 points) Let $f : [a\ ,\ b] \longrightarrow \mathbf{R}$ be a $\mathcal{C}^2$ function on $[a\ ,\ b]$.

   a. Show that $\forall\, x \in [a\ ,\ b], \forall\, h \in \mathbf{R}$ such that $x + 2h \in [a\ ,\ b], \exists\, \theta \in (0\ ,\ 1)$:

   $$f(x) - 2f(x+h) + f(x+2h) = h^2 f''(x + 2\theta h)$$

   b. Deduce the following limit

   $$\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

2. (30 points) Suppose that $x \in \mathbf{R}$, $x_n > 0$, and $\lim_{n\to\infty} x_n = x$. Prove that

   $$\lim_{n\to\infty} \sqrt{x_n} \;=\; \sqrt{x}$$

3. (30 points) Let $[a\ ,\ b]$ be a closed, bounded, and non degenerate interval. Find all functions f that satisfy the following conditions for some fixed $\alpha > 0$:

   - f is continuous and $1-1$ on $[a\ ,\ b]$
   - $f'(x) \neq 0$ for all $x \in (a\ ,\ b)$
   - $f'(x) = \alpha (f^{-1})'(f(x))$ for all $x \in (a\ ,\ b)$

4. (20 points) Apply the mean value theorem to the function $\log|\log|x||$ on a suitable interval to find the following limit

   $$\lim_{n\to\infty} \left(\frac{1}{2\log 2} + \frac{1}{3\log 3} + \cdots + \frac{1}{(n-1)\log(n-1)} + \frac{1}{n\log n} \right)$$

HEY, THERE'S MORE—TURN THE PAGE OVER!

5. (40 points) Let E be a nonempty subset of $\mathbf{R}$ and $f : E \to \mathbf{R}$ is uniformly continuous. Assume (x_n) is Cauchy. Prove that $(f(x_n))$ is Cauchy. What happens if f is continuous only?

6. (20 points) Using the Inverse Function Theorem, prove that

   a. $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$; $\quad \forall x \in (-1\ ,\ 1)$

   b. $(\arctan x)' = \dfrac{1}{1+x^2}$; $\quad \forall x \in (-\infty\ ,\ \infty)$

7. (30 points) Prove that

$$\sin x > x - \frac{x^3}{6}\ ; \quad \forall x \in (0\ ,\ 2\pi]$$