1. (30 points) Let \( f : [a , b] \rightarrow \mathbb{R} \) be a \( C^2 \) function on \([a , b]\).

   a. Show that \( \forall x \in [a , b], \forall h \in \mathbb{R} \) such that \( x + 2h \in [a , b], \exists \theta \in (0 , 1): \)
   \[
   f(x) - 2f(x + h) + f(x + 2h) = h^2 f''(x + 2\theta h)
   \]

   b. Deduce the following limit
   \[
   \lim_{h \to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2}
   \]

2. (30 points) Suppose that \( x \in \mathbb{R}, \ x_n > 0, \) and \( \lim_{n \to \infty} x_n = x \). Prove that
   \[
   \lim_{n \to \infty} \sqrt{x_n} = \sqrt{x}
   \]

3. (30 points) Let \([a , b]\) be a closed, bounded, and non degenerate interval. Find all functions \( f \) that satisfy the following conditions for some fixed \( \alpha > 0 \):
   
   • \( f \) is continuous and \( 1 - 1 \) on \([a , b]\)
   
   • \( f'(x) \neq 0 \) for all \( x \in (a , b) \)
   
   • \( f'(x) = \alpha (f^{-1})'(f(x)) \) for all \( x \in (a , b) \)

4. (20 points) Apply the mean value theorem to the function \( \log |\log |x|| \) on a suitable interval to find the following limit
   \[
   \lim_{n \to \infty} \left( \frac{1}{2\log 2} + \frac{1}{3\log 3} + \cdots + \frac{1}{(n - 1) \log(n - 1)} + \frac{1}{n \log n} \right)
   \]
5. (40 points) Let $E$ be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$ is uniformly continuous. Assume $(x_n)$ is Cauchy. Prove that $(f(x_n))$ is Cauchy. What happens if $f$ is continuous only?

6. (20 points) Using the Inverse Function Theorem, prove that

   a. $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}; \quad \forall x \in (-1, 1)$

   b. $(\arctan x)' = \frac{1}{1+x^2}; \quad \forall x \in (-\infty, \infty)$

7. (30 points) Prove that

   $$\sin x > x - \frac{x^3}{6}; \quad \forall x \in (0, 2\pi]$$