MATH 582C FINAL Spring 2005

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section (4) your instructor's name, and (5) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and **BOX IN YOUR FINAL ANSWERS**. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Textbooks, classnotes, crib sheets, or calculators are not permitted.

1. (30 points) Let f be an integrable function in $\mathbf{R}$. We define $\mathcal{L}(f)(s) = F(s)$, the Laplace transform of f, as follows

$$F(s) = \int_0^{+\infty} f(t)\, e^{-ts}\, dt$$

   a. Prove that $\mathcal{L}\left(H(t-a)f(t-a)\right)(s) = e^{-as}F(s)$; where H is the Heaviside step function and a is a real number.

   b. Use Laplace transform to solve the following boundary value problem

$$\text{(BVP)} \quad \begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0\ ; & x > 0\ ,\ t > 0 \\ u(0,t) = 100H(t-2)\ ; & t > 0 \\ u(x,0) = 0\ ; & x > 0 \end{cases} \qquad (1)$$

   You may use the following property to simplify the expression of u.

$$\mathcal{L}\left(\frac{x}{2c\sqrt{\pi}t^{3/2}} e^{-\dfrac{x^2}{4c^2t}}\right)(s) = e^{-\dfrac{x}{c}\sqrt{s}}$$

   where c is a positive number.

2. (40 points) Consider the following one-way wave equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = f$$

   and the associated *Crank-Nicolson* scheme

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^{n+1} - v_{m-1}^{n+1} + v_{m+1}^n - v_{m-1}^n}{4h} = \frac{f_m^{n+1} + f_m^n}{2}$$

   a. Find the order of accuracy of this scheme.

   b. Analyze the stability of Crank-Nicolson scheme.

3. (30 points) Let f be an integrable function in $\mathbf{R}$. We define $\mathcal{F}f$, the Fourier transform of f, as follows

$$\mathcal{F}f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{\mathrm{i}x\xi}\, dx \; ; \quad \xi \in \mathbf{R}$$

a. Find the Fourier transform of $f(x) = e^{-|x|}$; $\quad x \in \mathbf{R}$

b. let f be an even function. Prove that $\mathcal{F}\mathcal{F}f = f$ and deduce the Fourier transform of $f(x) = \dfrac{1}{1+x^2}$.

c. Use the Fourier transform to solve the following initial value problem

$$\text{(IVP)} \quad \begin{cases} \dfrac{\partial^2 u}{\partial t \partial x} - \dfrac{\partial^2 u}{\partial x^2} = 0 \; ; & -\infty < x < +\infty \; , \; t > 0 \\[2ex] u(x,0) = \sqrt{\dfrac{\pi}{2}} e^{-|x|} \; ; & -\infty < x < +\infty \end{cases} \tag{2}$$

Simplify the expression of the solution as much as you can.

4. (100 points) Consider the one-way wave equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = f \tag{3}$$

and the following Lax-Wendroff- like scheme

$$\begin{aligned} v_m^{n+1} = {} & v_m^n - ak\left(1 - \frac{h^2\delta^2}{6}\right)\delta_0 v_m^n \\ & + \frac{a^2k^2}{2}\left[\left(\frac{4}{3} + a^2\lambda^2\right)\delta^2 v_m^n - \left(\frac{1}{3} + a^2\lambda^2\right)\delta_0^2 v_m^n\right] \\ & + \frac{k}{2}\left(f_m^{n+1} + f_m^n\right) - \frac{ak^2}{2}\delta_0 f_m^n \end{aligned} \tag{4}$$

where δ_0 is the central (first) difference operator defined by

$$\delta_0 v_m^n = \frac{v_{m+1}^n - v_{m-1}^n}{2h} \tag{5}$$

and δ^2 is the central second difference operator defined by

$$\delta^2 v_m^n = \frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2} \tag{6}$$

We assume that the positive parameter $\lambda = \dfrac{k}{h}$ satisfies the following condition

$$|a\lambda| \leq \left(\frac{\sqrt{17}-1}{6}\right)^{\frac{1}{2}} \tag{7}$$

a. Analyze the consistency of the Lax-Wendroff scheme given by Eq.(4) with the one-way wave equation (3).

b. Is this Lax-Wendroff scheme stable?

c. What is the order of accuracy of this Lax-Wendroff scheme? (Note that $O(kh) \leq O(k^2) + O(h^4)$)

d. Show that this Lax-Wendroff scheme is dissipative of order 6.

e. Show that the phase speed $\alpha(h\xi)$ of this Lax-Wendroff scheme satisfies

$$\tan(\alpha(h\xi)k\xi) = \frac{ak\xi\left(1 + O(h\xi)^4\right)}{1 - \frac{1}{2}a^2(k\xi)^2\left[1 + O(h\xi)^4\right]} \tag{8}$$

and deduce its asymptotic behavior as $\xi \to 0$. Is this Lax-Wendroff scheme a dispersive scheme?