1. (30 points) Let \( f \) be an integrable function in \( \mathbb{R} \). We define \( \mathcal{L}(f)(s) = F(s) \), the Laplace transform of \( f \), as follows

\[
F(s) = \int_0^{+\infty} f(t) e^{-ts} \, dt
\]

a. Prove that \( \mathcal{L}(H(t-a)f(t-a))(s) = e^{-as}F(s) \); where \( H \) is the Heaviside step function and \( a \) is a real number.

b. Use Laplace transform to solve the following boundary value problem

\[
\begin{align*}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0 ; & x > 0, \ t > 0 \\
u(0,t) &= 100H(t-2) ; & t > 0 \\
u(x,0) &= 0 ; & x > 0
\end{align*}
\]

(1)

You may use the following property to simplify the expression of \( u \).

\[
\mathcal{L}\left(\frac{x^2}{2c\sqrt{\pi}t^{3/2}}e^{-\frac{x^2}{4c^2t}}\right)(s) = e^{-\frac{x}{c}\sqrt{s}}
\]

where \( c \) is a positive number.

2. (40 points) Consider the following one-way wave equation

\[
\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = f
\]

and the associated Crank-Nicolson scheme

\[
\frac{v^{n+1}_m - v^n_m}{k} + a \frac{v^{n+1}_{m+1} - v^{n+1}_{m-1} + v^n_{m+1} - v^n_{m-1}}{4h} = \frac{f^{n+1}_m + f^n_m}{2}
\]

a. Find the order of accuracy of this scheme.

b. Analyze the stability of Crank-Nicolson scheme.
3. (30 points) Let \( f \) be an integrable function in \( \mathbb{R} \). We define \( Ff \), the Fourier transform of \( f \), as follows

\[
Ff(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{ix\xi} \, dx ; \quad \xi \in \mathbb{R}
\]

a. Find the Fourier transform of \( f(x) = e^{-|x|} ; \quad x \in \mathbb{R} \)

b. Let \( f \) be an even function. Prove that \( F F f = f \) and deduce the Fourier transform of \( f(x) = \frac{1}{1 + x^2} \).

c. Use the Fourier transform to solve the following initial value problem

\[
\begin{cases}
\frac{\partial^2 u}{\partial t \partial x} - \frac{\partial^2 u}{\partial x^2} = 0 ; & -\infty < x < +\infty , \quad t > 0 \\
u(x, 0) = \sqrt{\frac{\pi}{2}} e^{-|x|} ; & -\infty < x < +\infty
\end{cases}
\] (IVP) (2)

Simplify the expression of the solution as much as you can.

4. (100 points) Consider the one-way wave equation

\[
\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = f
\] (3)

and the following Lax-Wendroff-like scheme

\[
v_{m}^{n+1} = v_{m}^{n} - ak \left( 1 - \frac{h^2 \delta^2}{6} \right) \delta_0 v_{m}^{n} \\
+ \frac{a^2 k^2}{2} \left[ \left( \frac{4}{3} + a^2 \lambda^2 \right) \delta^2 v_{m}^{n} - \left( \frac{1}{3} + a^2 \lambda^2 \right) \delta^2_0 v_{m}^{n} \right] \\
+ k \frac{1}{2} (f_{m}^{n+1} + f_{m}^{n}) - \frac{ak^2}{2} \delta_0 f_{m}^{n}
\] (4)

where \( \delta_0 \) is the central (first) difference operator defined by

\[
\delta_0 v_{m}^{n} = \frac{v_{m+1}^{n} - v_{m-1}^{n}}{2h}
\] (5)

and \( \delta^2 \) is the central second difference operator defined by

\[
\delta^2 v_{m}^{n} = \frac{v_{m+1}^{n} - 2v_{m}^{n} + v_{m-1}^{n}}{h^2}
\] (6)

We assume that the positive parameter \( \lambda = \frac{k}{h} \) satisfies the following condition

\[
|a\lambda| \leq \left( \frac{\sqrt{17} - 1}{6} \right) \frac{1}{2}
\] (7)
a. Analyze the consistency of the Lax-Wendroff scheme given by Eq.(4) with the one-way wave equation (3).

b. Is this Lax-Wendroff scheme stable?

c. What is the order of accuracy of this Lax-Wendroff scheme? (Note that $O(kh) \leq O(k^2) + O(h^4)$)

d. Show that this Lax-Wendroff scheme is dissipative of order 6.

e. Show that the phase speed $\alpha(h\xi)$ of this Lax-Wendroff scheme satisfies

$$\tan(\alpha(h\xi)k\xi) = \frac{ak\xi (1 + O(h\xi)^4)}{1 - \frac{1}{2}a^2(k\xi)^2 [1 + O(h\xi)^4]}$$

and deduce its asymptotic behavior as $\xi \to 0$. Is this Lax-Wendroff scheme a dispersive scheme?