MATH 582B FINAL Spring 2004

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section (4) your instructor's name, and (5) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and **BOX IN YOUR FINAL ANSWERS**. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Textbooks, classnotes, or crib sheets are not permitted.

1. (30 points) For a fixed positive number h, let $P_2(x)$ be the Lagrange polynomial interpolating $f(x)$ at $x = 0, h, 2h$.

   a. Use P_2 to derive a numerical integration formula $I_h(f)$ for $I(f) = \int_0^{3h} f(x)\,dx$.

   b. Find the sign of $\pi(t) = t(t-1)(t-2)$ for $t \in [0\ ,\ 3]$.

   c. Assume that $f \in \mathcal{C}^3([0\ ,\ 3h])$. Find $E_h(f) = I(f) - I_h(f)$

2. (40 points) Assume that $F : [\alpha\ ,\ \beta] \longrightarrow \mathbf{R}$ continuously differentiable. Our aim is to prove that

$$\lim_{\lambda \to \infty} \int_\alpha^\beta F(x)|\sin(\lambda x)|\,dx = \frac{2}{\pi}\int_\alpha^\beta F(x)\,dx \qquad (*)$$

   a. We set

$$f(x) = |\sin x|\ ; \quad \forall x \in \mathbf{R}$$

   Prove that the Fourier series corresponding to f converges uniformly to f.

   b. For $\mu > 0$, we set

$$J(\mu) = \int_\alpha^\beta F(x)\cos(\mu x)\,dx$$

   Prove the existence of a positive constant M, which does not depend on μ, such that

$$|J(\mu)| \le \frac{M}{\mu}\ ; \quad \forall \mu > 0$$

   c. Establish property $(*)$.

3. (60 points) Let $y(x)$ be the solution of the following initial value problem (IVP)

$$\text{(IVP)} \quad \left\{ \begin{array}{rcl} y'(x) &=& f(x\ ,\ y(x)) \\ \\ y(x_0) &=& \tilde{y}_0 \end{array} \right. \qquad (1)$$

   and let $(y_n)_{n\in\mathbf{N}}$ be the sequence obtained with Euler's algorithm (EA)

$$\text{(EA)} \quad \left\{ \begin{array}{l} y_{n+1} = y_n + hf(x_n\ ,\ y_n)\ ; \quad n \ge 0 \\ \\ y_0 = \tilde{y}_0 \end{array} \right. \qquad (2)$$

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Assume that:

i. $y(x)$, the solution of (IVP), is three times continuously differentiable.

ii. $\dfrac{\partial f}{\partial y}(x,y)$ and $\dfrac{\partial^2 f}{\partial y^2}(x,y)$ are continuous and bounded for $x_0 \leq x \leq b$ and $y \in \mathbf{R}$.

Our aim is to derive the following asymptotic estimate of the error in Euler's method:

$$Y(x_n) - y_n \;=\; D(x_n)h + O(h^2) \qquad (**)$$

where $D(x)$ is the solution of the following intermediate initial value problem

$$\text{(IIVP)} \quad \begin{cases} D'(x) \;=\; \dfrac{\partial f}{\partial y}(x\ ,\ y(x))D(x) + \dfrac{1}{2}Y^{(2)}(x) \\ \\ D(x_0) \;=\; 0 \end{cases} \tag{3}$$

a. Show that

$$\max_{0\leq n\leq N(h)} |y(x_n) - y_n| \;\leq\; \frac{e^{(b-x_0)K} - 1}{K}\frac{h}{2}||Y^{(2)}||_\infty \tag{4}$$

where K is the Lipschitz constant corresponding to f with respect to its second variable.

b. We set $e_n = y(x_n) - y_n$. Using Taylor's expansion, prove that

$$\begin{cases} e_{n+1} \;=\; \left[1 + +h\dfrac{\partial f}{\partial y}(x_n\ ,\ y(x_n))\right] e_n + \dfrac{h^2}{2}y^{(2)}(x_n) + B_n\ ; \quad n \geq 0 \\ \\ e_0 \;=\; 0 \end{cases} \tag{5}$$

where

$$B_n \;=\; \frac{h^3}{6}y^{(3)}(\xi_n) - \frac{h}{2}\frac{\partial^2 f}{\partial y^2}(x_n, t_n)e_n^2\ ; \quad n \geq 0 \tag{6}$$

for some ξ_n between x_n and x_{n+1} and for some t_n between $y(x_n)$ and y_n.

c. Show that $B_n \;=\; O(h^3)$, where B_n is defined in part (b) (See Eq.(6)).

d. Let g_n represents the dominant part of the error in Eq.(5). Hence, g_n is defined implicitely by

$$\begin{cases} g_{n+1} \;=\; \left[1 + h\dfrac{\partial f}{\partial y}(x_n\ ,\ y(x_n))\right] g_n + \dfrac{h^2}{2}y^{(2)}(x_n)\ ; \quad n \geq 0 \\ \\ g_0 \;=\; 0 \end{cases} \tag{7}$$

Show that $g_n \;=\; h\delta_n$ where δ_n is obtained by applying Euler's method for solving the intermediate initial value problem (IIVP) (See Eq.(3)).

e. Deduce that $g_n \;=\; D(x_n)h + O(h^2)\ ; \quad 0 \leq n \leq N(h)$

f. We set $k_n \;=\; e_n - g_n\ ; \quad 0 \leq n \leq N(h)$. Prove that $k_n \;=\; O(h^2)$

g. Conclude.

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h. Application: Consider the following initial value problem

$$\begin{cases} y'(x) &= -y(x) \\ \\ y(0) &= 1 \end{cases} \tag{8}$$

i. Find $y(x)$ the solution of the initial value problem given by Eq.(8).

ii. Find $D(x)$ defined by Eq.(3) corresponding to the initial value problem given by Eq.(8).

iii. Deduce the asymptotic formula for the error in Euler's method in the particular case of the initial value problem given by Eq.(8).

iv. Conclude.

4. (30 points) Let $f \in \mathcal{C}^m([0\ ,\ 1])$ with $m = 2$ or 3. We want to use the central difference quotient with step size h to approximate $f^{'}(x)$ i.e.

$$f^{'}(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

a. Evaluate the accuracy $\mathcal{E}_m(h)$ of this approximation defined by

$$\mathcal{E}_m(h) = \sup_{x \in [0\ ,\ 1]} \left| f^{'}(x) - \frac{f(x+h) - f(x-h)}{2h} \right|$$

Conclude.

b. We set $m = 3$. The function f is perturbed as follows

$$f_n^\delta(x) = f(x) + \delta \sin(\frac{nx}{\delta})$$

where $\delta \in]0\ ,\ 1[$ and $n \in \mathbf{N}$; $n \geq 2$.

i. Evaluate the error $\mathcal{E}_n^\delta(h)$ defined by

$$\mathcal{E}_n^\delta(h) = \sup_{x \in [0\ ,\ 1]} \left| f^{'}(x) - \frac{f_n^\delta(x+h) - f_n^\delta(x-h)}{2h} \right|$$

ii. For a given perturbation level δ, describe the behavior of the perturbed error $\mathcal{E}_n^\delta(h)$ as the step size h tends to zero.

iii. Assume that δ is known. What could be a "good" choice of the step size h to minimize the perturbed error $\mathcal{E}_n^\delta(h)$?

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5. (40 points) Consider the following multistep method

$$\text{(MM)} \quad \begin{cases} y_{n+1} = 3y_n - 2y_{n-1} + \dfrac{h}{2}\left[f(x_n\ ,\ y_n) - 3f(x_{n-1}\ ,\ y_{n-1})\right]\ ; \quad n \geq 0 \\ \\ y_0\ ,\ y_1 \text{ are given.} \end{cases} \tag{9}$$

a. Is this method consistent?

b. Is this method stable? Does the sequence $(y_n)_{n\in\mathbf{N}}$ converge?

c. Find the order of this method.

d. Deduce an explicit expression of the truncation error $T_n(y)$ assuming that y is three times continuously differentiable.

e. Consider solving the following initial problem (IVP)

$$\text{(IVP)} \quad \begin{cases} y'(x) = 0 \\ \\ y(x_0) = 0 \end{cases} \tag{10}$$

  i. Find the solution $y(x)$ of IVP.

  ii. Using $y_0 = y_1 = 0$, find the numerical solution $y_n\ ;\ n \geq 0$.

  iii. Perturb the initial data $z_0 = \dfrac{\epsilon}{2}$, $z_1 = \epsilon$ for some $\epsilon \neq 0$. Find the corresponding numerical solution $z_n\ ;\ n \geq 0$.

  iv. Evaluate $\max\limits_{0\leq n\leq N(h)} |y_n - z_n|$ and find the limit when $h \to 0$. Conclude.